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ON THE CONTINUOUS COHOMOLOGY OF THE LIE ALGEBRA OF VECTOR FIELDS ASSOCIATED WITH NON-TRIVIAL COEFFICIENTS

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§1. Let M be a smooth manifold and L_M the topological Lie algebra of all smooth vector fields on M. Recently Haefliger ([4]) proved the Bott conjecture, which states that the continuous cohomology of L_M with trivial coefficients is isomorphic to the singular cohomology of the space of cross-sections of a certain fibre bundle over M. As for the case associated with the Lie derivative action on a tensor space A on M, Losik ([5], [7]) has computed the cohomology of a certain subcomplex (called diagonal) of the standard cochain complex, and Reshetnikov ([9]) has announced partial results concerning the total continuous cohomology $H^*(L_M, A)$. In this note we state a theorem which reduces essentially the caluculation of $H^*(L_M, A)$ to that of the diagonal cohomology $H^*_\Delta(L_M, A)$ and the Gelfand-Fuks cohomology $H^*(L_M)$.

Details will be published elsewhere.

§2. Let W be a topological L_M-algebra.

Let $C^p(L_M, W)(p > 0)$ be the space of all continuous alternating p-forms on L_M with values in W and $C^0(L_M, W) = W$. For $\omega \in C^p(L_M, W)(p \geq 1)$, we define $d\omega \in C^{p+1}(L_M, W)$ by

$$d\omega(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^i X_i\, \omega(X_1, \cdots, \hat{X}_i, \cdots, X_{p+1})$$
$$+ \sum_{i<j} (-1)^{i+j-1} \omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{p+1})$$

$(X_1, \cdots, X_{p+1} \in L_M)$, and for $\omega \in C^0(L_M, W) = W$, $d\,\omega(X) = X\omega$ $(X \in L_M)$.

We also define $\omega \wedge \eta \in C^{p+q}(L_M, W)$ for $\omega \in C^p(L_M, W)$ and $\eta \in C^q(L_M, W)$ by

$$(\omega \wedge \eta)(X_1, \cdots, X_{p+q})$$
$$= \sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_q}} (-1)^{i_1+\cdots+i_p-\frac{p(p+1)}{2}} \omega(X_{i_1}, \cdots, X_{i_p})\eta(X_{j_1}, \cdots, X_{j_q})$$

$(X_1, \ldots, X_{p+q} \in L_M)$. Then $C^*(L_M, W) = \{\oplus C^p(L_M, W), d\}$ turns out to be a commutative differential graded algebra (DG-algebra for short). Putting $W = \mathbb{R}$, $W = C^\infty(M)$, we get two DG-algebras $C^*(L_M, \mathbb{R})$ and $C^*(L_M, C^\infty(M))$.

Furthermore, put

$$C^0_\Delta(L_M, C^\infty(M)) = C^\infty(M),$$

$$C^p_\Delta(L_M, C^\infty(M)) = \{\omega \in C^p(L_M, C^\infty(M));\ \operatorname{supp} \omega(X_1, \ldots, X_p)$$
$$\subset \bigcap_{i=1}^{p} \operatorname{supp} X_i\ (X_1, \cdots, X_p \in L_M)\} \quad (p > 0).$$

Then $C^*_\Delta(L_M, C^\infty(M)) = \oplus C^p_\Delta(L_M, C^\infty(M))$ is a DG-subalgebra of $C^*(L_M, C^\infty(M))$.

We note that the de Rham complex Ω^*_M of M can be naturally identified with a DG-subalgebra of $C^*_\Delta(L, C^\infty(M))$.

§3. Let $C^*(L_M, \Omega^*_M) = C^*(L_M, \mathbb{R}) \hat{\otimes} \Omega^*_M$ be the completed tensor product of DG-algebras, which is again a DG-algebra. Just as before we get a DG-subalgebra $C^*_\Delta(L_M, \Omega^*_M)$ of $C^*(L_M, \Omega^*_M)$ which consists of support preserving cochains. The inclusion map $\iota : \mathbb{R} \hookrightarrow C^\infty(M)$ being an L_M-homomorphism, there is a DG-algebra homomorphism $\iota^* : C^*(L_M, \mathbb{R}) \longrightarrow C^*(L_M, C^\infty(M))$. Consider $\kappa = \iota_* \hat{\otimes} j : C^*(L_M, \Omega^*_M) = C^*(L_M, \mathbb{R}) \hat{\otimes} \Omega^*_M \longrightarrow C^*(L_M, C^\infty(M))$, where $j : \Omega^*_M \hookrightarrow C^*(L_M, C^\infty(M))$. It is easy to see that the image of κ is contained in $C^*_\Delta(L_M, C^\infty(M))$. Thus we get the following commutative diagram of DG-algebra homomorphisms :

$$\begin{array}{ccc} C^*(L_M, C^\infty(M)) & \longleftarrow & C^*(L_M, \Omega^*_M) \\ \uparrow & & \uparrow \\ C^*_\Delta(L_M, C^\infty(M)) & \longleftarrow & C^*_\Delta(L_M, \Omega^*_M). \end{array} \tag{1}$$

From this there arises a natural homomorphism of graded algebras :

$$\alpha : \mathrm{Tor}^{C^*_\Delta(L_M, \Omega^*_M)}(C^*(L_M, \Omega^*_M), C^*_\Delta(L_M, C^\infty(M)) \longrightarrow H^*(L_M, C^\infty(M)),$$

where Tor denotes the differential torsion functor (cf [1]) and $H^*(L_M, C^\infty(M))$ the cohomology algebra of the DG-algebra $C^*(L_M, C^\infty(M))$.

<u>Theorem I.</u> α <u>is an isomorphism if</u> $\dim_{\mathbb{R}} H^*(M, \mathbb{R}) < \infty$.

§4. We recall the results of Losik ([5]), Guillemin ([3]) and Losik ([6]) and Haefliger ([4]), rewriting in more suitable form for our purpose.

Let $a(n)$ be the topological Lie algebra of formal vector fields of n-variables and $a_0(n)$ the subalgebra of $a(n)$ consisting of elements without constant terms in the coefficients. We get two DG-algebras $C^*(a(n), \mathbb{R})$ and $C^*(a_0(n), \mathbb{R})$ associated with the trivial module $\mathbb{R}$. Let S^*V and S^*U be minimal models for $C^*(a(n), \mathbb{R})$ and $C^*(a_0(n), \mathbb{R})$ respectively. Here $U = \mathbb{R}\theta_1 \oplus \cdots \oplus \mathbb{R}\theta_n$ (deg $\theta_i = 2i - 1$) and S^*U is the exterior algebra over U with trivial differentials. (As for S^*V, see [4]). Then

Theorem L ([5]). *There is a quasi-isomorphism*

$$\Omega^*_M \underset{\tau}{\otimes} S^*U \longrightarrow C^*_\Delta(L_M, C^\infty(M))$$

which is Ω^*_M*-linear. Here* $\Omega_M \underset{\tau}{\otimes} S^*U$ *is the twisted tensor product of* Ω^*_M *and* S^*U *defined by the twisting* $\tau(\theta_i) = p_i$, p_i *being the i-th Pontrjagin form of* M *with respect to a Riemannian metric.*

(For the notion of twisted tensor product, see [4].)

Recall that a DG-algebra homomorphism is said to be a quasi-isomorphism if it induces an isomorphism on cohomology level.

Theorem GL ([3], [6]). *There are a twisted tensor product* $\Omega_M^* \underset{\sigma}{\otimes} S^*V$ *and a quasi-isomorphism*

$$\beta \; : \; \Omega_M^* \underset{\sigma}{\otimes} S^*V \longrightarrow C_\Delta^*(L_M, \Omega_M^*),$$

which is Ω_M^**-linear*.

Let $\Omega_M^* \otimes V$ be the graded vector space such that $\deg(\omega \otimes v) = -\deg\omega + \deg v$. Let $S^*(\Omega_M^* \otimes V)$ be the graded algebra of graded commutative continuous forms on $\Omega_M^* \otimes V$.

Theorem H ([4]). *There are a DG-algebra structure on* $S^*(\Omega_M^* \otimes V)$ *and a quasi-isomorphism*

$$\gamma \; : \; S^*(\Omega_M^* \otimes V) \longrightarrow C^*(L_M, \mathbb{R}).$$

Let $\varepsilon : \Omega_M \underset{\sigma}{\otimes} S^*V \longrightarrow \Omega_M \hat{\otimes} S^*(\Omega_M \otimes V)$ be the algebraic evaluation map defined in [4]. Let $\lambda : S^*V \longrightarrow S^*U$ be the DG-algebra homomorphism corresponding to

$$\iota^* \; : \; C^*(a(n), \mathbb{R}) \longrightarrow C^*(a_0(n), \mathbb{R})$$

induced by the inclusion $\iota : a_0(n) \hookrightarrow a(n)$.

Remark. It is easy to see $\lambda(S^1V) = 0$.

Lemma 1. *We have the following commutative diagram of DG-algebra homomorphisms:*

$$\begin{array}{ccccc}
\Omega_M^* \hat{\otimes} S^*(\Omega^* \otimes V) & \xleftarrow{\ \varepsilon\ } & \Omega_M^* \underset{\sigma}{\otimes} S^*V & \xrightarrow{\ id \otimes \lambda\ } & \Omega_M \underset{\tau}{\otimes} S^*U \\
\downarrow{\scriptstyle id \hat{\otimes} \gamma} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} \\
\Omega_M^* \hat{\otimes} C^*(L_M, \mathbb{R}) & \longleftarrow & C_\Delta^*(L_M, \Omega_M^*) & \longrightarrow & C_\Delta^*(L_M, C^\infty(M)).
\end{array}$$

Recall the following

Proposition ([1]). *Suppose the following commutative diagram of DG-algebra homomorphisms is given*:

$$\text{(2)}\qquad \begin{array}{ccccc} M & \longleftarrow & U & \longrightarrow & N \\ \lambda\downarrow & & \mu\downarrow & & \nu\downarrow \\ M' & \longleftarrow & U' & \longrightarrow & N' \end{array}$$

where λ, μ *and* ν *are quasi-isomorphisms. Then the induced map* $\mathrm{Tor}^{U}(M, N) \longrightarrow \mathrm{Tor}^{U'}(M', N')$ *is an isomorphism.*

Thus we get

Theorem I'. *There is an isomorphism of graded algebras*:

$$H^*(L_M, C^\infty(M)) \cong \mathrm{Tor}^{\Omega^*_M \otimes_\sigma S^*V}(\Omega^*_M \hat{\otimes} S^*(\Omega^*_M \otimes V),\ \Omega_M \underset{\tau}{\otimes} S^*U).$$

§5. We give a geometric paraphrase of Theorem I'.

Let B be the principal $U(n)$-bundle associated to the complexification of the real tangent bundle of M. Let $\hat{E}U_n$ be the restriction of the universal principal $U(n)$-bundle to the 2n-skelton of the base BU_n with respect to the cell decomposition with even-dimensional cells. Put $E = B \underset{U(n)}{\times} \hat{E}U_n$. Fixing a fibre inclusion mapping $U(n) \hookrightarrow \hat{E}U_n$, we get an inclusion mapping: $B \hookrightarrow E$. Let $\Gamma(E)$ be the space of all continuous sections of E with the compact open topology. Let $\varepsilon : M \times \Gamma(E) \longrightarrow E$ be the evaluation mapping.

Let $X \longmapsto A^*(X)$ be any contravariant functor which associates to each topological space a commutative DG-algebra $A^*(X)$ over $\mathbb{R}$ such that $H(A^*(X)) \cong H^*(X, \mathbb{R})$ (cf. [11]). Corresponding to the diagram of topological spaces:

$$M \times \Gamma(E) \xrightarrow{\varepsilon} E \longleftarrow B$$

We get a diagram of DG-algebras:

$$A^*(M \times \Gamma(E)) \longleftarrow A^*E \longrightarrow A^*B.$$

We say that two triples of DG-algebras $T = \{M \leftarrow U \rightarrow N\}$ and $T' = \{M' \leftarrow U' \rightarrow N'\}$ are equivalent if there is a sequence of triples $T_0 = T, T_1, \cdots, T_{n-1}, T_n = T'$ such that for each i $(0 \leqq i \leqq n-1)$ there is a quasi-isomorphism $T_i \longrightarrow T_{i+1}$ or $T_{i+1} \longrightarrow T_i$. Here, a quasi-isomorphism $\{M \leftarrow U \rightarrow N\} \longrightarrow \{M' \leftarrow U' \rightarrow N'\}$ is simply a commutative diagram (2) such that λ, μ and ν are quasi-isomorphisms. Note that if T and T' are equivalent then $\mathrm{Tor}^U(M, N) \cong \mathrm{Tor}^{U'}(M', N')$.

<u>Lemma 2.</u> <u>Triples</u> $T_M = \{\Omega_M \hat{\otimes} S^*(\Omega_M \otimes V) \leftarrow \Omega_M \underset{\sigma}{\otimes} S^*V \longrightarrow \Omega_M \underset{\tau}{\otimes} S^*V\}$ <u>and</u> $\{A^*(M \times \Gamma(E)) \longleftarrow A^*E \longrightarrow A^*B\}$ <u>are equivalent</u>.

On the other hand, we can show the following

<u>Lemma 3</u>. $\varepsilon : M \times \Gamma(E) \longrightarrow E$ <u>is a Serre fibering</u>.

Recall the following

<u>Theorem</u> (Eilenberg-Moore-Gugenheim [1], [2].) <u>Let</u> $X \longrightarrow E$ <u>be a Serre fibering and</u> $\iota : B \longrightarrow E$ <u>a mapping</u>. <u>Let</u> $Y = \iota^* X$ <u>be the induced fibering</u>. <u>Assume that</u> $\pi_1(E) = 0$. <u>Then we have an isomorphism of graded algebras</u>:

$$\mathrm{Tor}^{A^*E}(A^*(M \times \Gamma(E)),\ A^*B) \cong H^*(Y,\ R).$$

Let Y be the fibering over B induced from $\varepsilon : M \times \Gamma(E) \longrightarrow E$ by $B \hookrightarrow E$. Then, in view of $\pi_1(\hat{E}U_n) = 0$, we have the following

<u>Theorem II</u>. If $\dim_R H^*(M, \mathbb{R}) < \infty$ and $\pi_1(M) = 0$, then

$$H^*(L_M,\ C^\infty(M)) \cong H^*(Y,\ \mathbb{R}).$$

§6. We consider examples.

Let $M = \mathbb{R}^n$. Since $\mathbb{R} \hookrightarrow \Omega^*_{\mathbb{R}^n}$ is a quasi-isomorphism, it is easy to see that the triple $T_{\mathbb{R}^n}$ is equivalent to $\{S^*V \xleftarrow{\ \mathrm{id}\ } S^*V \longrightarrow S^*U\}$. Hence

$$H^*(L_{\mathbb{R}^n},\ C^\infty(\mathbb{R}^n)) \cong \mathrm{Tor}^{S^*V}(S^*V,\ S^*U) \cong S^*U.$$

Thus

<u>Corollary 1</u>. $H^*(L_{\mathbb{R}^n},\ C^\infty(\mathbb{R}^n)) \cong H^*_\Delta(L_{\mathbb{R}^n},\ C^\infty(\mathbb{R}^n)) \cong S^*U.$

Let $M = S^1$. Then it is easy to see that the triple T_{S^1} can be replaced by

$$T' = \{S^*(t,\ \sigma,\ \xi) \xleftarrow{\ \alpha\ } S^*(t,\ \sigma) \xrightarrow{\ \beta\ } S^*(t,\ \theta)\}$$

where $\deg t = \deg \theta = 1$, $\deg \sigma = 3$, $\deg \xi = 2$, $dt = d\theta = d\sigma = d\xi = 0$, $\alpha(t) = t$, $\alpha(\sigma) = t\xi + \sigma$, $\beta(t) = t$, $\beta(\sigma) = 0$. Here $S^*(x, y, \cdots)$ denotes the free anti-commutative graded algebra generated by x, y, $\cdots$. We can check immediately that T' is equivalent to

$$T'' = \{ S^*(t, \sigma, \xi) \xleftarrow{\tilde{\alpha}} S^*(t,\sigma) \xrightarrow{\beta} S^*(t, \theta) \}$$

where $\tilde{\alpha}(t) = t$, $\tilde{\alpha}(\sigma) = \sigma$. Thus

$$\mathrm{Tor}^{S^*(t,\sigma)}(S^*(t, \sigma, \xi), S^*(t, \theta)) \cong S^*(t, \theta, \xi).$$

Corollary 2. $H^*(L_{S^1}, C^\infty(S^1)) \cong S^*(t, \theta, \xi)$, where $\deg t = \deg \theta = 1$, $\deg \xi = 2$.

§7. Finally we consider the general case.

Let $G^k (k \geq 1 \cdots)$ be the Lie group of k-jets at the origin 0 of diffeormorphisms of $\mathbb{R}^n$ fixing 0. Let A be a finite dimensional real G^k-module. For a smooth manifold M of dimension n, we denote by $S^k M$ the G^k-principal bundle canonically associated to M. Put $\alpha = S^k M \times_{G^k} A$. Then α is a Diff(M)-bundle over M. Hence $A_M = \Gamma^\infty(\alpha)$ can be naturally regarded as a topological L_M-module. We can then define $C^*(L_M, A_M)$, $C^*_\Delta(L_M, A_M)$, and $H^*(L_M, A_M)$ just as in §2. The natural pairing $C^\infty(M) \otimes A_M \longrightarrow A_M$ gives rise to a differential graded $C^*_\Delta(L_M, C^\infty(M))$-module structure on $C^*_\Delta(L_M, A_M)$. Using the DG-algebra homomorphism $\kappa : C^*_\Delta(L_M, \Omega^*_M) \longrightarrow C^*_\Delta(L_M, C^\infty(M))$, we regard $C^*_\Delta(L_M, A_M)$ as a differential greaded $C^*_\Delta(L_M, \Omega^*_M)$-module. On the other hand, the G^k-module A gives rise to an $a_0(n)$-module A canonically (cf [10]).

Theorem III. If $\dim_{\mathbb{R}} H^*(M, \mathbb{R}) < \infty$ and $\dim_{\mathbb{R}} H^i(a_0(n), A) < \infty$ $(i = 0, 1, \cdots)$, then there is an isomorphism of graded vector space:

$$H^*(L_M, A_M) \cong \mathrm{Tor}^{C^*_\Delta(L_M, \Omega^*_M)}(C^*(L_M, \Omega^*_M), C^*_\Delta(L_M, A_M)).$$

Remark. Let $G^1 \longrightarrow G^k$ be a lifting of $G^k \longrightarrow G^1$. Then A can be regarded as a G^1-module. If A is completely reducible G^1-module, it is east to see that $\dim_{\mathbb{R}} H^i(a_0(n), \mathbb{R}) < \infty$ $(i = 0, 1, 2, \cdots)$.

Under the hypotheses of Theorem II, we have the following corollaries.

Corollary 3. There is a spectral sequence converging to $H^*(L_M, A_M)$ whose E_2-term is

$$\mathrm{Tor}^{H(C^*_\Delta(L_M, \Omega^*_M))}(H(C^*(L_M, \Omega^*_M)), H(C^*_\Delta(L_M, A_M))).$$

Corollary 4. If $H^*(C^*_\Delta(L_M, A_M)) = 0$, then $H^*(L_M, A_M) = 0$. Especially, $H^*(L_M, L_M) = 0$, where L_M is the L_M-module defined by the adjoint action.

Corollary 5. $\dim_{\mathbb{R}} H^i(L_M, A_M) < \infty$ $(i = 0, 1, 2, \cdots)$. (cf [9]).

References

[1] Gugenheim, V.K.A.M., May, J.P. : On the theory and applications of differential torsion products. Memoirs of AMS, No. 142 (1974).

[2] Gugenheim, V.K.A.M. : On the multiplicative structure of the de Rham theory. J. Differential geometry 11, 309-314 (1976).

[3] Guillemin, V.W., Cohomology of vector fields on a manifold. Advances in Math. 10, 192-220 (1973).

[4] Haefliger, A. : Sur la cohomologie de Gelfand-Fucks. Differential topology and geometry, Dijon 1974, pp 121-152. Lecture notes in Math. Vol. 484 (1975).

[5] Losik, M.V. : On the cohomologies of infinite-dimensional Lie algebras of vector fields, Funkt. Anal. i Ego Pril., 4 (2), 43-53 (1970).

[6] Losik, M.V. : Cohomologies of Lie algebras of vector fields with coefficients in a trivial unitary representation. Funkt. Anal. i Ego Pril., 6 (1), 24-36 (1972).

[7] Losik, M.V. : On cohomologies of Lie algebras of vector fields with nontrivial coefficients. Funkt. Anal. i Ego Pril., 6 (4), 44-46 (1972).

[8] Reshetnikov, V.N. : Cohomology of Lie algebras of vector filelds on a manifold vanishing at a given point. Uspehi Mat. Nauk 27, No. 1 (163), 251-252 (1972).

[9] Reshetnikov, V.N. : On the cohomology of the Lie algebra of vector fields on a manifold with nontrivial coefficients. Dokl. Akad. Nauk SSSR Tom 208, No. 5, 234-240 (1973).

[10] Shiga, K. : Cohomology of Lie algebras over a manifold, I. J. Math. Soc. Japan Vol. 26, No.2, 324-361 (1974).

[11] Swan, R.G. : Thom's theory of differential forms on simplicial sets. Topology 14, 271-273 (1975).