

THE ENDOMORPHISM SEMIGROUP OF A SEMIGROUP  
AND ITS APPLICATION

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An endomorphism of a semigroup is one of important tools in the study of semigroups. In this paper, we represent the endomorphism semigroup by means of a wreath product and the greatest semilattice homomorphic image. And we apply to a left inverse extension of certain left regular band by a semilattice of groups.

1. The endomorphism semigroup of semigroups.

Let  $S$  be a semigroup. The set  $\text{End}(S)$  of all endomorphisms of  $S$  is a semigroup under the composition. The set  $\text{Aut}(S)$  of all automorphisms of  $S$  is a subgroup of  $\text{End}(S)$ .  $\text{End}(S)$  [ $\text{Aut}(S)$ ] is called the endomorphism semigroup [the automorphism group] of  $S$ . Now, let  $S = \bigcup \{S_\alpha : \alpha \in \Gamma\}$  be the greatest semilattice decomposition of  $S$ . Let  $H(S) = \bigcup \{\text{Hom}(S_\alpha, S_\beta) : \alpha, \beta \in \Gamma\} \cup \{\square\}$  be a semigroup under composition, where  $\text{Hom}(S_\alpha, S_\beta)$  is the set of all homomorphisms of  $S_\alpha$  to  $S_\beta$  and  $\square$  is the empty mapping. Consider the set  $H(S) \text{ wr } \text{End}(\Gamma) = \{(k, \theta) : k : \Gamma \rightarrow H(S), \theta \in \text{End}(\Gamma)\}$  and define a multiplication in  $H(S) \text{ wr } \text{End}(\Gamma)$  as follows:

$$(k, \theta)(k', \theta') = (k \cdot \theta k', \theta \theta'),$$

where  $\gamma k = k_\gamma$  and  $(k \cdot \theta k') = k_\gamma k'_\theta$  for all  $\gamma \in \Gamma$ . Then

$H(S) \text{ wr } \text{End}(\Gamma)$  is a semigroup and we call it the wreath product of  $H(S)$  and  $\text{End}(\Gamma)$ .

For each  $\phi \in \text{End}(S)$ , we define mappings  $k : \Gamma \rightarrow H(S)$  and  $\theta : \Gamma \rightarrow \Gamma$  as follows:

$$k_\alpha = \phi|_{S_\alpha}, \quad S_\alpha \phi \subseteq S_{\alpha\theta} \quad \text{for all } \alpha \in \Gamma.$$

Then a mapping  $f : \phi \mapsto (k, \theta)$  is an isomorphism of  $\text{End}(S)$  into  $H(S) \text{ wr } \text{End}(\Gamma)$ . Thus we have the following lemma.

Lemma 1.  $\text{End}(S) \cong_{\text{in}} H(S) \text{ wr } \text{End}(\Gamma)$ .

Next, we shall consider the condition that  $(k, \theta) \in H(S) \text{ wr } \text{End}(\Gamma)$  is contained in the image  $\text{Im}(f)$  of  $f$ .

Lemma 2. Let  $(k, \theta) \in H(S) \text{ wr } \text{End}(\Gamma)$ . Then  $(k, \theta) \in \text{Im}(f)$  if and only if

- (i)  $k_\alpha \in \text{Hom}(S_\alpha, S_{\alpha\theta})$  for all  $\alpha \in \Gamma$ ,
- (ii)  $(xy) \cdot k_{\alpha\beta} = [x \cdot k_\alpha][y \cdot k_\beta]$  for all  $x \in S_\alpha, y \in S_\beta$ .

Now, we shall call this condition an H-property and let  $H(S) \widetilde{\text{wr}} \text{End}(\Gamma) = \{(k, \theta) \in H(S) \text{ wr } \text{End}(\Gamma) : (k, \theta) \text{ has an H-property}\}$ . By above lemmas we obtain the following theorem.

Theorem 3. Let  $S = \bigcup \{S_\alpha : \alpha \in \Gamma\}$  be the greatest semilattice decomposition of a semigroup  $S$ . Then

$$\text{End}(S) \cong H(S) \widetilde{\text{wr}} \text{End}(\Gamma).$$

From now on, we identify elements of  $\text{End}(S)$  with elements of  $H(S) \widetilde{\text{wr}} \text{End}(\Gamma)$ .

Remark 1. Since  $H(S) \text{ wr } \mathcal{J}_\Gamma$  is isomorphic with the semigroup of all row monomial  $\Gamma \times \Gamma$ -matrices over  $H(S)$ ,  $\text{End}(S)$  can represent by row monomial matrices.

2. Let  $S = \bigcup \{S_\alpha : \alpha \in \Gamma\}$  be a strong semilattice of semigroups  $S_\alpha$  determined by a transitive system  $\{\phi_{\alpha, \beta} : \alpha \geq \beta\}$  of homomorphisms. If each  $S_\alpha$  satisfy certain conditions (e.g.,  $S_\alpha$  has an identity and no other idempotent or  $S_\alpha$  is weakly cancellative), then an H-property is equivalent to that the following diagram is commutative:

$$\begin{array}{ccc} S_\alpha & \xrightarrow{k_\alpha} & S_{\alpha\theta} \\ \phi_{\alpha, \beta} \downarrow & & \downarrow \phi_{\alpha\theta, \beta\theta} \\ S_\beta & \xrightarrow{k_\beta} & S_{\beta\theta} \end{array} \quad (\alpha \geq \beta, \alpha, \beta \in \Gamma)$$

(c.f., Petrich[2]).

2. The endomorphism semigroup of a successively annihilating chain of semigroups.

To each  $\alpha$  in a chain  $Y$  assign a pairwise disjoint semigroup  $S_\alpha$ , and on  $S = \bigcup \{S_\alpha : \alpha \in Y\}$  define a multiplication  $\cdot$  by

$$a \cdot b = b \cdot a = b \quad \text{if } a \in S_\alpha, b \in S_\beta, \alpha > \beta,$$

$$a \cdot b = ab \quad \text{if } a, b \in S_\alpha.$$

Then an semigroup  $S(\cdot)$  is called a successively annihilating chain  $Y$  of semigroups  $S_\alpha$  (c.f., Petrich [2]).

We shall first consider an endomorphism of a successively annihilating chain of completely simple semigroups.

Let  $Y$  be a chain. To each  $\alpha \in Y$  assign a pairwise disjoint completely simple semigroup  $S_\alpha = M(G_\alpha : I_\alpha, \Lambda_\alpha : P_\alpha)$  where  $P_\alpha = (p_{\lambda, i}^{(\alpha)})$  has been normalized. Let  $S = \bigcup \{S_\alpha : \alpha \in Y\}$  be a successively annihilating chain of  $S_\alpha$ .

Theorem 4. Let  $\theta$  be an endomorphism of  $Y$  (i.e., an isotone transformation of  $Y$ ). We define a mapping  $k: Y \rightarrow H(S)$  as follows:

(i) The case  $|\alpha\theta\theta^{-1}| > 1$ .

Let  $i_0$  and  $\lambda_0$  be fixed elements of  $I_{\alpha\theta}$  and  $\Lambda_{\alpha\theta}$ , respectively.

Let  $\beta$  be an element of  $Y$  such that  $\beta\theta = \alpha\theta$ . Let  $\omega$  be a homomorphism of  $G_\beta$  in  $G_{\alpha\theta}$  such that  $p_{\lambda, i}^{(\beta)}\omega = e_{\alpha\theta}$ , the identity element of  $G_{\alpha\theta}$ ,

for all  $i \in I_\beta$  and  $\lambda \in \Lambda_\beta$ . Define  $k_\beta: S_\beta \rightarrow S_{\alpha\theta}$  by, for all  $i_\beta \in I_\beta$ ,  $\lambda_\beta \in \Lambda_\beta$  and  $g_\beta \in G_\beta$ ,

$$(i_\beta, g_\beta, \lambda_\beta) \cdot k_\beta = \begin{cases} (i_0, (g_\beta \omega) (p_{\lambda_0, i_0}^{(\alpha\theta)})^{-1}, \lambda_0) & \text{if } \beta \text{ is the least element} \\ & \text{in } \alpha\theta\theta^{-1}, \\ (i_0, (p_{\lambda_0, i_0}^{(\alpha\theta)})^{-1}, \lambda_0) & \text{otherwise.} \end{cases}$$

(ii) Other case (i.e.,  $|\alpha\theta\theta^{-1}| = 1$ ).

Define  $k_\alpha$  by any homomorphism of  $S_\alpha$  in  $S_{\alpha\theta}$  (c.f., [1]).

Then  $(k, \theta) \in \text{End}(S) = H(S) \widetilde{\text{wr}} \text{End}(Y)$  and conversely every endo-

morphism of  $S$  can be obtained in this fashion.

We shall use the following corollary in next section.

Corollary 5. Let  $L = \bigcup \{L_\alpha : \alpha \in Y\}$  be a successively annihilating chain  $Y$  of left zero semigroups  $L_\alpha$ . Let  $\theta$  be an endomorphism of  $Y$ . We define a mapping  $k: Y \rightarrow H(L)$  as follows:

(i) Case  $|\alpha\theta\theta^{-1}| > 1$ .

Let  $l_0$  be a fixed element of  $L_{\alpha\theta}$ . Let  $\beta$  be an element of  $Y$  such that  $\beta\theta = \alpha\theta$ . Define  $k_\beta: L_\beta \rightarrow L_{\alpha\theta}$  by

$$xk_\beta = l_0 \quad \text{for all } x \in L_\beta.$$

(ii) Case  $|\alpha\theta\theta^{-1}| = 1$ .

Define  $k_\alpha$  by any mapping of  $L_\alpha$  in  $L_{\alpha\theta}$ .

Then  $(k, \theta) \in \text{End}(L) = H(L) \widetilde{\text{Wr}} \text{End}(Y)$  and conversely every endomorphism of  $L$  can be constructed in this manner.

Under some condition we next show that the endomorphism semigroup is a direct product of endomorphism semigroups of subsemigroups.

Let  $\Gamma = \bigcup \{\Gamma_i : i \in Y\}$  be a semilattice which is a chain  $Y$  of semilattice  $\Gamma_i$ . Let  $\{T_i : i \in Y\}$  be a family of semilattice indecomposable semigroups  $T_i$  such that  $\text{Hom}(T_i, T_j) = \emptyset$   $[T_i \not\cong T_j]$  for all  $i \neq j \in Y$ . Let  $S = \bigcup \{S_\alpha : \alpha \in \Gamma\}$  be a semigroup such that  $\Gamma$  is the structure semilattice of  $S$  and  $S_\alpha \cong T_i$  if  $\alpha \in \Gamma_i$ .

In Proposition 6 and Corollary 7, let  $S = \bigcup \{S_i : i \in Y\}$  be a chain of semigroups  $S_i$  where  $S_i = \bigcup \{S_\alpha : \alpha \in \Gamma_i\}$ , and assume that  $T_i$ 's satisfy conditions in the brackets for a statement on an automorphism group.

Proposition 6.  $\text{End}(S) \cong_{\text{in}} \prod_{i \in Y} \text{End}(S_i)$ .

$$[\text{Aut}(S) \cong_{\text{in}} \prod_{i \in Y} \text{Aut}(S_i)].$$

Corollary 7. Moreover, let  $S$  be a successively annihilating chain  $Y$  of semigroups  $S_i$ . Then

$$\begin{aligned} \text{End}(S) &\cong \prod_{i \in Y} \text{End}(S_i). \\ [\text{Aut}(S) &\cong \prod_{i \in Y} \text{Aut}(S_i)]. \end{aligned}$$

By the above result we need to consider the endomorphism semigroup as follows. Let  $S$  be a semigroup such that  $S = \bigcup \{S_\alpha : \alpha \in \Gamma\}$  and  $\phi_\alpha : S_\alpha \cong S_0$  for all  $\alpha \in \Gamma$ , where  $S_0$  is a semi-lattice indecomposable semigroup.

Let  $k$  be a mapping of  $\Gamma$  into  $\text{End}(S_0)$  and let  $\theta \in \text{End}(\Gamma)$ . Then we say that  $(k, \theta)$  has an H-property if  $(k, \theta)$  satisfy the following condition:

$$[a(\phi_\alpha k_\alpha \phi_\alpha^{-1})][b(\phi_\beta k_\beta \phi_\beta^{-1})] = (ab)(\phi_{\alpha\beta} k_{\alpha\beta} \phi_{\alpha\beta}^{-1})$$

for all  $a \in S_\alpha, b \in S_\beta$ ,

and  $\text{End}(S_0) \overline{\text{wr}} \text{End}(\Gamma) = \{(k, \theta) \in \text{End}(S_0) \text{ wr } \text{End}(\Gamma) : (k, \theta) \text{ has an H-property}\}$ .

Proposition 8. Let  $S$  be the above semigroup. Then

$$\begin{aligned} \text{End}(S) &\cong \text{End}(S_0) \overline{\text{wr}} \text{End}(\Gamma) \text{ and} \\ \text{Aut}(S) &\cong \text{Aut}(S_0) \overline{\text{wr}} \text{Aut}(\Gamma). \end{aligned}$$

Corollary 9. Let  $S_0$  be a weakly cancellative semigroup or  $S_0$  has an identity and no other idempotent. Then

$$\begin{aligned} \text{End}(S_0 \times \Gamma) &\cong \text{End}(S_0) \times \text{End}(\Gamma) \text{ and} \\ \text{Aut}(S_0 \times \Gamma) &\cong \text{Aut}(S_0) \times \text{Aut}(\Gamma). \end{aligned}$$

### 3. An application to left inverse semigroup

A regular semigroup  $S$  is called a left inverse semigroup if the set  $E$  of all idempotents in  $S$  satisfy the identity  $xyx = xy$ . A structure theorem for left inverse semigroups have been given by M. Yamada [3] as follows:

Structure Theorem. Let  $\Omega$  be an inverse semigroup and  $Y$  its

basic semilattice. Let  $L = \bigcup \{L_\alpha : \alpha \in Y\}$  be a left regular band (i.e.,  $L$  is a semilattice  $Y$  of left zero semigroups  $L_\alpha$ ). Let  $\mathcal{P}$  be a mapping of  $\Omega$  into  $\text{End}(L)$  such that the family  $\{\mathcal{P}_\omega : \omega \in \Omega\}$ ,  $\mathcal{P}_\omega = \mathcal{P}(\omega)$ , satisfies the following  $(C_1)$  and  $(C_2)$ :

$(C_1)$  Each  $\mathcal{P}_\omega$  is an endomorphism on  $L$  such that  $\mathcal{P}_\omega(L_\alpha) \subseteq L_{\omega\alpha(\omega\alpha)^{-1}}$  for all  $\alpha \in Y$ . In particular, for  $\tau \in Y$ ,  $\mathcal{P}_\tau$  is an inner endomorphism on  $L$ .

$(C_2)$   $\delta_e \delta_f \mathcal{P}_\alpha \mathcal{P}_\beta = \delta_e \delta_f \mathcal{P}_{\alpha\beta}$  for  $e \in L_{\alpha\alpha^{-1}}$ ,  $f \in L_{(\alpha\beta)(\alpha\beta)^{-1}}$ ,  $\alpha, \beta \in \Omega$  (where  $\delta_e$  is the inner endomorphism on  $L$  induced by  $e$ ).

Consider the set  $L \underset{\mathcal{P}}{\otimes} \Omega = \{(e, \omega) : \omega \in \Omega, e \in L_{\omega\omega^{-1}}\}$  and define multiplication in  $L \underset{\mathcal{P}}{\otimes} \Omega$  as follows:

$$(e, \omega)(f, \tau) = (e \mathcal{P}_\omega(f), \omega\tau).$$

Then  $L \underset{\mathcal{P}}{\otimes} \Omega$  is a left inverse semigroup and conversely every left inverse semigroup can be constructed in this manner.

We shall call this  $L \underset{\mathcal{P}}{\otimes} \Omega$  the left inverse extension of  $L$  by  $\Omega$  and a mapping  $\mathcal{P} : \Omega \rightarrow \text{End}(L)$  which satisfies  $(C_1)$  and  $(C_2)$  is called a factor system of  $\Omega$  into  $L$  and we shall denote the set of all factor systems of  $\Omega$  into  $L$  by  $\Phi(\Omega, L)$ . We shall define a relation  $\sim$  on  $\Phi(\Omega, L)$  as follows; let  $\mathcal{P}_1, \mathcal{P}_2 \in \Phi(\Omega, L)$ ,

$$\mathcal{P}_1 \sim \mathcal{P}_2 \text{ if and only if } L \underset{\mathcal{P}_1}{\otimes} \Omega \cong L \underset{\mathcal{P}_2}{\otimes} \Omega.$$

The purpose of this section is to investigate  $\Phi(\Omega, L)/\sim$  in case  $\Omega$  is a chain of groups and  $L$  is a successively annihilating chain of left zero semigroups.

Let  $L = \bigcup \{L_\alpha : \alpha \in Y\}$  be a successively annihilating chain  $Y$  of left zero semigroups  $L_\alpha$ . Let  $\Omega = \bigcup \{G_\alpha : \alpha \in Y\}$  be a chain  $Y$  of groups  $G_\alpha$  determined by a transitive system  $\{f_{\beta, \alpha} : \beta \leq \alpha\}$  of a homomorphism  $f_{\beta, \alpha} : G_\alpha \rightarrow G_\beta$ .

First of all, we shall investigate a factor system of  $\Omega$  in

L.

Theorem 10. Let  $k = (k_{\beta, \alpha})$  be an element of  $\prod_{\beta < \alpha} \text{Hom}(G_\alpha, S(L_\beta))$  such that if  $\gamma < \beta < \alpha$  then  $k_{\gamma, \beta} \circ k_{\beta, \alpha} = k_{\gamma, \alpha}$  where  $S(L_\beta)$  is the symmetric group on  $L_\beta$ . Let  $\phi$  be a mapping of  $\Omega$  in  $L$  such that  $\phi(G_\alpha) \subseteq L_\alpha$  for all  $\alpha \in Y$  where if  $\alpha$  is the greatest element in  $Y$  then assume that  $\phi(G_\alpha) \subseteq \mathcal{J}(L_\alpha)$ , the full transformation semigroup on  $L_\alpha$  and  $\phi(e_\alpha)$  is a constant mapping ( $e_\alpha$  is an identity of  $G_\alpha$ ). Define  $\mathcal{F}: \Omega \rightarrow \text{End}(L)$  by

$$x \cdot \mathcal{F}(g_\alpha) = \begin{cases} x \cdot k_{\beta, \alpha}(g_\alpha) & \text{if } \beta < \alpha, \\ \phi(g_\alpha) & \text{if } \beta \geq \alpha, \alpha \text{ is not greatest,} \\ x \cdot \phi(g_\alpha) & \text{if } \beta = \alpha, \alpha \text{ is greatest,} \end{cases}$$

for all  $x \in L_\beta$ ,  $g_\alpha \in G_\alpha$ ,  $\alpha, \beta \in \Omega$ .

Then  $\mathcal{F}$  is a factor system of  $\Omega$  in  $L$ . Conversely any factor system of  $\Omega$  in  $L$  can be obtained in this fashion.

Hence we shall denote a factor system  $\mathcal{F}$  by  $\mathcal{F} = ((k, \phi))$ .

Theorem 11. Let  $((k^{(1)}, \phi^{(1)}))$  and  $((k^{(2)}, \phi^{(2)}))$  be factor systems of  $\Omega$  in  $L$ . Then  $((k^{(1)}, \phi^{(1)})) \sim ((k^{(2)}, \phi^{(2)}))$  if and only if there are an automorphism  $(\psi, \theta) \in \text{Aut}(\Omega)$  and bijections  $p_\beta: L_\beta \rightarrow L_{\beta\theta}$  ( $\beta \in Y$ ) such that  $k_{\beta, \alpha}^{(1)} = \rho_{p_\beta} k_{\beta\theta, \alpha\theta}^{(2)} \psi_\alpha$  for any  $\alpha, \beta \in Y$  such that  $\beta < \alpha$ , where  $\rho_{p_\beta}: S(L_{\beta\theta}) \rightarrow S(L_\beta)$  is defined by  $\rho_{p_\beta}(p) = p_\beta^{-1} p p_\beta$ .

Corollary 12. Let  $\Omega = G \times Y$  be a direct product of a group  $G$  and a well-ordered chain  $Y$ . Let  $L = \bigcup \{L_\alpha: \alpha \in Y\}$  be a successively annihilating chain  $Y$  of left zero semigroups  $L_\alpha$ .

On the set  $\prod_{\beta \in Y^*} \text{Hom}(G, S(L_\beta))$ , define a relation  $\approx$  by,

$k^{(1)} \approx k^{(2)}$  if and only if there are an automorphism  $\psi \in \text{Aut}(G)$  and  $p_\beta \in S(L_\beta)$  ( $\beta \in Y^*$ ) such that  $k_\beta^{(1)} = \rho_{p_\beta} k_\beta^{(2)} \psi$  for all  $\beta \in Y^*$

where  $Y^* = Y \setminus \alpha_1$ ,  $\alpha_1$  is the greatest element of  $Y$ . Then there is a one-to-one correspondence between the classes of isomorphic left inverse extensions of  $L$  by  $\Omega$  and the elements of  $\prod_{\beta \in Y^*} \text{Hom}(G, S(L_\beta)) / \approx$ .

Example. Let  $G$  be a cyclic group of order 2,  $Y = \{0, 1\}$  and  $L = L_0 \cup L_1$ . If  $|L_0| = 2n$ , then  $|\mathfrak{E}(\Omega, L) / \sim| = n+1$ . If  $|L_0| = \infty$  then  $|\mathfrak{E}(\Omega, L) / \sim| = \infty$ .

Corollary 13. Let  $Y$  be a dense chain (i.e., if  $\alpha > \beta$ , there is an element  $\gamma$  such that  $\alpha > \gamma > \beta$ ). Let  $\Omega$  and  $L$  be successively annihilating chains  $Y$  of groups  $G_\alpha$  and left zero semigroups  $L_\alpha$ , respectively. Then there exist the only one left inverse extension of  $L$  by  $\Omega$ , up to isomorphism.

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