

ORTHODOX SEMIGROUPS ON WHICH GREEN'S RELATIONS ARE  
COMPATIBLE, I. H-COMPATIBLE ORTHODOX SEMIGROUPS

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A semigroup  $S$  is said to be  $H$   $[L,R,D]$ -compatible if the Green's  $H$   $[L,R,D]$ -relation on  $S$  is a congruence. A strictly inversive semigroup introduced by the previous paper [5] of the author is of course  $H$ -compatible, and the structure of strictly inversive semigroups has been clarified by [5]. As a generalization of [5], the structure of  $H$ -compatible orthodox semigroups is studied in this paper. Further, we investigate  $L$   $[R]$ -compatible orthodox semigroups.

For an orthodox semigroup  $S$ , it is easily verified that the following conditions are equivalent:

Let  $E_S$  be the band of idempotents of  $S$ .

- (C1) If  $e, f \in E_S$ ,  $f \leq e$ ,  $x \in S$  and  $xx^* = x^*x = e$  (where  $x^*$  denotes an inverse of  $x$ ) then  $xf = fx$ . (Note.  $f \leq e$  means  $ef = fe = f$ .)
- (C2) If  $e, f \in E_S$  and  $f \leq e$ , then  $S(e) \cup S(f)$  (where  $S(h)$  denotes the maximal subgroup of  $S$  containing  $h$  for each  $h \in E_S$ ) is a subsemigroup of  $S$ .
- (C3) The union  $G_S$  of all maximal subgroups of  $S$  is a strictly inversive semigroup (that is, a (orthodox) band of groups; see [5]).
- (C4) The Green's  $H$ -relation (see [2]) on  $S$  is compatible (hence, it is a congruence; see [1]).

If  $S$  satisfies one of these conditions, then  $S$  is said to be  $H$ -compatible. If every maximal subgroup (that is,  $H$ -class containing an idempotent) of an orthodox semigroup  $A$  consists of a single element, then  $A$  is of course  $H$ -compatible. In this case,  $A$  is particularly called an  $H$ -degenerated orthodox semi-

group (abbrev., an H-d.o. semigroup). In this paper, as a generalization of the previous paper [5], we shall investigate the structure of H-compatible orthodox semigroups. The complete proofs are omitted and will be given in detail by [9].

### § 1. Spined product decomposition.

Let  $S$  be an H-compatible orthodox semigroup. Since  $G_S$  is a strictly inversive semigroup, there exist a semilattice  $\Lambda$  and a rectangular subgroup  $S_\lambda$  of  $G_S$  for each  $\lambda \in \Lambda$  such that

$$G_S = \Sigma \{ S_\lambda : \lambda \in \Lambda \} \text{ (disjoint sum), and}$$

$$S_\alpha S_\beta \subset S_{\alpha\beta} \text{ for } \alpha, \beta \in \Lambda.$$

That is,  $G_S$  is decomposed into a semilattice  $\Lambda$  of rectangular groups  $S_\lambda$ . Further, such a decomposition of  $G_S$  is uniquely determined and is called the structure decomposition (see [5]); and this decomposition is denoted by  $G_S \sim \Sigma \{ S_\lambda : \lambda \in \Lambda \}$ .

Now, define a relation  $\sigma_S$  on  $S$  as follows:

(C5)  $a \sigma_S b$  if and only if for any  $e \in E_S^1$  (where  $E_S^1$  denotes the adjunction of an identity 1 to  $E_S$ ) there exist  $\alpha, \beta \in \Lambda$  such that both  $aea^*$  and  $beb^*$  are contained in  $S_\alpha$  and both  $a^*ea$  and  $b^*eb$  are contained in  $S_\beta$ .

Then,

Theorem 1. (1)  $\sigma_S$  is the finest H-degenerated inverse semigroup (abbrev., an H-d.i. semigroup) congruence on  $S$ ,

(2) for the natural homomorphism  $\phi: S \rightarrow S/\sigma_S$ ,  $\text{Ker } \phi \equiv \{ \bar{e} \phi^{-1} : \bar{e} \in E_{S/\sigma_S} \} = \{ S_\lambda : \lambda \in \Lambda \}$ , where  $\bar{e}$  is the  $\sigma_S$ -class containing  $e$ , and  $\bigcup \text{Ker } \phi = G_S$ ,

(3) for the congruence  $\nu_\phi$  on  $S$  induced by  $\phi$ , the restriction  $\nu_\phi|_{G_S}$

gives the structure decomposition of  $G_S$ .

Next, we extend the concept of spined products introduced by [5] (see also [6]) from strictly inversive semigroups to H-compatible orthodox semigroups. Let  $\phi_1: A \rightarrow \Gamma$  and  $\phi_2: B \rightarrow \Gamma$  be homomorphisms of H-compatible orthodox semigroups A and B onto an H-d.i. semigroup  $\Gamma$  such that the congruences  $\vee_{\phi_1}, \vee_{\phi_2}$  induced by  $\phi_1, \phi_2$  give the finest H-d.i. semigroup decompositions of A, B respectively. Put  $\gamma\phi_1^{-1} = A_\gamma, \gamma\phi_2^{-1} = B_\gamma$  for  $\gamma \in \Gamma$ . Define multiplication in  $C = \bigcup \{A_\gamma \times B_\gamma : \gamma \in \Gamma\}$  as follows:

$$(a_\alpha, b_\alpha)(c_\beta, d_\beta) = (a_\alpha c_\beta, b_\alpha d_\beta)$$

for  $(a_\alpha, b_\alpha) \in A_\alpha \times B_\alpha$  and  $(c_\beta, d_\beta) \in A_\beta \times B_\beta$ .

Then, C is also an H-compatible orthodox semigroup. This C is called the spined product of A and B with respect to  $(\Gamma, \phi_1, \phi_2)$ , and denoted by  $(A \bowtie B; \Gamma, \phi_1, \phi_2)$ .

Under this definition, we have the following:

Lemma 2. Let S be an H-compatible orthodox semigroup, and  $\sigma_S$  the finest H-d.i. semigroup congruence on S. If H-compatible orthodox semigroup congruences  $\sigma_1$  and  $\sigma_2$  on S satisfy

- (1)  $\sigma_1 \subset \sigma_S$  and  $\sigma_2 \subset \sigma_S$ , and  $\sigma_1, \sigma_2$  are permutable,
- (2)  $\sigma_1 \wedge \sigma_2 = \iota_S$  (the identity congruence on S), and
- (3)  $\sigma_1 \vee \sigma_2 = \sigma_S$  (where  $\sigma_1 \vee \sigma_2$  denotes the least congruence on S containing both  $\sigma_1$  and  $\sigma_2$ ),

then S is isomorphic to  $(S/\sigma_1 \bowtie S/\sigma_2; S/\sigma_S, \phi_1, \phi_2)$ , where  $\phi_1, \phi_2$  are the homomorphisms of  $S/\sigma_1, S/\sigma_2$  to  $S/\sigma_S$  respectively defined by  $(x\sigma_1)\phi_1 = x\sigma_S$  and  $(x\sigma_2)\phi_2 = x\sigma_S$ .

Now, define two relations  $\sigma_1, \sigma_2$  on an H-compatible orthodox

semigroup  $S$  as follows:

- (1)  $a \sigma_1 b$  if and only if  $a \sigma_S b$  and  $ab^*, b^*a \in E_S$  for some inverse  $b^*$  of  $b$ ,
- (2)  $a \sigma_2 b$  if and only if  $a \sigma_S b$  and  
 $aea^*beb^* = beb^*$ ,  $beb^*aea^* = aea^*$  for all  $e \in E_S^1$  and for some inverses  $a^*$  and  $b^*$ ;  
 $a^*eab^*eb = a^*ea$ ,  $b^*eba^*ea = b^*eb$  for all  $e \in E_S^1$  and for some inverses  $a^*$  and  $b^*$ .

Then, these  $\sigma_1, \sigma_2$  satisfy the conditions (1), (2) and (3) of Lemma 2. Hence,  $S$  is isomorphic to  $(S/\sigma_1 \bowtie S/\sigma_2; S/\sigma_S, \phi_1, \phi_2)$ , where  $\phi_1, \phi_2$  are the homomorphisms of  $S/\sigma_1, S/\sigma_2$  to  $S/\sigma_S$  respectively defined by  $(x\sigma_1)\phi_1 = x\sigma_S$  and  $(x\sigma_2)\phi_2 = x\sigma_S$ . In this case, it is easily seen that  $S/\sigma_1$  and  $S/\sigma_2$  are an H-compatible inverse semigroup and an H-d.o. semigroup respectively.

Thus, we have the following theorem.

**Theorem 3.** An orthodox semigroup  $S$  is H-compatible if and only if  $S \cong (A \bowtie B; \Gamma, \phi_1, \phi_2)$ , where  $A$  is an H-compatible inverse semigroup,  $B$  an H-d.o. semigroup,  $\Gamma$  an H-d.i. semigroup, and  $\phi_1, \phi_2$  homomorphisms of  $A, B$  onto  $\Gamma$  respectively.

## § 2. Construction theorems.

The concept of a regular extension of an inversive semigroup [a band] by an inverse semigroup has been introduced by [8] [by [7]] as follows:

If a regular semigroup  $R$  contains an inversive semigroup [a band]  $T$  as its subsemigroup and if there exists a homomorphism  $\phi$  of  $R$  onto an inverse semigroup  $\Gamma$  such that

- (1)  $\bigcup \text{Ker } \phi \cong \bigcup \{\lambda\phi^{-1} : \lambda \in E_\Gamma\} = T$ , and

(2) the structure decomposition of  $T$  is  $T \sim \Sigma \{ \lambda \phi^{-1} : \lambda \in E_\Gamma \}$ , then  $R$  is called a regular extension of  $T$  by  $\Gamma$ .

Further, the previous paper [8] [[7]] has given a description of all possible regular extensions of  $T$  by  $\Gamma$  for a given inversive semigroup [band]  $T$  and for a given inverse semigroup  $\Gamma$ .

Since every H-d.o. semigroup  $B$  can be obtained as a regular extension of a band by an H-d.i. semigroup (see [7]), the construction of H-compatible orthodox semigroups is reduced to the following problems:

- I. Construction of H-d.i. semigroups.
- II. Construction of H-compatible inverse semigroups.

Now, we shall consider about these two problems.

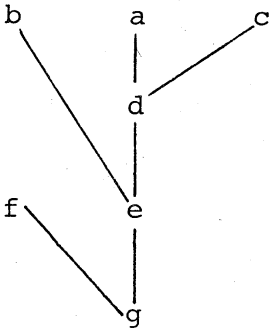
I. Let  $E$  be a semilattice, and put  $\{(e, f) : eE \cong fE\} = \Delta$ . Let  $\Omega$  be a subset of  $\Delta$  and  $\phi_{(e, f)}$  an isomorphism of  $eE$  onto  $fE$  for each  $(e, f) \in \Omega$ . Assume that  $\Omega$  and the family  $F_\Omega(E) = \{\phi_{(e, f)} : (e, f) \in \Omega\}$  satisfy the following:

- (C6) (1)  $(e, e) \in \Omega$  for all  $e \in E$ ,
- (2)  $(e, f) \in \Omega$  implies  $(f, e) \in \Omega$ ,
- (3)  $\phi_{(e, e)}$  is the identity mapping on  $eE$  for each  $e \in E$ ,
- (4)  $(e, f), (h, t) \in \Omega$  implies  $((fh)\phi_{(f, e)}, (fh)\phi_{(h, t)}) \in \Omega$   
 and  $\phi_{((fh)\phi_{(f, e)}, (fh)\phi_{(h, t)})} = \phi_{(e, f)} \phi_{(h, t)} \Big|_{(fh)\phi_{(f, e)} E}$ .

For example, consider the case where  $E$  satisfies the following condition:

- (C7) For any  $e \in E$ ,  $eE$  is a totally ordered set (with respect to the usual ordering); and when  $e, f \in E$ ,  $e \neq f$  and  $eE \cong fE$ , an isomorphism  $\theta$  of  $eE$  onto  $fE$  is unique. (For example, see the

diagram below.)



In this case, select  $\Omega$  and  $\phi_{(e,f)}$  as follows:

- (1)  $\Omega = \Delta$ ,
- (2)  $\phi_{(e,e)}$  is the identity mapping on  $eE$  for each  $e \in E$ ; and  $\phi_{(e,f)}$  is the unique isomorphism of  $eE$  onto  $fE$  for each  $(e,f) \in \Delta$  such that  $e \neq f$ .

Then, the system  $\{\Delta, \phi_{(e,f)}\}$  satisfies (1)-(4) of (C6).

Now,  $F_\Omega(E)$  above is an H-d.i. subsemigroup of the symmetric inverse semigroup  $\mathcal{I}_E$  on  $E$ . This  $F_\Omega(E)$  is denoted by the symbol  $HD(E; \Omega, \phi_{(e,f)})$ .

Then, we have the following result:

**Theorem 4.** Any H-d.i. semigroup  $A$  is isomorphic to some  $HD(E; \Omega, \phi_{(e,f)})$ . In this case, the semilattice  $E_A$  can be selected as  $E$ . In particular, if  $E_A$  satisfies (C7) then  $A$  is isomorphic to a full regular subsemigroup of  $HD(E_A; \Delta, \phi_{(e,f)})$  (where  $\Delta = \{(e,f) : e, f \in E_A, eE \cong fE\}$ ;  $\phi_{(e,e)}$  = the identity mapping on  $eE$  for each  $e \in E_A$ ; and  $\phi_{(e,f)}$  = the unique isomorphism of  $eE$  onto  $fE$  for each  $(e,f) \in \Delta$  such that  $e \neq f$ ).

**Remark.** This result can be obtained by slightly modifying the results given by [3].

II. It is easily verified that an inverse semigroup is H-compatible if and only if it is a regular extension of a weakly C-inversive semigroup (that is, a semilattice of groups; see [6])

by an H-d.i. semigroup. Hence, hereafter we consider the problem of constructing all possible regular extensions of  $G$  by  $\Gamma$  for any given H-d.i. semigroup  $\Gamma$  having  $\Lambda$  as the semilattice of its idempotents and for any given weakly C-inversive semigroup  $G$  having  $G \sim \Sigma \{ G_\lambda : \lambda \in \Lambda \}$  as its structure decomposition (that is,  $G$  is a semilattice  $\Lambda$  of groups  $G_\lambda$ ).

Let  $e_\lambda$  be the identity element of  $G_\lambda$ .

The regular extensions of  $G$  by  $\Gamma$  are obtained by slightly modifying the concept of a Schreier extension in the group theory.

Let  $\Psi: \Gamma \rightarrow \text{End}(G)$  (the semigroup of endomorphisms on  $G$ ) and  $\phi: \Gamma \times \Gamma \rightarrow G$  be mappings such that each  $\bar{\gamma} = \gamma\Psi$  maps  $G_\lambda$  into  $G_{\gamma\lambda(\gamma\lambda)^*}$ , especially maps  $G_{\gamma^*\gamma}$  onto  $G_{\gamma\gamma^*}$ , and each  $C(\gamma, \tau) = (\gamma, \tau)\phi$  is an element of  $G_{\gamma\tau(\gamma\tau)^*}$ . If  $\{\bar{\gamma} : \gamma \in \Gamma\}$  and  $\{C(\gamma, \tau) : (\gamma, \tau) \in \Gamma \times \Gamma\}$  satisfy

- (C8) (1)  $C(\lambda, \delta) = e_{\lambda\delta}$  for  $\lambda, \delta \in E_\Gamma$ , and  $C(\gamma\gamma^*, \gamma) = C(\gamma, \gamma^*\gamma) = e_{\gamma\gamma^*}$  for each  $\gamma \in \Gamma$ ,
- (2)  $C(\tau, \xi)\bar{\gamma}C(\gamma, \tau\xi) = C(\gamma, \tau)C(\gamma\tau, \xi)$ , where  $x\bar{\gamma} = x\bar{\gamma}$ ,
- (3)  $\bar{\gamma}\bar{\tau} = \overline{\tau\gamma C(\tau, \gamma)}$  (where, for each  $a \in G$ ,  $\bar{a}$  denotes the endomorphism on  $G$  defined by  $z\bar{a} = aza^*$ ,  $z \in G$ ),
- (4)  $e_\lambda b = b\bar{\lambda}$  for  $\lambda, \delta \in E_\Gamma$  and  $b \in G_\delta$ ,

then the system  $\{\bar{\gamma}, C(\gamma, \xi)\}$  is called a (normalized) factor set of  $G$  belonging to  $\Gamma$ .

Now, under this definition we have the following theorem:

Theorem 5. Let  $G$ ,  $\Gamma$  and  $\Lambda$  be as above. Let  $\{\bar{\gamma}, C(\tau, \xi)\}$  be a factor set of  $G$  belonging to  $\Gamma$ . If multiplication is defined in  $G \rtimes \Gamma = \{(a, \gamma) : a \in G_{\gamma\gamma^*}, \gamma \in \Gamma\}$  by

$$(a, \gamma)(b, \tau) = (ab\bar{\gamma}C(\gamma, \tau), \gamma\tau),$$

then

- (1)  $\underline{G} = \{ (a, \lambda) : a \in G_\lambda, \lambda \in E_\Gamma \}$  is a weakly C-inversive semigroup which is isomorphic to  $G$ , and
- (2)  $G \rtimes \Gamma$  is a regular extension of  $\underline{G}$  by  $\Gamma$ .

Further, every regular extension of  $G$  by  $\Gamma$  is obtained in this fashion, up to isomorphism.

### § 3. L-compatible regular semigroups.

A semigroup  $S$  is said to be L-compatible [R-compatible] if the Green's L-relation [R-relation] on  $S$  is compatible (that is, a congruence). Firstly, we have the following theorem.

**Theorem 6.** A regular semigroup  $S$  is L-compatible if and only if it is a right regular band of left groups. Of course, in this case  $S$  is a union of groups.

A band  $B$  is called a regular [ left regular, right regular ] band if  $B$  satisfies the identity  $xyxzx = xyzx$  [ $xyx = xy$ ,  $yx = yx$ ] (see [4]). If a semigroup  $S$  is an orthodox semigroup and is a right regular band of left groups, then  $S$  is called an orthodox right regular band of left groups. An orthodox band of groups, an orthodox regular band of groups, etc. are similarly defined.

Under these definitions and by using Theorem 6, we have the following result:

**Corollary 7.** A semigroup is an L-compatible orthodox semigroup if and only if it is an orthodox right regular band of left groups.

By using Theorem 6 and its dual result, the following is also obtained:



Theorem 8. A regular semigroup is both L-compatible and R-compatible if and only if it is a regular band of groups.

Corollary 9. A semigroup is a both L- and R-compatible orthodox semigroup if and only if it is both a semilattice of rectangular groups and a regular band of groups.

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