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ORTHODOX SEMIGROUPS ON WHICH GREEN'S RELATIONS ARE COMPATIBLE, II.

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This paper is a continuation of the previous paper [11]. Krishna Iyengar [2] has shown that a regular semigroup is D-compatible if and only if it is a semilattice of bisimple semigroups. In this paper, the structure of bisimple orthodox semigroups, especially that of H-compatible bisimple orthodox semigroups, is clarified. Further, we investigate the structure of orthodox semigroups S on which some of the Green's relations $\mathcal{H}_S$, $\mathcal{L}_S$, $\mathcal{R}_S$ and $\mathcal{D}_S$ are compatible.

A semigroup S is said to be H [L,R,D]-compatible if the Green's H [L,R,D]-relation $\mathcal{H}_S$ [$\mathcal{L}_S$, $\mathcal{R}_S$, $\mathcal{D}_S$] on S is a congruence.

In the previous paper [11], one of the authors has clarified the structure of H [L,R]-compatible orthodox semigroups. On the other hand, it has been shown by Krishna Iyengar [2] that a regular semigroup is D-compatible if and only if it is a semilattice of bisimple semigroups. Accordingly, it is obvious that an orthodox semigroup is D-compatible if and only if it is a semilattice of bisimple orthodox semigroups. In the first half of this paper, the structure of bisimple orthodox semigroups, especially that of H-compatible bisimple orthodox semigroups, will be clarified. By using the results obtained in the first half, we shall next investigate the structure of orthodox semigroups S on which some of the Green's relations $\mathcal{H}_S$, $\mathcal{L}_S$, $\mathcal{R}_S$ and $\mathcal{D}_S$ are compatible. The complete proofs are omitted and will be given in detail elsewhere. Throughout the whole paper, the set [the band] of idempotents of a regular [an orthodox] semigroup S will be denoted by E_S.

§ 1. H-compatible bisimple orthodox semigroups.

If f is a homomorphism of a regular semigroup A onto a regular semigroup B, then the collection $\{ef^{-1}: e \in E_B\}$ of subsemigroups ef^{-1} $(e \in E_B)$ of A is called the <u>kernel</u> of f and is denoted by Ker f.

Let T be an inversive semigroup (that is, an orthogroup), and Γ an inverse semigroup. If a regular semigroup S contains T and if there exists a surjective homomorohism $\xi: S \rightarrow \Gamma$ such that

(C1) $\bigcup \text{Ker } \xi \equiv \bigcup \{ \lambda\xi^{-1} : \lambda \in E_\Gamma \} = T$ and

(C2) the structure decomposition (see [7],[11]) of T is given as $T \sim \Sigma\{ \lambda\xi^{-1} : \lambda \in E_\Gamma\}$,

then S is called a <u>regular extension</u> of T by Γ (see [11]).

The following results have been given by the previous papers [8] and [11]:

A. An orthodox semigroup is a regular extension of a band by an inverse semigroup, and vice-versa.

B. An H-compatible orthodox semigroup is a regular extension of a strictly inversive semigroup (that is, an orthodox band of groups; see [7],[11]) by an H-degenerated inverse semigroup, and vice-versa.

Now, let S be a regular extension of a stricly inversive semigroup T by an inverse semigroup Γ. By the definition of a regular extension, it follows that $S \supset T$ and there exists a surjective homomorphism $\xi: S \rightarrow \Gamma$ such that $\bigcup \text{Ker } \xi \equiv \bigcup\{\lambda\xi^{-1}:\lambda \in E_\Gamma\} = T$ and the structure decomposition of T is given as $T \sim \Sigma\{\lambda\xi^{-1} : \lambda \in E_\Gamma \}$. (That is, T is a semilattice E_Γ of the rectangular groups $\lambda\xi^{-1}$.)

For each $a \in S$, put $a\xi = \bar{a}$. Then, the following result can be proved by slightly modifying the proof of Lemma 1 of [9]:

Lemma 1. $a \mathcal{D}_S b$ if and only if $\bar{a} \mathcal{D}_\Gamma \bar{b}$.

By using Lemma 1 and the results A,B above, we can obtain the following theorem.

Theorem 2. (1) A bisimple orthodox semigroup is a regular extension of a band by a bisimple inverse semigroup, and vice-versa.
(2) An H-compatible bisimple orthodox semigroup is a regular extension of a strictly inversive semigroup by an H-degenerated bisimple inverse semigroup, and vice-versa.

Remark. A method of constructing all possible regular extensions of T by Γ for a given strictly inversive semigroup T and a given inverse semigroup Γ has been given by [10]; in particular for the case where T is a band, see also [8]. The structure of bisimple inverse semigroups has been also clarified by Reilly [4] and Reilly and Clifford [5]. Hence, we can know the gross structure of bisimple orthodox semigroups from Theorem 2,(1). A somewhat different construction of bisimple orthodox semigroups has been also given in Clifford [1], by extending Reilly's construction (see [4]) of bisimple inverse semigroups to bisimple orthodox semigroups.

By Theorem 2,(2) and Remark above, the problem of describing all H-compatible bisimple orthodox semigroups is reduced to that of describing all H-degenerated bisimple inverse semigroups. Therefore, we shall investigate the construction of H-degenerated bisimple inverse semigroups from now on.

Let E be a uniform semilattice, that is, a semilattice satisfying the following condition (C3):

(C3) For any $e,f \in E$, eE is isomorphic to fE; $eE \cong fE$.

Put $E \times E = \Delta$, and take an isomorphism $\xi_{(e,f)}$ of eE onto fE for each $(e,f) \in \Delta$. Assume that $F_\Delta(E) = \{\xi_{(e,f)} : (e,f) \in \Delta\}$ satisfies the conditions (3),(4) of (C6) of [11], that is, the conditions

(C4) (1) $\xi_{(e,e)}$ is the identity mapping on eE for each $e \in E$,

(2) for $(e,f),(h,t) \in \Delta$,

$$\xi_{((fh)\xi_{(f,e)},(fh)\xi_{(h,t)})} = \xi_{(e,f)}\xi_{(h,t)} \mid (fh)\xi_{(f,e)}E.$$

Then, it is easily seen from [11] that $F_\Delta(E)$ is an H-degenerated inverse subsemigroup of the symmetric inverse semigroup $\mathscr{I}_E(*)$ on E. Further, we have $\xi_{(e,f)} * \xi_{(f,e)} = \xi_{(e,e)}$ and $\xi_{(f,e)} * \xi_{(e,f)} = \xi_{(f,f)}$ for any $(e,f) \in \Delta$. Hence, any two idempotents $\xi_{(e,e)}$ and $\xi_{(f,f)}$ are contained in the same $\mathscr{D}_{F_\Delta(E)}$-class. This implies that $F_\Delta(E)$ is bisimple.

Remark. This result is closely related with Theorem 3.2 of Munn [3].

Now, we have the following main theorem.

Theorem 3. Any H-degenerated bisimple inverse semigroup is isomorphic to some $F_\Delta(E)$ constructed as above.

§ 2. Relationship between Green's relations; and some remarks.

By using Krishna Iyengar [2] and [7],[11], firstly we have the following theorem which shows the structure of orthodox semigroups S on which some of the Green's relations $\mathcal{H}_S$, $\mathcal{L}_S$, $\mathcal{R}_S$ and $\mathcal{D}_S$ are compatible.

Theorem 4. Let S be an orthodox semigroup.

(1) If S is L [R]-compatible, then S is D-compatible.

(2) If S is both L-compatible and R-compatible, then S is H-compatible.

(3) S is both H-compatible and L [R]-compatible if and only if S is a strictly inversive semigroup in which the set E_S of idempotents is a right [left] semiregular band (that is, a band satisfying the identity xyzx = xyzxyxzx [xyx = xyxzxyzx]).

(4) S is both H-compatible and D-compatible if and only if S is a semilattice of H-compatible bisimple orthodox semigroups and the union of maximal subgroups of S is a strictly inversive subsemigroup.

Remarks. 1. An orthodox semigroup which is a semilattice of H-compatible bisimple orthodox semigroups is not necessarily H-compatible. For example, an inversive semigroup (that is, an orthogroup) S is a semilattice of rectangular groups (accordingly, a semilattice of H-compatible bisimple orthodox semigroups), but not necessarily H-compatible. S is H-compatible only when S is strictly inversive.

2. An H-compatible orthodox semigroup is not necessarily D-compatible. Let A,B be two sets such that $A \cap B = \square$ and $|A| = |B|$ (where $|X|$ means the cardinality of X). For X,Y = A or B, let $H_{X,Y}$ be the set of all 1-1 mappings of X onto Y. Put $H_{A,A} \cup H_{B,B} \cup H_{A,B} \cup H_{B,A} \cup \{0\}$ (where 0 is a symbol which is different from any element of $H_{X,Y}$, X,Y = A or B) = S. For $\delta, \xi \in S$, define the

product $\delta * \xi$ as follows:

$$\delta * \xi = \begin{cases} 0 & \text{if } (1)\ \delta \in H_{A,A}, \xi \in H_{B,B}\ ;\ (2)\ \xi \in H_{A,A},\ \delta \in H_{B,B}; \\ & (3)\ \delta, \xi \in H_{A,B} \text{ or } \delta, \xi \in H_{B,A};\ \text{or } (4)\ \delta = 0 \text{ or } \xi = 0, \\ \text{resultant composition,} & \text{otherwise} \end{cases}$$

Then, in the resulting system S(*), the $\mathcal{D}_S$-classes are $H_{A,A} \cup H_{B,B} \cup H_{A,B} \cup H_{B,A}$ and {0}. On the other hand, the $\mathcal{H}_S$-classes are $H_{A,A}$, $H_{A,B}$, $H_{B,A}$, $H_{B,B}$ and {0}. Now, we can easily see that this semigroup S(*) is H-compatible but not D-compatible.

3. The full transformation semigroup $\mathcal{T}_X$ on the set X = {a,b} is an orthodox semigroup which is D-compatible but not H-compatible.

4. A band B is H-compatible but not necessarily L-compatible [R-compatible]. It has been shown by [6] that B is L [R]-compatible if and only if B is a left [right] semiregular band.

5. Consider $\mathcal{T}_X$ above. $\mathcal{T}_X$ consists of four transformations $\begin{pmatrix} a & b \\ a & b \end{pmatrix}$, $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, $\begin{pmatrix} a & b \\ a & a \end{pmatrix}$ and $\begin{pmatrix} a & b \\ b & b \end{pmatrix}$; that is $\mathcal{T}_X = \left\{\begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \begin{pmatrix} a & b \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ b & b \end{pmatrix}\right\}$. The set $\left\{\begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & b \\ b & a \end{pmatrix}\right\} = R_1$ is a subgroup of $\mathcal{T}_X$ and the set $\left\{\begin{pmatrix} a & b \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ b & b \end{pmatrix}\right\} = R_0$ is a right zero semigroup. Further, $\mathcal{T}_X$ is a semilattice {0,1} of the $\mathcal{R}_{\mathcal{T}_X}$-classes R_0 and R_1. Hence, $\mathcal{T}_X$ is R-compatible but not H-compatible. Similarly, there exists an orthodox semigroup which is L-compatible but not H-compatible.

6. A bicyclic semigroup is both D-compatible and H-compatible but neither L-compatible nor R-compatible.

7. A left semiregular band B is both H-compatible and L-compatible but not necessarily R-compatible. In fact, B is R-compa-

tible if and only if B is a regular band. Similarly, there exists an orthodox semigroup which is R-compatible but not L-compatible.

Problem. Determine the structure of H-compatible regular semigroups.

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