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ONE METHOD OF REPRESENTATIONS OF INVERSE SEMIGROUPS

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MOTIVATION.

Any inverse semigroup is isomorphic with the inverse semigroup of partial injective transformations of a set. Generalizing this characterization we study categories of partial injective mappings between sets (closed also under inverse partial mappings). Such categories (called inverse categories) can be defined by the same way as inverse semigroups - categories with unique generalized inverses. But the feature of categories can conversely bring a new view of inverse semigroups.

ABSTRACT.

Analogously to the inverse semigroups, inverse categories are such categories with generalized inverses the idempotents of which commute. We may transform any inverse category (inverse semigroup, in particular) to the category of isomorphisms and build a representation of the inverse category from a functor F of this iso-category into sets. Such representation of inverse category by the injective mappings is called a canonical F-representation, or canonical regular representation for a special type of the functor F . It holds that any representation of the inverse category by partial injective mappings is a factorization of its canonical regular representation. This factorization is specified by the relations on sets $\propto$ that we can roughly describe as quasiorderings preserved by the partial mappings from the canonical representation.

I. INVERSE CATEGORIES

In this paper we shall use the nonobjective definition of category.

A category $(X,.)$ is a class X with the partial operation $.$ satisfying the following axioms.

1) For any $x,y,z \in X$ the whole equality $(x.y).z = x.(y.z)$ is defined and holds true if either at least one side of it is defined or the both expressions $x.y$, $y.z$ are defined.

2) For any $x \in X$ units d_x, r_x of partial operation $.$ exist such that $d_x.x.r_x$ is defined.

3) Only the set of elements $x \in X$ that $j_1.x.j_2$ is defined exists for arbitrary units j_1, j_2 of $.$.

/ j is a unit of partial operation $(X,.)$ iff

for $\forall x,y \in X$ $\quad j.x$ is defined $\Longrightarrow j.x = x$,
$\quad y.j$ is defined $\Longrightarrow y.j = y$.

The class of all units of $(X,.)$ is denoted by $J(X,.)$. /

Note there exist unique units d_x, r_x for any morphism x of the category $(X,.)$.

For $j \in J(X,.)$ $\; d_j = j = r_j$; for $\forall x,y \in X$ $\; x.y$ is defined iff $r_x = d_y$; $d_{x.y} = d_x$, $r_{x.y} = r_y$ -- [5] .

A morphism f is called an idempotent if $f.f = f$ holds true. The class of all idempotents of $(X,.)$ is denoted by $I(X,.)$ / $J(X,.) \subseteq I(X,.)$ /.

DEFINITION.

A morphism z of the category $(X,.)$ is called a generalized inverse of $x \in X$ provided that $x.z.x = x$

and $z.x.z = z$ (are defined and) hold true.

A category $(X,.)$ is called an inverse category provided that any morphism $x \in X$ has a unique generalized inverse $\bar{x}$.

/ Let us note $d_{\bar{x}} = r_x$, $r_{\bar{x}} = d_x$. /

The inverse categories $(X,.)$ where $.$ is a total operation on the set X are exactly the inverse semigroups with unit.

PROPOSITION 1.

A category $(X,.)$ is an inverse category if and only if any morphism $x \in X$ has a generalized inverse and the idempotents of $(X,.)$ commute.

Proof. Let $(X,.)$ be an inverse category. For any idempotent $f \in X$ $\bar{f} = f$ holds. Let the composition of two idempotents $f.g$ be defined in $(X,.)$. / $d_f = r_f = d_g = r_g$ / The morphism $b = g.\overline{fg}.f$ is an idempotent. Namely, $g.\overline{fg}.f.g.\overline{fg}.f = g.\overline{fg}.f$. And $b.fg.b = g\overline{fg}ffgg\overline{fg}f = g\overline{fg}fg\overline{fg}f =$ $= b$, $fg.b.fg = fgg\overline{fg}ffg = fg$, so $b = fg$ is the unique generalized inverse of the idempotent b. $fgfg = fg = gbf =$ $= gfgf$. By the same reason $gfgf = gf$.

Conversely, let us suppose that the category $(X,.)$ has generalized inverses and if for any $f,g \in I(X,.)$ $f.g$ is defined, then $f.g = g.f$. Notice, if z is a generalized inverse of x, xz, zx are idempotents. For generalized inverses z_1, z_2 of one morphism x we may write: $z_1 =$ $= z_1xz_1 = z_1xz_2xz_1xz_2xz_1 = z_2xz_1x.z_1.xz_1xz_2 = z_2xz_2 = z_2$.

It holds in the inverse category $\bar{\bar{x}} = x$, $\overline{x.y} = \bar{y}.\bar{x}$.

In arbitrary category $(X,.)$ we can define Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{D}$ quite analogously to the semigroups.

a, b -- morphisms of $(X,.)$,

$$a \mathcal{L} b \iff \exists x,y \in X \quad x.a = b , \quad y.b = a ,$$
$$a \mathcal{R} b \iff \exists z,v \in X \quad a.z = b , \quad b.v = a .$$

$\mathcal{L}, \mathcal{R}$ are equivalences on X which commute. (For any $a,b \in X$) $\exists c \in X \ a\mathcal{L}c\mathcal{R}b \iff \exists d \in X \ a\mathcal{R}d\mathcal{L}b$. Actually, $a\mathcal{L}c\mathcal{R}b$ signifies $xa = c = bz$, $yc = a$, $cv = b$, then $ycv = yb = av$, $ycvz = ybz = a$, $xycv = xav = b$ signifies $a\mathcal{R}ycv\mathcal{L}b$. Analogously for the converse implication.

$\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ is also an equivalence on X .

Let $(X,.)$ be an inverse category. For idempotents $f,g \in I(X,.)$, $f\mathcal{L}g \iff f = g$. Namely, $f\mathcal{L}g \Longrightarrow xf = g$, $g = xff = gf$, but also $f = fg = gf$.

Because for any morphism $a \in X$ $a\mathcal{L}\bar{a}a$, it holds $a\mathcal{L}b \iff \bar{a}a = \bar{b}b$. Analogously $a\mathcal{R}b \iff a\bar{a} = b\bar{b}$. Then in the inverse category $a\mathcal{D}b \iff \exists d \ \ a\bar{a} = d\bar{d}$, $\bar{d}d = \bar{b}b$.

We can generalize also the partial ordering of the inverse semigroup for an inverse category $(X,.)$

For $a,b \in X$ $a \leq b$ iff $a\bar{b} = a\bar{a}$ (is defined and holds).

Let us note $a\bar{b} = a\bar{a} \Longrightarrow \bar{b}a = \bar{a}a$ / $d_a = d_b$, $r_a = r_b$ /. $a\bar{b} = a\bar{a} = \overline{(a\bar{b})} = b\bar{a}$ and then $\bar{a}a = \bar{a}a\bar{b}b\bar{a}a = \bar{b}b\bar{a}a = \bar{b}a\bar{a}a = \bar{b}a$ ($= \bar{a}b$) .

Then we can easily prove $a \leq b \leq c \Longrightarrow a \leq c$, $a \leq b \leq a \Longrightarrow a = b$. $\leq$ is a reflexive, transitive and antisymmetric relation on X .

For idempotents $f,g \in I(X,.)$, $f \leq g \iff fg = f$.

<u>LEMMA 1.</u>

The following holds for the elements of an inverse category $(X,.)$:

1) $a \leq b \Longrightarrow \bar{a} \leq \bar{b}$

2) $a \leq b,\ r_x = d_a,\ d_y = r_a \Longrightarrow x.a.y \leq x.b.y$

3) $a \leq f,\ f \in I(X,.) \Longrightarrow a \in I(X,.)$

4) For $\forall a \leq x\bar{x}\ \exists!\ y \leq x$ such that $a = y\bar{y}$

<u>Proof.</u> 3) $a = a\bar{a}a = affa = a\bar{a}fa = a\bar{a}aa = aa$;
4) $a \in I(X,.)$ $ax \leq x\bar{x}x$, $ax.\overline{ax} = ax\bar{x}\bar{a} = a$ and any other y is $y = y\bar{y}y = y\bar{y}x = ax$.

The category of all partial mappings $(PM,.)$ is the class of all tripls (f,A,B) , where f is a partial mapping from the set A into the set B . The composition $(f,A,B).(g,C,D)$ is defined if and only if $B = C$, and it is equal to $(f \circ g,A,D)$. ($f \circ g$ is the usual composition of relations.)

The subcategory of all injective partial mappings $(PIM,.)$ is an inverse category -- $\overline{(f,A,B)} = (f^{-1},B,A)$. Note that $(f,A,B) \leq (g,C,D)$ iff $A = C,\ B = D,\ f \subseteq g$.

II. CANONICAL REPRESENTATIONS

A functor F from the category $(X,.)$ into the category $(Y,.)$ is a mapping of the class X into Y satisfying:

1) For any $j \in J(X,.)$ $(j)F$ is a unit of $(Y,.)$.

2) If $x.y$ is defined, then $(x.y)F = (x)F.(y)F$ is defined and holds ($x,y \in X$).

Let us note that the functor F satisfies $(d_x)F = d_{(x)F}$,

$(r_x)F = r_{(x)F}$ / $x \in X$ /. A functor from the inverse category into the inverse category preserves generalized inverses $(\bar{x})F = \overline{(x)F}$, and also $x \leq y \Longrightarrow (x)F \leq (y)F$.

If F is a functor from $(X,.)$ into $(PM,.)$ we can describe it fully by the pair of mappings Fm, Fo from X or from $J(X,.)$, respectively, into sets such that

1) $\forall x \in X \quad (x)Fm \subseteq (d_x)Fo \times (r_x)Fo$,
2) $\forall j \in J(X,.) \quad (j)Fm = id_{(j)Fo}$,
3) $\forall x,y \in X$, $x.y$ is def. $\quad (x.y)Fm = (x)Fm \circ (y)Fm$.

$(x)F = ((x)Fm, (d_x)Fo, (r_x)Fo)$

Definition of iso-category $(X,*)$

Let $(X,.)$ be an inverse category. We will define a new partial operation $*$ on the class X .

$x*y$ is defined if and only if $\bar{x}.x = y.\bar{y}$, and then $x*y = x.y$ ($x,y \in X$) / $\bar{x}x = y\bar{y} \Longrightarrow r_x = d_y$ /.

The equalities $(x*y).\overline{(x*y)} = x.y.\bar{y}.\bar{x} = x\bar{x}x\bar{x} = x\bar{x}$, $\overline{(x*y)}.(x*y) = \bar{y}y$ show that $*$ is also a partial associative operation. $x\bar{x} * x = x$, $x * \bar{x}x = x$. It is to see that $(X,*)$ is a category and $J(X,*) = I(X,.)$. $(X,*)$ is an iso-category because in $(X,*)$, $\bar{x}$ is an inverse morphism of x -- $x*\bar{x}$, $\bar{x}*x \in J(X,*)$.

From the previous we can easily conclude that for units $f,g \in J(X,*)$ there exists a morphism x that $f*x*g$ is defined, if and only if $f\mathcal{D}g$. So, the Green's $\mathcal{D}$-classes devide $(X,*)$ into connected subcategories.

PROPOSITION 2.

Let $(X,.)$ be an inverse category and F be a functor from $(X,*)$ into $(PM,.)$ which satisfies

$\forall f,g \in J(X,*) \qquad (f)Fo \cap (g)Fo \neq \emptyset \iff f = g$.

The following formulas define a functor ϕ from $(X,.)$ into $(PIM,.)$:

$$(x)\phi m = \bigcup_{y \leq x} (y)Fm \qquad /x \in X/$$

$$(j)\phi o = \bigcup_{z \leq j} (z)Fo \qquad /j \in J(X,.)/$$

We call the functor ϕ a canonical F-representation of inverse category $(X,.)$.

Note that ϕ satisfies the condition

$\forall x,y \in X \qquad (x)\phi = (y)\phi \implies (x)F = (y)F$.

<u>Proof.</u> Any $(y)Fm$ is a bijection of the set $(y\bar{y})Fo$ onto the set $(\bar{y}y)Fo$. According to Lemma 1 and disjunctive character of F , $(x)\phi m$ is a bijection of set $\bigcup_{z \leq x\bar{x}} (z)Fo$ onto $\bigcup_{u \leq \bar{x}x} (u)Fo$.

For any $j \in J(X,.)$ $\quad (j)\phi m = \bigcup_{f \leq j} (f)Fm = \bigcup_{f \leq j} id_{(f)Fo} = id_{(j)\phi o}$.

Let $x,y \in X$ now, and $x.y$ be defined. $(x)\phi m \circ (y)\phi m = \bigcup_{v \leq x, w \leq y} (v)Fm \circ (w)Fm$. Because $(v)Fm \circ (w)Fm = \emptyset$ for any v,w , $\bar{v}v \neq w\bar{w}$, $(x)\phi m \circ (y)\phi m = \bigcup_{\substack{v \leq x, w \leq y, \\ v*w \text{ is def.}}} (v)Fm \circ (w)Fm = \bigcup_{\substack{v \leq x, w \leq y \\ v*w \leq x.y}} (v*w)Fm$.

But any $t \leq x.y$ can be composed from $t\bar{y}*\bar{x}t$, where $t\bar{y} \leq xy\bar{y} \leq x$ and $\bar{x}t \leq y$. Namely, $\bar{x}t\bar{y}.\bar{x}t\bar{y} = \bar{x}t\bar{t}t\bar{y} = \bar{x}t\bar{y} \in I(X,.)$, then we can easily compute $\overline{(t\bar{y})}t\bar{y} = y\bar{t}t\bar{y} = y\bar{y}\bar{x}t\bar{y} = \bar{x}t\bar{y} = \bar{x}t\bar{t}x = \bar{x}t\overline{(\bar{x}t)}$, $t\bar{y}.\bar{x}t = t\bar{t}t = t$.

Consequently $(x)\phi m \circ (y)\phi m = \bigcup_{t \leq x.y} (t)Fm = (x.y)\phi m$.

ϕ is a functor of $(X,.)$ into $(PIM,.)$. Let us suppose now that $\exists x,y \in X$ such that $(x)\phi = (y)\phi$, $(x)F \neq (y)F$. The equality of partial bijections gives

$$\bigcup_{z \leq x\bar{x}} (z)Fo = \bigcup_{u \leq y\bar{y}} (u)Fo , \quad \bigcup_{z \leq \bar{x}x} (z)Fo = \bigcup_{u \leq \bar{y}y} (u)Fo .$$

Because $\emptyset \neq (x\bar{x})Fo \subseteq \bigcup_{u \leq y\bar{y}} (u)Fo$, there exists $u' \leq y\bar{y}$ $(x\bar{x})Fo \cap (u')Fo \neq \emptyset$, $u' = x\bar{x} \leq y\bar{y}$; symmetrically $y\bar{y} \leq x\bar{x}$. Analogously also $\bar{x}x = \bar{y}y$. It shows $(x)Fm \neq (y)Fm$, which gives the contradiction $(x)\phi m \neq (y)\phi m$.

DEFINITION.

A functor $G:(X,.) \longrightarrow (PM,.)$ is a factorization of a functor $F:(X,.) \longrightarrow (PM,.)$ if there exist partial mappings b_j from $(j)Fo$ onto $(j)Go$ ($j \in J(X,.)$) that satisfy

$\forall x \in X$ $(\alpha)(x)Gm$ is defined (in the point $\alpha \in (d_x)Go$) iff $\exists \xi \in (d_x)Fo$ $(\xi)b_{d_x} = \alpha$ and $(\xi)(x)Fm$ is defined, and then $(\alpha)(x)Gm = ((\xi)(x)Fm)b_{r_x}$ holds (and has since).

Notice that the definition of factorization is independent on the natural equivalence of functors $G' \sim G''$ and $F' \sim F''$.

It is also easy to show the relationship of factorization between two functors is transitive.

PROPOSITION 3.

Let ϕ be a canonical F-representation of an inverse category $(X,.)$. Let $\propto$ be a relation on the sets $(j)\phi o$ satisfying

0) $\xi_1 \propto \xi_2 \Longrightarrow \exists j \in J(X,.)$ $\xi_1, \xi_2 \in (j)\phi o$,

1) $\xi_1 \propto \xi_2,\ \xi_2 \propto \xi_3 \Longrightarrow \xi_1 \propto \xi_3$,

2) $\xi_1 \propto \xi_2,\ \xi_3 \propto \xi_2 \Longrightarrow \exists \xi_4$ $\xi_4 \propto \xi_1,\ \xi_4 \propto \xi_3$,

3) $\xi_1 \propto \xi_2$ and if for some $x \in X$ $(x)\phi m$ is defined in the point ξ_2 $\Longrightarrow$ $(x)\phi m$ is defined also for ξ_1 and $(\xi_1)(x)\phi m \propto (\xi_2)(x)\phi m$.

On any set $(j)\phi o$ we can define a partial equivalence

ρ_j by $\eta_1 \rho_j \eta_2$ iff $\exists \xi \ \xi \prec \eta_1 , \xi \prec \eta_2$ $(\eta_1, \eta_2 \in (j)\phi o)$.

Let $(j)\Gamma o$ be the set of all nonempty equivalence classes of ρ_j / $j \in J(X,.)$ /. Then we can define for every $x \in X$ a partial mapping $(x)\Gamma m$ from $(d_x)\Gamma o$ into $(r_x)\Gamma o$ by following:

$$[\xi](x)\Gamma m = [(\xi')(x)\phi m] \quad \text{iff} \quad (x)\phi m \text{ is defined}$$

for some $\xi' \in [\xi]$.

Γ is a functor from the inverse category $(X,.)$ into $(PIM,.)$ and it is a factorization of the canonical F-representation ϕ .

Any factorization of ϕ can be obtained (up to a natural equivalence) by the described method.

Proof. The partial mapping $(x)\Gamma m$ is defined correctly. If $(x)\phi m$ is defined for $\xi', \xi'' \in [\xi]$, there exists ξ_4 $\xi' \succ \xi_4 \prec \xi''$ and (by 3)) $(\xi')(x)\phi m \succ (\xi_4)(x)\phi m \prec (\xi'')(x)\phi m$, especially $[(\xi')(x)\phi m]$ is a nonempty class of the partial equivalence ρ_{r_x} .

Moreover, $(x)\Gamma m$ is an injective partial mapping. If we have $(\xi')(x)\phi m \ \rho_{r_x} (\xi'')(x)\phi m$, then $((\xi')(x)\phi m)(\bar{x})\phi m = \xi'$ is defined and holds because $(\bar{x})\phi m = ((x)\phi m)^{-1}$. Hence $[\xi'] = [(\xi')(x)\phi m](\bar{x})\Gamma m = [\xi'']$.

From the definition of Γm it is easy to see that $(j)\Gamma m = id_{(j)\Gamma o}$ for any $j \in J(X,.)$ and $(x.y)\Gamma m \subseteq (x)\Gamma m \circ (y)\Gamma m$ for $x,y \in X$, $x.y$ - def.. To prove that Γ is a functor into $(PIM,.)$ we need only to show $(x)\Gamma m \circ (y)\Gamma m \subseteq (x.y)\Gamma m$.

Let us consider $[\xi](x)\Gamma m = [(\xi)(x)\phi m] = [\eta]$, $[\eta](y)\Gamma m = [(\eta)(y)\phi m]$. There exists η' $(\xi)(x)\phi m \succ \eta' \prec \eta$. Then $(\eta')(y)\phi m \prec (\eta)(y)\phi m$, $(\eta')(\bar{x})\phi m \prec ((\xi)(x)\phi m)(\bar{x})\phi m = \xi$

and it is defined $((\eta')(\bar{x})\Phi m)(x.y)\Phi m =$
$= ((\eta')(\bar{x})\Phi m)((x)\Phi m \circ (y)\Phi m) = (\eta')(y)\Phi m$. Actually
$[\xi](x.y)\Gamma m = [\xi]((x)\Gamma m \circ (y)\Gamma m)$.

Finally, Γ was defined as a factorization of Φ with b_j from $(j)\Phi o$ onto $(j)\Gamma o$ sending $\xi \longmapsto [\xi]$.

Conversely, if the functor Γ' is a factorization of the canonical F-representation Φ , we may define on every set $(j)\Phi o$ the relation $\prec$ as follows.

$\xi \prec \eta$ iff $\exists j \in J(X,.)$ such that $(\xi)b_j = (\eta)b_j$ (is defined and holds) and $f_\xi \leq f_\eta$, where $f_\xi, f_\eta \in I(X,.)$ are idempotents that $\xi \in (f_\xi)Fo$, $\eta \in (f_\eta)Fo$.

We shall show firstly that for any $\eta_1, \eta_2 \in (j)\Phi o$ $(\eta_1)b_j = (\eta_2)b_j$ iff $\exists \xi \ \ \eta_1 \succ \xi \prec \eta_2$.

It is to prove the implication when $(\eta_1)b_j = (\eta_2)b_j = \alpha$. In this case $(\eta_1)(f_{\eta_1})\Phi m = \eta_1$ and $(\eta_2)(f_{\eta_2})\Phi m = \eta_2$. By the definition of factorization we have $(\alpha)(f_{\eta_1})\Gamma'm = \alpha =$ $= (\alpha)(f_{\eta_2})\Gamma'm$. Consequently, $\alpha = (\alpha)(f_{\eta_1}.f_{\eta_2})\Gamma'm$. But it shows: $\xi = (\xi)(f_{\eta_1}.f_{\eta_2})\Phi m$ must exist that $(\xi)b_j = \alpha$. So $f_\xi \leq f_{\eta_1}.f_{\eta_2}$ and actually $\eta_1 \succ \xi \prec \eta_2$.

We see easily that the relation $\prec$ satisfies the conditions 0),1) and also 2). We shall prove 3). If $\xi \prec \eta$ and $(\eta)(x)\Phi m$ is defined, it means $f_\xi \leq f_\eta = y\bar{y}$, $(\eta)(x)\Phi m =$ $= (\eta)(y)Fm$ where $y \leq x$. By Lemma 1 there exists $v \leq y$, $v\bar{v} = f_\xi$ and then $(\xi)(v)Fm = (\xi)(x)\Phi m$. $(\xi)(x)\Phi m \in (\bar{v}v)Fo$, $(\eta)(x)\Phi m \in (\bar{y}y)Fo$ / $\bar{v}v \leq \bar{y}y$ /. Because $(\xi)b_{d_x} = (\eta)b_{d_x}$ it must be $((\xi)(x)\Phi m)b_{r_x} = ((\eta)(x)\Phi m)b_{r_x}$. Really, $(\xi)(x)\Phi m \prec (\eta)(x)\Phi m$ holds true.

According to the proved part of the proposition, $\prec$ defines factorization Γ of Φ . We have also shown that a_j

defined by $[\xi] \longmapsto (\xi)b_j$ is a bijection of $(j)\Gamma o$ onto $(j)\Gamma'_o$ / $j \in J(X,.)$ /. Comparing the definitions we see that Γ is the functor Γ' with "renamed" morphisms $(x)\Gamma$ -- for $\forall x \in X$ $(x)\Gamma . (a_{r_x},(r_x)\Gamma o,(r_x)\Gamma'_o) =$

$$= (a_{d_x},(d_x)\Gamma o,(d_x)\Gamma'_o) . (x)\Gamma'.$$

This brings the proof to a close.

PROPOSITION 4.

Any functor from inverse category $(X,.)$ into $(PIM,.)$ is the factorization of some canonical F-representation.

Proof. Let H be a functor from $(X,.)$ into $(PIM,.)$. We shall define the functor F from the iso-category $(X,*)$ into $(PM,.)$ by following:

1) if $(x)Hm \neq \emptyset$,

$(x)F = ((x)Fm, \mathrm{Im}(x\bar{x})Hm \times \{x\bar{x}\}, \mathrm{Im}(\bar{x}x)Hm \times \{\bar{x}x\})$, where $\mathrm{Im}A$ means $\{\eta ; \exists \xi \ (\xi)A = \eta\}$ and $(x)Fm$ is defined by $(\xi,x\bar{x}) \longmapsto ((\xi)(x)Hm,\bar{x}x)$.

2) if $(x)Hm = \emptyset$,

$(x)F = ((x)Fm, \{\varepsilon\}\times\{x\bar{x}\},\{\varepsilon\}\times\{\bar{x}x\})$, where $(x)Fm$ maps $(\varepsilon,x\bar{x})$ to $(\varepsilon,\bar{x}x)$ and ε is one fixed element such that $\varepsilon \notin \mathrm{Im}(f)Hm$ for every $f \in I(X,.)$.

Let us note to this definition that $(x)Hm = \emptyset$ iff $(x\bar{x})Hm = \emptyset$.

The equality $x = x.\bar{x}.x$ also gives $\mathrm{Im}(x)Hm = \mathrm{Im}(\bar{x}x)Hm$. Consider $(x)Hm^{-1} = (\bar{x})Hm$ / and $\bar{\bar{x}} = x$ /, we see that $(x)Hm$ is a bijection of the set $\mathrm{Im}(x\bar{x})Hm$ onto $\mathrm{Im}(\bar{x}x)Hm$ (identity, especially, for $x \in I(X,.)$). It shows easily that F is a functor from $(X,*)$ into $(PM,.)$.

The functor F satisfies the condition from Proposition 2.

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We can build the functor Φ -- canonical F-representation of $(X,.)$ and its factorization will be H . Let us define on the sets $\bigcup_{z \leq j} (z)Fo$ relation $\prec$ as

$$(\xi,f) \prec (\eta,g) \quad \text{iff} \quad \xi = \eta \neq \varepsilon \quad \text{and} \quad f \leq g \ .$$

$\prec$ evidently satisfies assumptions 0) and 1) from Proposition 3.

Consider that $f,g \in I(X,.)$ are idempotents, $f.g$ is defined. If there is $\xi \in Im(f)Hm \cap Im(g)Hm$ then $\xi \in Im(f.g)Hm$. Namely, $(f.g)Hm = (f)Hm \circ (g)Hm$ is the equality for partial identities warranted by property of the functor H . Then $(\xi,f) \succ (\xi,f.g) \prec (\xi,g)$.

This proves the assumption 2) (and will show that a_j is injective).

Assumption 3). Suppose $(\xi,f) \prec (\xi,g)$, $(\xi,g)(x)\Phi m$ is defined, / $x \in X$ /, i.e. $(\xi,g)(x)\Phi m = (\xi,g)(gx)Fm =$ $= ((\xi)(gx)Hm, \bar{x}gx)$ / $g \leq x\bar{x}$ /. Because $(\xi)(f)Hm = \xi$, $(\xi)(gx)Hm = (\xi)(f.gx)Hm = (\xi)(fx)Hm$ (and also $= (\xi)(x)Hm$). Hence $(\xi,f)(x)\Phi m = ((\xi)(fx)Hm , \bar{x}fx)$ is defined and $(\xi,f)(x)\Phi m \prec (\xi,g)(x)\Phi m$.

According to Proposition 3, $\prec$ defines factorization Γ of Φ . This functor Γ is natural equivalent with H .

Define the bijections a_j / $j \in J(X,.)$ / of $(j)\Gamma o$ onto $(j)Ho$ by $[(\xi,f)] \longmapsto \xi$. Then $[(\xi,g)]\,((x)\Gamma m \circ a_{r_x}) =$ $= ((\xi)(x)Hm , \bar{x}g'x)a_{r_x} = (\xi)(x)Hm$ -- defined iff

$\exists\, g' \leq x\bar{x}.g$ and $\xi \in Im(g')Hm$; equivalently $\xi \in Im(x\bar{x})Hm$ (and $(\xi,g) \in (g)Fo$).

But under the same condition, it is defined

$$[(\xi,g)]\,(a_{d_x} \circ (x)Hm) = (\xi)(x)Hm \ .$$

We have verified $(x)\Gamma m \circ a_{r_x} = a_{d_x} \circ (x)Hm$ and the proof is complete.

III. CANONICAL REGULAR REPRESENTATIONS

A special case of inverse category is a connected category of isomorphisms. Small connected iso-category is a Brandt's groupiod; thin connected iso-category is the category $(X,.)$ satisfying $\forall j_1, j_2 \in J(X,.)\ \exists !\ x \in X$ $d_x = j_1,\ r_x = j_2$.

LEMMA 2.

Let $(X,.)$ be a category. $(X,.)$ is a connected iso-category iff it is isomorphic with the cartesian product of a group and a thin connected iso-category.

Proof. Let us fix one unit q of the connected iso-category $(X,.)$. Choose for any $j \in J(X,.)$ one morphism $m_j \in X$ satisfying $d_{m_j} = q$, $r_{m_j} = j$. The category $(X,.)$ is isomorphic with cartesian product of the group $(G,.)$, $G = \{ x \in X \ ;\ d_x = r_x = q \}$, . is the operation of $(X,.)$, and of the thin connected iso-category $(T,^\bullet)$, where $T = \{ (j_1,j_2) \ ;\ j_1,j_2 \in J(X,.) \}$ and $(j_1,j_2)^\bullet(j_3,j_4) = (j_1,j_4)$ is defined iff $j_2 = j_3$.

Injective functor H of $(X,.)$ onto $(G,.)\times(T,^\bullet)$ can be defined by $(x)H = (m_{d_x}.x.m_{r_x}^{-1},(d_x,r_x))$.

Conversely, it is trivial to conclude that the cartesian product of connected iso-categories is also a connected iso-category.

Call the representation C of a group $(G,.)$ by inner right translations the Cayley's representation of group -- $(g)C = (\rho_g,G,G)$ / ρ_g maps $g_1 \mapsto g_1.g$ /. Let S be a functor from thin connected iso-category $(T,^\bullet)$ into $(PM,.)$

which is injective (injective on $J(T,^{\bullet})$) and $(\iota)So \neq \emptyset$ / $\iota \in J(T,^{\bullet})$ /.

We can define a product functor $C\times S$ from the category $(G,.)\times(T,^{\bullet})$ into $(PM,.)$ by following

for units $(1,\iota)(C\times S)o = G\times(\iota)So$ / $\iota \in J(T,^{\bullet})$ /,

for morphisms $(g,\tau)(C\times S)m = \rho_g\times(\tau)Sm$ / $g\in G, \tau\in T$ /.

DEFINITION.

Let H be an isomorphism of connected iso-category $(X,.)$ onto product $(G,.)\times(T,^{\bullet})$ of a group and a thin iso-category. The composition of H and the described product functor -- $R = H\circ(C\times S)$ -- is called a regular representation of $(X,.)$

LEMMA 3.

Any functor F from a connected iso-category $(X,.)$ into $(PM,.)$ is factorization of a regular representation of $(X,.)$.

(If $(j)Fo \neq \emptyset$ for $j\in J(X,.)$, the factorization is total -- b_j are mappings.)

Proof. Let us suppose that the iso-category $(X,.)$ is directly $(G,.)\times(T,^{\bullet})$ and that F satisfies $(1,j)Fo \neq \emptyset$.

Choose one unit $\varkappa \in J(T,^{\bullet})$ and define an equivalence η on the set $(1,\varkappa)Fo$ by

$z_1 \,\eta\, z_2$ iff $\exists g\in G \;\; (z_1)(g,\varkappa)Fm = z_2$.

Denote $Y = \{[z] \; ; \; z\in(1,\varkappa)Fo\}$ / $\neq \emptyset$ / and fix for any $y\in Y$ one $z_y \in y$.

We shall define an injective functor S from $(T,^{\bullet})$ into $(PM,.)$ as $(\iota)So = Y\times\{\iota\}$ for $\iota\in J(T,^{\bullet})$ and $(\tau)Sm$, for $\tau\in T$, will map $(y,d_\tau) \longmapsto (y,r_\tau)$.

The functor F will be a total factorization of the regular representation $R = C\times S$.

Let us define / for any $\iota \in J(T,.)$ / a mapping b_ι from $(1,\iota)Ro = G\times(\iota)So$ into $(1,\iota)Fo$ as $(g,(y,\iota))b_\iota =$ $= (z_y)(g,\tau_\iota)Fm$, where τ_ι is a (unique) morphism from T such that $d_{\tau_\iota} = \varkappa$, $r_{\tau_\iota} = \iota$.

Any b_ι is onto. Namely, if $u \in (1,\iota)Fo$ then $y_u =$ $= [(u)(1,\tau_\iota^{-1})Fm] \in Y$ and there exists $g_u \in G$ $(z_{y_u})(g_u,\varkappa)Fm = (u)(1,\tau_\iota^{-1})Fm$. Consequently $(g_u,(y_u,\iota))b_\iota =$ $= ((u)(1,\tau_\iota^{-1})Fm)(1,\tau_\iota)Fm = u$.

It holds for any morphism $(f,\omega) \in G\times T$ $b_{d_\omega} \circ (f,\omega)Fm =$ $= (f,\omega)(C\times S)m \circ b_{r_\omega}$, which shows that F is a total factorization of $C\times S$. For any element (g,y,d_ω) namely $(g,y,d_\omega)(b_{d_\omega} \circ (f,\omega)Fm) = (z_y)(g.f,\tau_{r_\omega})Fm$ and also $(g,y,d_\omega)((f,\omega)(C\times S)m \circ b_{r_\omega}) = (g.f,y,r_\omega)b_r = (z_y)(g.f,\tau_{r_\omega})Fm$.

DEFINITION.

A canonical F-representation ϕ of inverse category $(X,.)$ is called a canonical regular representation provided that the restriction F_D of the functor F on every connected subcategory $(D,*)$ is a regular representation of $(D,*)$.

Note that it is easy, for any inverse category $(X,.)$, to construct (injective) functor F from $(X,*)$ into $(PM,.)$ satisfying the condition of Proposition 2 and whose restrictions F_D are regular representations of $(D,*)$. Because any canonical regular representation is injective (Proposition 2) every inverse category is isomorphic with an inverse category of partial bijections between sets. Easily, any inverse category is concrete.

By Proposition 4, any functor H from inverse category

(X,.) into (PIM,.) is a factorization of a canonical F-representation ϕ . According to Lemma 3, there exists (D,*)-regular representation R of (X,*) the total factorization of which is the functor F . It is easy to verify that the mappings b_f from the factorization R onto H build mappings $\bigcup_{f\leq j} b_f$ of $\bigcup_{f\leq j}(f)R\circ$ onto $(j)\phi\circ$ factorizing the canonical R-representation onto ϕ . Consequently, H is a factorization of canonical regular representation. So, we have proved the following proposition.

PROPOSITION 5.

Any representation of inverse category (X,.) by partial injective mappings between sets is a factorization of canonical regular representation of (X,.) .

According to Proposition 3 the factorizations of canonical regular representation are described by relations $\propto$. In particular, it describes any representation of the inverse semigroup with unit by partial injective transformations (characterized in [2] – §7.3) as a factorization of its canonical regular representation.

PROBLEM.

Is there a simple connection between relations $\propto$ and closed inverse subsemigroups (for the effective transitive representations) ?

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