数理解析研究所講究録
第 297 巻 1977 年 24-43

Several Topics on The Quadratic Forms of 3-Manifolds

By

Akio Kawauchi

Department of Mathematics, Osaka City University

In [7] the author defined a quadratic form of a compact, connected, oriented 3-manifold with non-zero first Betti number. (Also see a preliminary report [6].) This paper will consist of the applications of this quadratic form to several unrelated topics.

We consider a finite complex pair (X,A) (It is possible $A=\emptyset$.) with X connected and the non-zero first Betti number $\beta_1(X) \neq 0$. Let $(\tilde{X},\tilde{A})$ be the infinite cyclic cover of (X,A) associated with an epimorphism $\gamma: \pi_1(X) \to \langle t\rangle$. Define $T_*(\tilde{X},\tilde{A})=\mathrm{Tor}_{Q[t]}H_*(\tilde{X},\tilde{A};Q)$ and $T^*(\tilde{X},\tilde{A})=\mathrm{Hom}_Q[T_*(\tilde{X},\tilde{A}),Q]$.

<u>Definitions</u>. The ideal order $A_\gamma(t)$ of $T_1(\tilde{X})$ over $Q[t]$ is the Alexander polynomial of X with γ. Also, we define $\beta_1^\gamma(X,A) = \mathrm{rank}_{Q[t]}H_1(\tilde{X},\tilde{A};Q)$.

Remember the definition of the quadratic form: For a compact, connected, oriented 3-manifold M with an epimorphism $\gamma: \pi_1(M) \to \langle t\rangle$, a t-isometric, symmetric bilinear form $\langle\ ,\ \rangle: T^1(\tilde{M},\partial\tilde{M})\times T^1(\tilde{M},\partial\tilde{M}) \to Q = T^2(\tilde{M},\partial\tilde{M})$ is defined by the identity $\langle x,y\rangle = xUty + yUtx$ for all $x, y \in T^1(\tilde{M},\partial\tilde{M})$.

<u>Definitions</u>. The pair $(\langle\ ,\ \rangle,t)$ is the quadratic form of M with γ. Further, let $\sigma_\gamma(M)=$ signature$\langle\ ,\ \rangle$ and $n_\gamma(M)=$nullity $\langle\ ,\ \rangle$.[1)]

1)The definition of the nullity is the same as that of [7], but different from that of [6] in the case of bounded manifolds.

Definition. The ideal order $h_\gamma(t)$ of $T_1(\tilde{M})/\mathrm{Im}[T_1(\partial\tilde{M}) \to T_1(\tilde{M})]$ is the Hosokawa polynomial.

For the quadratic form of a bounded 3-manifold, the Hosokawa polynomial should be considered as well as the Alexander polynomial. (cf.[7].)

Main results of [7] may be stated as follows:

1.Theorem.(1) Let M be closed with an epimorphism $\gamma: \pi_1(M) \to \langle t \rangle$. If M is the boundary of a compact, connected, oriented 4-manifold W with an epimorphism $\bar{\gamma}: \pi_1(W) \to \langle t \rangle$ extending γ and such that the sequence $T_2(\tilde{W},\tilde{M}) \xrightarrow{\partial} T_1(\tilde{M}) \xrightarrow{i_*} T_1(\tilde{W})$ is exact at $T_1(\tilde{M})$, then $n_\gamma(M)$ is even and the induced non-singular form $(\langle\ ,\ \hat{\rangle}, t): \hat{T}^1(\tilde{M}) \times \hat{T}^1(\tilde{M}) \to Q$ is null-cobordant. In particular, $\sigma_\gamma(M) = 0$. Further, $A_\gamma(t) \doteq f(t)f(t^{-1})$ for some $f(t)$ in $Q[t]$.

(2) Let X_i, $i = 0,1$, be finite connected complexes. If there exists a finite connected complex Y which contains X_i and such that $H_j(Y,X_i;Q) = 0$, $j=1,2$, then we have $\beta_1^{\gamma_0}(X_0) = \beta_1^{\gamma_1}(X_1)$ for all compatible epimorphisms γ_i, $i = 0,1$. Moreover, if $H_j(Y,X_i;Z) = 0$, then $A_{\gamma_0}(t)$ and $A_{\gamma_1}(t)$ are equal up to units of $Q[t]$ and integral polynomials $f(t)$ with $|f(1)| = 1$. In case $X_i = M_i$, an orientable 3-manifold, $n_{\gamma_0}(M_0) = n_{\gamma_1}(M_1)$,and $h_{\gamma_0}(t)$ and $h_{\gamma_1}(t)$ are equal up to units of $Q[t]$ and integral polynomials $f(t)$ with $|f(1)| = 1$, provided that $H_j(Y,M_i;Z) = H_j(Y',\partial M_i;Z) = 0$, $j= 1,2$, for a subcomplex Y' of Y.

Illustration of (1). Let $M = F_g \times S^1$, F_g a closed surface of genus g and $W = T_g \times S^1$, T_g a solid torus of genus g. By taking the epimorphism γ determined by the projection to S^1, we have

$H_1(\tilde{M}) = H_1(F_g \times R^1) = \oplus[Q[t]/t-1]^{2g}$ and $H_2(\tilde{W},\tilde{M}) = T_2(\tilde{W},\tilde{M})$. Hence the conditions of (1) are satisfied. The form $< , >: T^1(\tilde{M}) \times T^1(\tilde{M}) \to Q$ satisfies $<x,y> = [t^{-1}(t-1)(t+1)x] \cup y = 0$ for all x, y. We have $n_\gamma(M) = 2g$, $\sigma_\gamma(M) = 0$, $A_\gamma(t) = (t-1)^{2g}$.

Illustration of (2). Consider two oriented links ℓ_0, $\ell_1 \subset S^3$ which are PL link concordant, that is, bound possibly non-locally flat PL annuli A in $S^3 \times [0,1]$ provided $\ell_i \subset S^3 \times i$, $i = 0, 1$. Let $Y = S^3 \times [0,1] - \mathrm{Int}N(A)$ for a regular neighborhood $N(A)$ of A in $S^3 \times [0,1]$ meeting the boundary $S^3 \times 0 \cup -S^3 \times 1$ regularly. Further, let $X_i = Y \cap S^3 \times i$. The triple (Y, X_0, X_1) satisfies the conditions of (2). By taking the epimorphism $\pi_1(Y) \to <t>$ determined by the unique epimorphism $\pi_1(S^3 - \ell_0) \to <t>$ sending each oriented meridian curve to t, we have $\beta(\ell_0) = \beta(\ell_1)$, $n(\ell_0) = n(\ell_1)$, $A_0(t) \equiv A_1(t)$ and $h_0(t) \equiv h_1(t)$ modulo $f(t) \in Z[t]$ with $|f(1)| = 1$ and units of $Q[t]$. In particular, since every knot k is PL knot cobordant to a trivial knot, it follows that $\beta(k) = 0$, $n(k) = 0$ and $|A(1)| = 1$, where we choose $A(t)$ to be primitive.

As a standard corollary of Theorem 1, we have the following:

2. Corollary. The polynomial $A_\gamma(t)$ modulo $f(t)f(t^{-1})$ for $f(t) \in Z[t]$ with $|f(1)| = 1$ and the integers $\beta_1^\gamma(M)$, $n_\gamma(M)$, $\sigma_\gamma(M)$ are the invariants of the homology cobordisms of a closed and oriented 3-manifold M with epimorphism $\gamma: \pi_1(M) \to <t>$.

In [7] we have obtained the following three consequences: [In fact, in [7] one can find a more general principle on each topic.]

1. <u>The Baumslag-Solitar group</u> $G_{(p,q)} = (a, b \mid a^{-1}b^{p}a = b^{q})$ <u>is a 3-manifold group if and only if</u> $|p| = |q|$ <u>or</u> $pq = 0$.

2. <u>Consider the orientable torus bundle</u> M <u>over</u> S^1 <u>with bundle projection</u> $p: M \to S^1$. <u>Regard</u> $S^1 = S^1 \times * \subset S^1 \times S^3$. <u>The projection</u> p <u>is not homotopic to a piecewise-linear embedding except for the possible case that</u> M <u>is homeomorphic to</u> $S^1 \times S^1 \times S^1$ <u>or the boundary</u> $\partial N(S^1 \times_\tau S^1; S^4)$ <u>of the regular neighborhood</u> $N(S^1 \times_\tau S^1; S^4)$ <u>of a "standardly" embedded Klein bottle</u> $S^1 \times_\tau S^1$ <u>in</u> S^4. [To be precise, a standardly embedded Klein bottle in S^4 means the boundary of a solid Klein bottle in S^4.]

3. (<u>A counterexample to PL Whitney lemma</u>) <u>There exists a simply connected, compact 4-manifold</u> W <u>with</u> $H_2(W;Z) = Z \oplus Z$ <u>and such that</u>

(1) <u>Each homology class of</u> $H_2(W;Z)$ <u>can be represented by a piecewise-linearly embedded</u> 2-<u>sphere in</u> W,

(2) <u>Each pair of the homology classes of</u> $H_2(W;Z)$ <u>has the intersection number</u> 0,

(3) <u>Each pair of the homology classes of</u> $H_2(W;Z)$ <u>forming a basis can not be represented by mutually disjoint</u>, <u>piecewise-linearly embedded 2-spheres</u>.

<u>Application A</u>. <u>Remarks on a connected linear graph in</u> S^3.

Let $L^n \subset S^3$ be a connected linear graph with $H_1(L^n;Z) = \oplus Z^n$. We let $X = S^3 - \mathrm{Int}N(L^n)$ for the regular neighborhood $N(L^n)$ of L^n in S^3. For an epimorphism $\gamma: \pi_1(X) \to \langle t \rangle$ the Alexander polynomial $A_\gamma(t)$ of X with γ is called the Alexander polynomial of L^n with γ. Also, let $\beta^\gamma(L^n) = \beta_1^\gamma(X)$. <u>It follows that</u> $\beta^\gamma(L^n) = n-1$ <u>and the primitive Alexander polynomial</u> $A_\gamma(t)$ <u>necessarily</u>

<u>satisfies</u> $|A_\gamma(1)| = 1$ <u>for all epimorphisms</u> γ.

Proof. It is easy to construct a finite connected 2-complex $K \subset S^3 \times [0,1]$ with $S^3 \times 0 \cap K = L \times 0$ and $L' = S^3 \times 1 \cap K$, a standard n-leafed rose in $S^3 \times 1$, and such that $H_*(S^3 \times [0,1] - K; Z) = H_*(S^3 \times 0 - L \times 0; Z) = H_*(S^3 \times 1 - L'; Z)$ by the inclusion isomorphisms. Take a regular neighborhood $N(K)$ of K in $S^3 \times [0,1]$ meeting $S^3 \times 0 \cup S^3 \times 1$ regularly. Let $Y = S^3 \times [0,1] - \mathrm{Int} N(K)$, $X_i = Y \cap S^3 \times i$, $i = 0,1$. By applying Theorem 1,(2) to the triple (Y, X_0, X_1), we have that $\beta^\gamma(L^n) = n-1$ and $|A_\gamma(1)| = 1$, since $\pi_1(S^3 \times 1 - L')$ is a free group of rank n. This completes the proof.

This assertion can be also derived from results of S.Kinoshita [9, Theorems 8 and 9]. It seems that the above proof clarifies the geometric meaning to some extent.

The Hosokawa polynomial $h_\gamma(t)$ of X with γ is called the Hosokawa polynomial of L^n with γ. As a simple consequence, the Hosokawa polynomial of L^n with γ is equal to the Alexander polynomial of L^n with γ. In [10] S.Kinoshita showed that, for any integral polynomial $f(t)$ with $|f(1)| = 1$, there exists a θ-curve whose Alexander polynomial is $f(t)$. From this one can derive the following existence assertion: <u>For each</u> $n \geq 2$ <u>and any primitive</u>(<u>integral</u>) <u>polynomial</u> $f(t)$ <u>with</u> $|f(1)| = 1$, <u>there exists a knotted n-leafed rose</u> $L^n \subset S^3$ <u>with an epimorphism</u> $\gamma: \pi_1(S^3 - L^n) \to \langle t \rangle$ sending suitably oriented meridian curves of L^n to t and such that the Alexander polynomial $A_\gamma(t)$ of L^n with γ is $f(t)$. (See S.Suzuki[14] for a discussion.)

<u>Application B. An elementary proof of Y.Matsumoto's example of a spineless 4-manifolds.</u>

In [12] Y.Matsumoto announced the following assertion:

Assertion. *There exists a compact, connected orientable 4-manifold W which is homotopy equivalent to a torus $F_1=S^1\times S^1$ and such that no homotopy equivalence of F_1 to W is homotopic to a piecewise-linear embedding.*

We shall give it an elementary proof.

The construction of W is as follows: Take the embedding $h:S^1 \to S^1\times D^2$ illustrated in Fig. 1. Then extend h to a framed

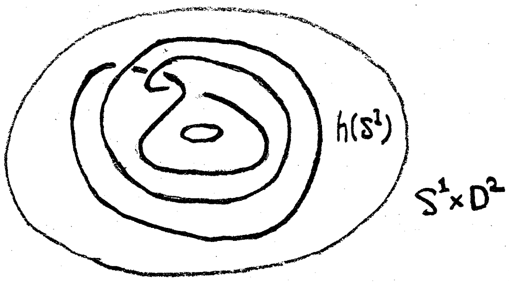


Fig. 1

embedding $\bar{h}:S^1\times D^2 \to \mathrm{Int}S^1\times D^2$ so that the framing is trivial in S^3 via inclusion $S^1\times D^2\subset S^3$. Take as W the mapping torus of $\bar{h}$.

Proof of Assertion. Suppose there exists a piecewise-linear embedding $F_1\subset W$ that is also a homotopy equivalence. We can assume that F_1 has just one locally-knotted point in W.(See R.H.Fox and J.W.Milnor[2].) Let $k\subset S^3$ be a knot representing this local knot type. Take a regular neighborhood $N = N(F_1)$ of F_1 in W and let $W'= W-\mathrm{Int}N$. Since the embedding $F_1\subset W$ induces an isomorphism $H_*(F_1;Z)\approx H_*(W;Z)$, it follows that W' is a homology cobordism between ∂W and ∂N. Note that $\partial N=\overline{F-\Delta}\times S^1\cup E(k)$

, where Δ is a 2-cell in F_1 and $E(k)$ is the knot exterior of the knot $k \subset S^3$. Choose a basis $\{x_1, x_2, x_3\}$ for $H_1(\partial W; Z) \approx Z \oplus Z \oplus Z$ such that x_1 is the image of a generator of $H_2(W, W-F_1; Z) \approx Z$ under the composite monomorphism $H_2(W, W-F_1; Z) \xrightarrow{\partial} H_1(W-F_1; Z) \underset{\approx}{\xrightarrow{i_*^{-1}}} H_1(\partial W; Z)$. The remaining two generators x_2 and x_3 are chosen so that x_2 represents a generator of $H_1(S^1 \times D^2; Z)$ and x_3 is a new generator resulting from the mapping torus of $\bar{h}$. Let γ_1: $\pi_1(\partial W) \rightarrow \langle t \rangle$ be the epimorphism sending x_1 to t and x_2, x_3 to 1. We have $H_1(\partial\widetilde{W}) = Q[t]/(t-1) \oplus Q[t]/(t-1)$ and $H_1(\partial\widetilde{N}) = Q[t]/(t-1) \oplus Q[t]/(t-1) \oplus T_1(\widetilde{E}(k))$ for the epimorphism $\pi_1(\partial N) \rightarrow \langle t \rangle$ corresponding to γ_1. By Corollary 2, the knot polynomial $\Delta_k(t)$ of $k \subset S^3$ [, that is, the ideal order of $T_1(\widetilde{E}(k))$] must be of a slice type, i.e., $\Delta_k(t) \doteq f(t)f(t^{-1})$. Next, let $\gamma_2: \pi_1(\partial W) \rightarrow \langle t \rangle$ be the epimorphism sending x_1, x_2, x_3 to t. Then we have $H_1(\partial\widetilde{W}) = Q[t]/(t-1) \oplus Q[t]/(t-1) \oplus Q[t]/(2t^2-3t+2)$ and $H_1(\partial\widetilde{N}) = Q[t]/(t-1) \oplus Q[t]/(t-1) \oplus T_1(\widetilde{E}(k))$. By Corollary 2 again, $2t^2-3t+2$ must be of a slice type, since $\Delta_k(t)$ is of a slice type. This contradicts the irreducibility of $2t^2-3t+2$. This completes the proof.

Remark 1. The conclusion of the proof can be also reached by using the signatures instead of the Alexander polynomials. In fact, from the above proof we can see that $\sigma(k) = \sigma_{\gamma_1}(\partial W) = 0$ and $\sigma(k) = \sigma_{\gamma_2}(\partial W) = \pm 2$, which is a contradiction.

Remark 2. In [8] Y. Nakagawa and the author showed that the n-variable Alexander polynomial $A_\gamma(t_1,..,t_n)$ modulo $f(t_1,..,t_n) f(t_1^{-1},..,t_n^{-1})$ for an integral $f(t_1,..,t_n)$ with $|f(1,..,1)| = 1$ is a homology cobordism invariant of a closed orientable 3-manifold M with epimorphism $\gamma: \pi_1(M) \rightarrow \langle t_1,..,t_n \rangle$. Y. Matsumoto suggested to the author that the above assertion can be also shown by applying this to the three-variable Alexander polynomial $A_\gamma(t_1,t_2,t_3)$ of ∂W with γ sending x_i to t_i, i=1,2,3.

Now consider a compact, orientable 4-manifold W with $H_*(W;Z) \approx H_*(F_g;Z)$ and $H_*(\partial W;Z)\approx H_*(F_g \times S^1;Z)$, where F_g is a closed surface of genus g. Let x_0 be an element of $H_1(\partial W;Z)$ that is the image of a generator of $H_2(W,\partial W;Z)\approx Z$ by the monomorphism $\partial: H_2(W,\partial W;Z) \to H_1(\partial W;Z)$. The principle used in the proof of the above assertion is formulated as follows:

Theorem. Assume there exists a piecewise-linear embedding $e:F_g \to W$ such that $e_*:H_*(F_g;Z)\approx H_*(W;Z)$. Then for all epimorphisms $\gamma: \pi_1(\partial W) \to \langle t\rangle$ sending x_0 to t we have $A_\gamma(t) \equiv (t-1)^{2g} \Delta_k(t)$ modulo $f(t)f(t^{-1})$ for an integral f(t) with $|f(1)| = 1$ and units of Q[t], $n_\gamma(\partial W) = 2g$, $\sigma_\gamma(\partial W) = \sigma(k)$ and $\beta_1^\gamma(\partial W) = 0$. Here, $\Delta_k(t)$ is the knot polynomial of the local knot type $\langle k \subset S^3\rangle$ of the embedded surface $e(F_g)$ in W, provided that $e(F_g)$ is modified to have just one locally knotted point.

Application C. A piecewise-linear embedding of a closed 3-manifold in 4-sphere S^4 (or a homology 4-sphere $\bar{S}^4$).

It suffices to consider a closed, connected and orientable 3-manifold M^3, since any closed non-orientable 3-manifold cannot be embedded in a homology 4-sphere $\bar{S}^4$.

The only known classical result is due to W.Hantzsch[3].

Hantzsch's Theorem. If M is embeddable to S^4, then the torsion part $T_1(M;Z)$ of $H_1(M;Z)$ splits into two copies of a torsion group T, i.e., $T_1(M;Z)\approx T\oplus T$.

This follows from the Alexander duality and the Mayer-Vietoris sequence. The 4-sphere S^4 can be replaced by a homology 4-sphere $\bar{S}^4$ and the embedding may be topological. As a direct consequence of Hantzsch's Theorem, the lens space $L(p,q)$ cannot be embedded in $\bar{S}^4$.

We are now interested in a piecewise-linear embedding $e:M \to \bar{S}^4$. The only advantage of this is that $e(M)$ separates $\bar{S}^4$ into two connected compact piecewise-linear manifolds W_1, W_2 with common boundary $e(M)$ by the Alexander's theorem. [Here, by a homology 4-sphere, we mean a triangulated 4-manifold having the integral homology group of S^4.]

We shall consider only pairs (M,γ) satisfying the following hypothesis:

Hypothesis. The epimorphism $\gamma: \pi_1(M) \to \langle t \rangle$ satisfies $\beta_1^{\gamma}(M) = \beta_1(M)-1$.

Note. If for M with $\beta_1(M) = n \geq 1$ there exists an epimorphism from $\pi_1(M)$ to the free group of rank n, then such a manifold M satisfies $\beta_1^{\gamma}(M)= \beta_1(M)-1$ for all epimorphisms $\gamma: \pi_1(M) \to \langle t \rangle$. An example is the manifold $M(\ell)$ obtained from S^3 by a surgery along a (homology) boundary link $\ell \subset S^3$ with null-homologous framing. (cf. N.Smythe[13].)

For an embedding $e:M \subset \bar{S}^4$, we let $\bar{S}^4 = W_1 \cup_M W_2$. From the Mayer-Vietoris sequence, we obtain an isomorphism $i_1^*+i_2^*$: $H^1(W_1;Z) \oplus H^1(W_2;Z) \approx H^1(M;Z)$.

Definition. An embedding $e: M \subset \bar{S}^4$ is γ-essential, if γ, regarded as an element of $H^1(M;Z)$, is either in $\operatorname{Im} i_1^*$ or in $\operatorname{Im} i_2^*$.

Theorem. If there exists a γ-essential embedding $e:M \to \overline{S}^4$, then $A_\gamma(t) \doteq f(t)f(t^{-1})$, $\sigma_\gamma(M) = 0$ and $n_\gamma(M)$ is even.

Proof. Assume $\gamma \in \operatorname{Im} i_1^*$. Then there exists an epimorphism $\overline{\gamma}: \pi_1(W_1) \to \langle t \rangle$ with the following commutative triangle:

$$\begin{array}{ccc} \pi_1(M) & \xrightarrow{i_\#} & \pi_1(W_1) \\ & {}_{\gamma}\searrow \quad \swarrow_{\overline{\gamma}} & \\ & \langle t \rangle & \end{array}$$

Let $\beta_1(M) = n$, $\beta_1(W_1) = m$ and $\beta_2(W_1) = n-m$. From the duality theorem on $Q[t]$-ranks (See [7].), it follows that

$$\beta_3^{\overline{\gamma}}(W_1) = \beta_1^{\overline{\gamma}}(W_1,M) = 0,$$
$$\beta_3^{\overline{\gamma}}(W_1,M) = \beta_1^{\overline{\gamma}}(W_1) \leq m-1,$$
$$\beta_2^{\gamma}(M) = \beta_1^{\gamma}(M) = n-1$$

and

$$\beta_2^{\overline{\gamma}}(W_1) \leq n-m.$$

Consider the following exact sequence of the pair $(\tilde{W}_1, \tilde{M})$:

$$\cdots \to H_3(\tilde{W}_1) \to H_3(\tilde{W}_1,\tilde{M}) \to H_2(\tilde{M}) \to H_2(\tilde{W}_1) \xrightarrow{j_*} H_2(\tilde{W}_1,\tilde{M})$$
$$\xrightarrow{\partial} H_1(\tilde{M}) \xrightarrow{i_*} H_1(\tilde{W}_1) \to \cdots$$

This sequence asserts that $\beta_2^{\gamma}(M) - \beta_3^{\overline{\gamma}}(W_1,M) \leq \beta_2^{\overline{\gamma}}(W_1)$. Hence $\beta_2^{\gamma}(M) - \beta_3^{\overline{\gamma}}(W_1,M) = \beta_2^{\overline{\gamma}}(W_1)$. Thus, the image of $j_*: H_2(\tilde{W}_1;Q) \to H_2(\tilde{W}_1,\tilde{M};Q)$ is contained in $T_2(W,M)$. This implies that the sequence $T_2(\tilde{W}_1,\tilde{M}) \xrightarrow{\partial} T_1(\tilde{M}) \xrightarrow{i_*} T_1(\tilde{W}_1)$ is exact. The result now follows from Theorem 1,(1). This completes the proof.

Example 1. Consider the link $3_1 \cup -3_1^*$ in S^3, illustrated in Fig. 2. Let $M = M(3_1 \cup -3_1^*)$ be the 3-manifold obtained from S^3 by a surgery along the link $3_1 \cup -3_1^*$ with null-homologous framing. M is not embeddable to $\overline{S}^4$.

Proof. Choose a basis $\{x, y\}$ for $H_1(M;Z) \approx Z \oplus Z$ as in Fig.2.

Suppose $e: M \subset \overline{S}^4$. For simplicity identify $H_1(M;Z)=H_1(W_1;Z)\oplus H_1(W_2;Z)$.

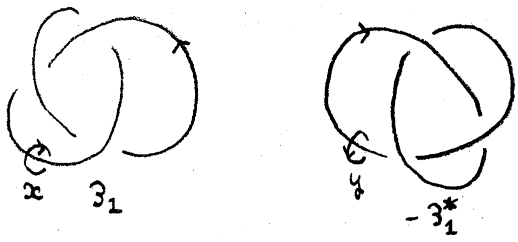


Fig. 2.

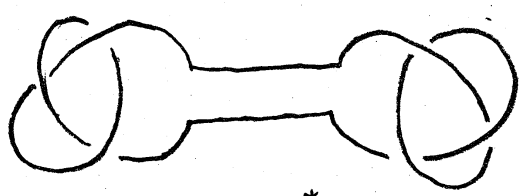
Case(1) $H_1(W_1;Z)=0$ (or $H_1(W_2;Z)=0$). In this case, e is γ-essential for all epimorphisms $\gamma: \pi_1(M) \to \langle t \rangle$. By taking the epimorphism γ sending x to t and y to 1, we obtain $A_\gamma(t)=t^2-t+1$ (and $\sigma_\gamma(M) = \pm 2$), which is a contradiction.

Case(2) $H_1(W_1;Z)\approx H_1(W_2;Z)\approx Z$. Let $mx+ny$ and $m'x+n'y$ be the generators of $H_1(W_1;Z)$ and $H_1(W_2;Z)$, respectively. Then we may assume $mn'-m'n = 1$. Consider the epimorphism $\gamma_i: \pi_1(W_i) \to \langle t \rangle$,$i= 1,2$, as follows: $\gamma_1(mx+ny)= t$ and $\gamma_1(m'x+n'y)= 1$, i.e., $\gamma_1(x) = t^{n'}$ and $\gamma_1(y)= t^{-m'}$, and $\gamma_2(mx+ny)= 1$ and $\gamma_2(m'x+n'y)=t$,i.e., $\gamma_2(x) = t^{-n}$ and $\gamma_2(y) = t^{m}$. Notice that $A_{\gamma_1}(t) = [(t^{n'})^2-t^{n'}+1][(t^{-m'})^2-t^{-m'}+1]$. In case $m'+n'$ is odd, then $A_{\gamma_1}(-1)=3$. Hence $A_{\gamma_1}(t)$ is not of a slice type; i.e., $f(t)f(t^{-1})$. In case $m'+n'$ is even, then $m+n$ is odd, since $mn'-m'n = 1$. Then, $A_{\gamma_2}(t)=[(t^{-n})^2-t^{-n}+1][(t^{m})^2-t^{m}+1]$ is not of a slice type. This contradicts to the fact that e is γ_1- and γ_2-essential. This completes the proof.

<u>Remark</u> 1. In the above example, let γ be the epimorphism

sending both x and y to t. One can see that $A_\gamma(t)=(t^2-t+1)^2$ and $\sigma_\gamma(M)=0$ and $n_\gamma(M)=0$. More strongly, the quadratic form $(\langle\ ,\ \rangle,t)$ of (M,γ) is null-cobordant.(cf.[5].)

Remark 2. If we perform a knot sum operation on $3_1 \cup -3_1^*$, then the manifold $M(3_1\#-3_1^*)$ resulting from the knot $3_1\#-3_1^*$ (the square knot) (See Fig. 3 below.) with null-homologous framing is embeddable in S^4, since the knot $3_1\#-3_1^*$ is a slice knot. (See, for example,[5,Corollary 2.5].)



$3_1\#-3_1^*$

Fig. 3.

Question. Is $M(3_1)$ - Int Δ^3 embeddable to S^4 ? That is, do there exist a 2-knot $k^2 \subset S^4$ having $M(3_1)$-Int Δ^3 as a Seifert surface ? (D.B.A.Epstein[1] observes that if such an embedding exists, then there will do exist a degree one map from S^4 to the suspension of $M(3_1)$.)

Example 2. Consider the link $\ell \subset S^3$ with two components illustrated in Fig. 4. Choose a basis $\{x, y\}$ for $H_1(M(\ell);Z)$ specified in Fig. 4 and let $\gamma_{m,n}: \pi_1(M(\ell)) \twoheadrightarrow \langle t\rangle$ be the epimorphism sending x to t^m and y to t^n, where m and n are coprime integers. It can be shown that there esists a $\Upsilon_{0,1}$- (or $\Upsilon_{1,0}$-) essential embedding of $M(\ell)$ to S^4. In fact, divide

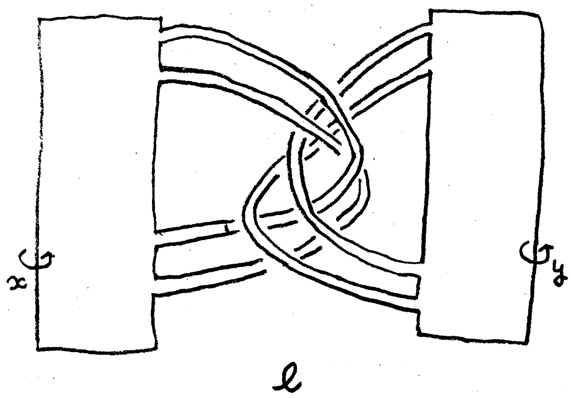


Fig 4.

S^4 into 4-cells D_1^4, D_2^4 with common boundary S^3: $S^4 = D_1^4 \cup D_2^4$. Let $D^2 \times D_1^2$, $D^2 \times D_2^2$ be the two 2-handles used to obtain $M(\ell)$ from S^3. Embed $D^2 \times D_1^2$ to D_1^4 and $D^2 \times D_2^2$ to D_2^4. (See Fig. 5.) This shows that $M(\ell)$ is embeddable to S^4 as a $\Upsilon_{0,1}$-(or $\Upsilon_{1,0}$-) essential embedding.

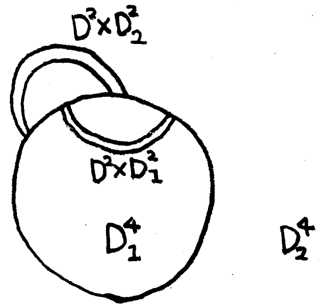


Fig. 5.

On the other hand, we can show that there is no $\Upsilon_{m,n}$-essential embedding of $M(\ell)$ to $\overline{S}^4$ at least in the case that both m and n <u>are odd</u>.(For other cases, the assertion is still undetermined.)

Proof. $A_{\tau_{m,n}}(t) = (t^m-1)^2(t^n-1)^2+t^{m+n}$. Since both m and n are odd, we have $A_{\tau_{m,n}}(-1) = 17$. This implies that $A_{\tau_{m,n}}(t)$ is not of a slice type.

<u>Application</u> D. <u>Cobordism of 3-manifolds</u>.

A closed connected orientable 3-manifold M is <u>distinguished</u> ,if there are given an epimorphism $\gamma: \pi_1(M) \twoheadrightarrow \langle t \rangle$ and a generator ι of $H_3(M;Z) \approx Z$. Denote the distinguished manifold by $M(\gamma,\iota)$. If $H^1(M;Z) = 0$, then we take the empty $\emptyset$ as γ.

<u>Definition</u>. Two distinguished manifolds $M(\gamma,\iota)$ and $M'(\gamma',\iota')$ are <u>cobordic</u>, if there exists a compact, connected, oriented 4-manifold W with $\partial W = M(\gamma,\iota) \cup M'(\gamma',-\iota')$ and an epimorphism $\bar{\gamma}: \pi_1(W) \twoheadrightarrow \langle t \rangle$ extending γ,γ' such that the sequence $T_2(\tilde{W},\partial\tilde{W}) \to T_1(\partial\tilde{W}) \xrightarrow{i_*} T_1(\tilde{W})$ is exact at $T_1(\partial\tilde{W})$.

One can easily checked that the class of distinguished 3-manifolds modulo this cobordic relation forms an abelian group under the usual (oriented) connected sum operation. We say that this group is <u>the</u> (<u>3-dimensional</u>) <u>rational cobordic group</u> and denoted by $\Omega^3(Q)$. The subclass of distinguished 3-manifolds with free abelian first homology groups modulo the cobordic relation forms a subgroup of $\Omega^3(Q)$ denoted by $\Omega^3(Z)$ and called <u>the integral cobordic group</u>.

1.<u>Lemma</u>. <u>If</u> $M(\gamma,\iota)$ <u>is homology cobordant to</u> $M'(\gamma',\iota')$

<u>with compatible</u> γ <u>and</u> γ' <u>accompanied</u>, <u>then</u> $M(\gamma,\iota)$ <u>is</u> <u>cobordic to</u> $M'(\gamma',\iota')$.

Proof. Let W be a homology cobordism between $M(\gamma,\iota)$ and $M'(\gamma',\iota')$. It is immediate to see that the sequence $T_2(\tilde{W},\partial\tilde{W}) \xrightarrow{\partial} T_1(\partial\tilde{W}) \xrightarrow{i_*} T_1(\tilde{W})$ is exact. This completes the proof.

The following shows that the converse of Lemma 1 is not true.

<u>Example</u>. $S^1\times S^2 \# S^1\times S^2 \# S^1\times S^2$ and $S^1\times S^1\times S^1$ (with arbitrary epimorphisms) are cobordic (, in fact, represent the zero element of $\Omega^3(Z)\subset\Omega^3(Q)$) and have the same (integral) homology group. However, these are never homology cobordant.[This follows from Corollary 2, since $n_\gamma(S^1\times S^2\# S^1\times S^2\# S^1\times S^2) = 0$ and $n_{\gamma'}(S^1\times S^1\times S^1) = 2$ for arbitrary epimorphisms γ and γ'.]

2. <u>Lemma</u>. <u>If</u> $H^1(M;Z) = 0$, <u>that is</u>, M <u>is a rational homology 3-sphere</u>, <u>then</u> M <u>represents the zero element of</u> $\Omega^3(Q)$.

Proof. Take a simply connected, compact 4-manifold W whose boundary is M. Let Σ be the boundary of a 3-cell in the interior of W and N be the regular neighborhood of Σ in W. Write $W_1 = W - \mathrm{Int}N$. Then $\partial W_1 = M \cup S^1\times S^2$ and the inclusion $S^1\times S^2 \subset W_1$ induces an isomorphism $\pi_1(S^1\times S^2) \approx \pi_1(W_1) \approx \langle t\rangle$. Since $T_1(\partial\tilde{W}_1) = 0$, M is cobordic to $S^1\times S^2$ and hence represents the zero element of $\Omega^3(Q)$.

Now we consider two distinguished homology orientable handles $M(\gamma,\iota)$ and $M'(\gamma',\iota')$ (, that is, two distinguished manifolds having the integral homology group of $S^1 \times S^2$).

3. Lemma. The connected sum $M(\gamma,\iota)\# M'(\gamma',\iota')$ is cobordic to the circle union $(M \circ M')(\gamma'',\iota'')$. (See [5].)

Proof. Construct a 4-manifold W that is the adjunction space $M\times[0,1] \cup B^3\times[0,1] \cup M'\times[0,1] \cup S^1\times B^2\times[0,1] \cup$ with boundary ∂W homeomorphic to the disjoint union $M(\gamma,\iota)\# M'(\gamma',\iota')\cup(M\circ M')(\gamma'',-\iota'')$. (See Fig. 6.) Notice that $\beta_2^{\bar{\gamma}}(W) = 0$ for the specified

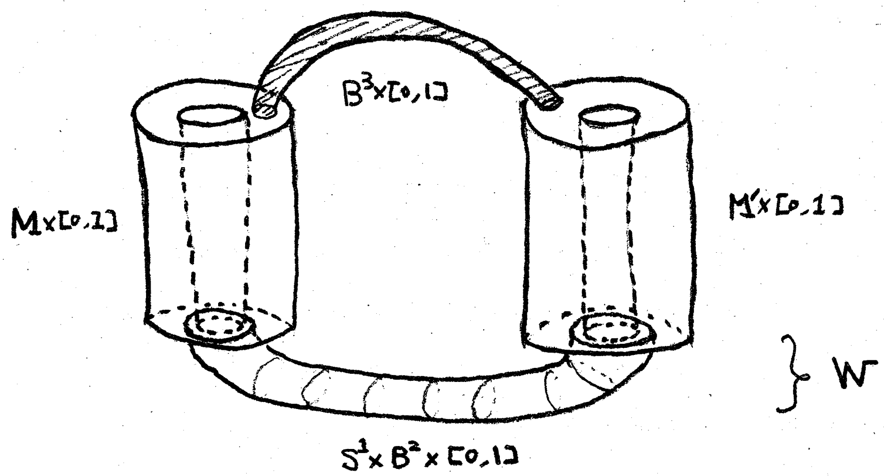


Fig. 6.

epimorphism $\bar{\gamma}: \pi_1(W) \to \langle t\rangle$. Hence $\beta_2^{\bar{\gamma}}(W,\partial W)= 0$ by Duality Theorem II of [7]. This implies that $T_2(\tilde{W},\partial\tilde{W}) \xrightarrow{\partial} T_1(\partial\tilde{W}) \xrightarrow{i_*} T_1(\tilde{W})$ is exact. This completes the proof.

From Lemma 3 it follows that the class of distinguished homology orientable handles modulo the cobordic relation forms a subgroup $\Omega_1^3(Z)$ of $\Omega^3(Z)$ (and hence of $\Omega^3(Q)$) under the circle union. From the construction of the $\tilde{H}$-cobordism group $\Omega(S^1\times S^2)$ (See [5].), we can easily see the following:

4. *Lemma. There is the following commutative triangle of epimorphisms*:

$$\begin{array}{ccc} \Omega(S^1\times S^2) & \longrightarrow & \Omega_1^3(Z) \\ & \varphi\searrow \quad \swarrow\Psi & \\ & G_- & \end{array}$$

, *where* G_- *is the Levine's integral matrix cobordism group*.

We shall show the following:

5. *Theorem. The epimorphism* $\Psi:\Omega_1^3(Z) \to G_-$ *is an isomorphism. Thus,* $\Omega_1^3(Z)$ *is isomorphic to the direct sum of infinite copies of* Z, Z_2 *and* Z_4.

Proof. Let $M = M(\Upsilon,\iota)$ be a distinguished homology orientable handle with $\Psi[M]= 0$. Take a closed connected oriented surface $F_g \subset M$ of genus g transversal to the generator of $H_1(M;Z)$ dual to Υ. (See [4].) Let θ: $H_1(F_g;Z)\times H_1(F_g;Z) \to Z$ be the linking pairing defined by $\theta(x,y) = L(x,i_*(y))$ for cycles x and y in F_g, where $i_*(y)$ is the cycle in M translating y off F_g in the positive normal direction and $L(x,i_*(y))$ is the linking number of x and $i_*(y)$ in M. [Notice that both x and $i_*(y)$ are homologous to 0 in M.] (cf. [5, 2.19].) $\Psi[M] = 0$ asserts that there exists a basis $\alpha_1,\dots, \alpha_g, \beta_1,\dots, \beta_g$ for $H_1(F_g;Z)$ with $\theta(\alpha_i,\alpha_j) = 0$ for all i, j. In particular, $\alpha_i\cdot\alpha_j = \theta(\alpha_i,\alpha_j)- \theta(\alpha_j,\alpha_i) = 0$, where $\cdot$ denotes the intersection number in F_g.

6. *Lemma. By suitable choices of* $\beta_1,\dots, \beta_g$, *we can assume that* $\alpha_i\cdot\alpha_j = \beta_i\cdot\beta_j = 0$ *and* $\alpha_i\cdot\beta_j = \delta_{ij}$ *for all* i, j.

Proof. Let $\alpha_1^\#$ be the element of $H_1(F_g;Z)$ dual to α_1, i.e., $\alpha_i\cdot\alpha_1^\# = \delta_{i1}$, $\beta_i\cdot\alpha_1^\# = 0$ by the non-singular skewsymmetric pairing $H_1(F_g;Z)\times H_1(F_g;Z) \xrightarrow{\cdot} Z$ w.r.t. the basis $\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g$. $H_1(F_g;Z)$ has the orthogonal decomposition $H_1(F_g;Z)=\{\alpha_1,\alpha_1^{\#}\}\perp X_1$. Notice that $\alpha_2,\dots,\alpha_g$ are contained in X_1. Since $H_1(F_g;Z)/\{\alpha_1,\dots,\alpha_g\}\approx\{\alpha_1,\alpha_1^{\#}\}/\{\alpha_1\}\oplus X_1/\{\alpha_2,\dots,\alpha_g\}$ is a free abelian group of rank g, $X_1/\{\alpha_2,\dots,\alpha_g\}$ is free abelian of rank $g-1$. By induction, X_1 has an orthogonal decomposition $X_1=\{\alpha_2,\alpha_2^{\#}\}\perp\dots\perp\{\alpha_g,\alpha_g^{\#}\}$ and hence $H_1(F_g;Z)=\{\alpha_1,\alpha_1^{\#}\}\perp\dots\perp\{\alpha_g,\alpha_g^{\#}\}$. Take $\alpha_1^{\#},\dots,\alpha_g^{\#}$ as $\beta_1,\dots,\beta_g$. This proves Lemma 6.

By lemma 6, assume $\alpha_i\cdot\alpha_j=\beta_i\cdot\beta_j=0$ and $\alpha_i\cdot\beta_j=\delta_{ij}$. Let $\alpha_1^o,\dots,\alpha_g^o,\beta_1^o,\dots,\beta_g^o$ be the standard basis for $H_1(F_g;Z)$. As pointed out by H.Terasaka(See, also, [11 ,p178].), we can obtain an auto-homeomorphism $h:F_g \to F_g$ such that $h_*(\alpha_i^o)=\alpha_i$ and $h_*(\beta_j^o)=\beta_j$. As a result, $\alpha_1,\dots,\alpha_g$ are represented by mutually disjoint, simple closed curves $S_1^1\subset F_g,\dots, S_g^1\subset F_g$. Thicken $S_1^1,\dots,S_g^1$ to mutually disjoint annuli $S_1^1\times[0,1]\subset F_g,\dots, S_g^1\times[0,1]\subset F_g$ and then thicken these annuli to mutually disjoint solid tori $S_1^1\times[0,1]^2,\dots, S_g^1\times[0,1]^2$ along the collar of F_g in M. Identify S_i^1 with the boundary ∂D_i of a 2-cell D_i, $i=1,\dots,g$. Let $W=M\times[0,1]\cup(\cup_{i=1}^g D_i\times[0,1]^2)\times 1$ be the adjunction 4-manifold. $\theta(\alpha_i,\alpha_i)=0$ implies that the framing of $S_i^1\times[0,1]^2$ is the null-homologous framing. W is homeomorphic to the disjoint union $M\cup S^1\times S^2\# N$, where $H_1(N;Z)\approx\oplus Z^g$, since $\theta(\alpha_i,\alpha_j)=0$ for all i,j. Now consider the following exact sequence $H_3(\tilde W,\partial\tilde W)\to H_2(\partial\tilde W)\to H_2(\tilde W)\xrightarrow{i_*} H_2(\tilde W,\partial\tilde W)$, where the cover $\tilde W$ is associated with the epimorphism $\bar\gamma:\pi_1(W)\to\langle t\rangle$ extending $\gamma:\pi_1(M)\to\langle t\rangle$. Notice that the epimorphism $\gamma':\pi_1(S^1\times S^2\# N)\to\langle t\rangle$ induced from $\bar\gamma$ is in fact defined by the projection of $\pi_1(S^1\times S^2\# N)=\pi_1(S^1\times S^2)*\pi_1(N)$ to the factor $\pi_1(S^1\times S^2)=\langle t\rangle$. Since $H_1(W;Z)\approx Z$, we obtain $\beta_1^{\bar{\gamma}}(W) = \beta_3^{\bar{\gamma}}(W,\partial W)=0$. Also, we have $\beta_2^{\bar{\gamma}}(\partial W)= g$ and $\beta_2^{\bar{\gamma}}(W)\leq g$. [Note that $H_2(W;Z) = \oplus Z^g$.] The above sequence, then, implies Image $j_* \subset T_2(\tilde{W},\partial\tilde{W})$. Hence the sequence $T_2(\tilde{W},\partial\tilde{W}) \xrightarrow{\partial} T_1(\partial\tilde{W}) \xrightarrow{i_*} T_1(\tilde{W})$ is exact. This completes the proof of Theorem 5.

References

1.D.B.A.Epstein: Embedding punctured manifolds,Proc.A.M.S. 16 (1965),175-176.

2.R.H.Fox and J.W.Milnor:Singularities of 2-spheres in 4-space and cobordism of knots,Osaka J. Math. 3(1966) , 257-267.

3.W.Hantzsch:Einlagerung von Mannigfaltigkeiten in euklidische Räume, Math. Z. 43(1938), 38-58.

4.A.Kawauchi: Three dimensional homology handles and circles, Osaka J. Math. 12(1975),565-581.

5.________: $\tilde{H}$-cobordism, part I, Osaka J. Math.(to appear).

6.________: On isometric structures of 3-manifolds,Report on a meeting at R.I.M.S. Kyoto Univ., Jan. 1976.

7.________: On quadratic forms of 3-manifolds,(to appear)

8.________ and Y.Nakagawa:On the Alexander polynomials of cobordant links,(to appear)

9.S.Kinoshita:On elementary ideals of polyhedra in the 3-sphere, Pacific J. Math. 42(1972), 89-98.

10.________: On elementary ideals of θ-curves in the 3-sphere and 2-links in the 4-sphere,Pacific J. Math. 49 (1973),127-134.

11.W.Magnus and A.Karrass and D.Solitar:Combinatorial Group Theory, Interscience, 1966.

12.Y.Matsumoto:A 4-manifold which admits no spine,Bull.A.M.S. 81(1975) 467-470.

13.N.Smythe:Boundary links,Topology Seminar Wisconsin,1966.

14.S.Suzuki:On linear graphs in 3-sphere,Osaka J.Math.7(1970).