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8

A COMPOSITE OBJECTIVE FUNCTION FOR THE MULTI-OBJECTIVE PROGRAMMINGS

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9

Abstract

In the multi-objective mathematical programming problems, we usually need to consider compromise of several objective functions because they are often conflicting each other. While the concept of compromise is defined by several ways, the way of compromise due to J.F. Nash in the bargaining games seems to be interesting and introduced to the multi-objective programming theory in this paper. For this purpose we construct one composite objective function composed of several different objective functions, in which all objectives are treated impartially. Our result shows that the Nash concept seems to have a wide applicability beyond the current scope of the bargaining game theory.

1. INTRODUCTION

In the multi-objective mathematical programming problems, it is required that several different objective functions are simultaneously optimized in some sense. But as the objectives are often conflicting each other, we usually need to consider compromise of them in order to get an optimum solution. This concept of compromise is defined by several different ways in the literature: By giving a hierarchy to the objectives, by methods of utility measurement or by methods of heuristic analyses, etc.

On the other hand, the concept of compromise has been extensively studied in game theory. Especially the way of compromise due to J.F. Nash [5] in the bargaining games seems to be most interesting and basic in the multi-objective programming problems. Motivated by this we introduce the Nash-type solution to the multi-objective programming theory. For this purpose we will construct one composite objective function which is composed of several different objective functions. In the composition, in which our way of compromise is reflected, we restrict ourselves to the case in which all objectives are treated impartially, viz., we will construct a well balanced composite objective function. This case seems to be fundamental to the multi-objective programming theory. Then the compromise solution will be obtained as a maximum solution of the composite objective function. Consequently we have a utility function integrating several objective functions, but our approach is quite different from the methods of utility measurement mentioned above.

Let us consider a multi-objective mathematical programming problem. Let K be a nonempty compact and convex set in euclidean space R^m of appropriate dimensions m, which means the constraint for the problem and is usually given by a system of linear or nonlinear inequalities. An element of K is said to be an alternative. There are n different objective functions f_i (i = 1,..,n), each of which is continuous and concave on K. Here we assume $n \geq 2$. A value of an objective function is said to be an outcome. We want to maximize these objective functions on K. We denote $f = (f_1,...,f_n)$. Any order relation for vectors is considered to be coordinate-wise. Each $f_i(x)$ is measured with each unit of measurement. These units of measurement are various and need not to have a common unit of measurement. Even when there exists a common unit of measurement for some objective functions $f_i(x)$, the sum of them may not be used by some reason. Indeed if the sum were possible to use, we could reduce the number of objective functions originally. Thus we may say that we consider the case in which no direct comparison of the units of measurement is admitted.

When K and f are given, it is natural to assume that the decision maker sets a floor $p \in R^n$, below which the outcome of final alternatives should not be fallen. The floor is considered as a minimal requirement. It is not so difficult to give the floor because we usually put it as a status-quo point. Then we assume that p is given a priori and that there is an $x \in K$ such that $f_i(x) \geq p_i$ for all $i = 1,\ldots,n$. Here if the equality always holds for some j, then f_j becomes unnecessary and may be removed from the first. Then we assume without loss of generality that:

$$\text{There is an } x \in K \text{ such that } f(x) > p. \qquad (1.1)$$

As we consider an optimum solution which balances the several objectives impartially, it may be reasonable to restrict K to the following set of alternatives: $K_p = \{ x \in K \mid f(x) > p \}$.

We also assume that the decision maker sets a goal $g \in R^n$, which indicates an ideal outcome of the programing problem. Whereas a floor is determined rather easily, it may be difficult to set a goal a priori. Because the realization of the goal belongs to future events and the determination of it directly influences optimum solutions. Then we only assume an appropriate region to which a goal belongs and we will investigate an optimum solution which is invariant under the selection of a goal.

Let m_i be the maximum value of $f_i(x)$ on K_p, and we call $m = (m_1,\ldots,m_n)$ an ideal point. This is often adopted as a candidate for a goal in the literature, though it is usually infeasible. Let E(K,f,p) be the set of efficient points of K_p, i.e., the set of alternatives $x \in K_p$ for which there is no $y \in K_p$ with $f(y) \geq f(x)$ and $f(y) \neq f(x)$. An efficient point is naturally considered as a candidate for a goal. Let us define a goal set by

$$G(K,f,p) = \{ g \in R^n \mid f(x) \leq g \leq m \text{ for some } x \in E(K,f,p) \} .$$

In fact we can use any goal set G(K,f,p) with the property that $f(x) \in G(K,f,p)$, if $x \in E(K,f,p)$. In this case, the results in this paper remain true.

We will measure a degree of accomplishment of $f_i(x)$, $x \in K_p$, by $d_g(x) \in R^n$, where

$$d_{g_i}(x) = \frac{f_i(x) - p_i}{g_i - p_i}, \quad i = 1,..,n. \qquad (1.2)$$

We say that a quardruple (K,f,p,g) is a problem, if (1.1) is satisfied. For a given problem A = (K,f,p,g), x and y in K_p are said to be equivalent if f(x) = f(y). This equivalence relation partitions K_p into the equivalence sets, each of which is denoted by $[x]_A$.

2. THE COMPOSITION

We now would like to construct a composite objective function which gives a unique optimum solution $[x^*]_A$ to a given problem A. Let us define a composition rule G as a mapping which carries a problem A to a real-valued function $G_A(x)$ defined on $x \in K_p$. This function is called a composite objective function. If once the composition rule G is determined, we seek to maximize $G_A(x)$ subject to $x \in K_p$ for a given problem A = (K,f,p,g). If the maximum is attained at an $[x^*]_A$, we call it a solution for A with respect to G_A.

In order to have our desired composition rule, we impose some reasonable conditions which the rule should satisfy.

The first condition guarantees that our composition rule must solve any problem in such a manner that the outcome of optimum alternatives is uniquely determined.

Condition I (Existence and Uniqueness): For each problem $A = (K,f,p,g)$, a solution $[x^*]_A$ always exists and unique.

The second condition states that our composite objective function must respond positively to an increase of an outcome.

Condition II (Monotonicity): Let a problem $A = (K,f,p,g)$ be given. If, for x and y in K_p, f(x) and f(y) satisfy $f(x) \geq f(y)$ and $f(x) \neq f(y)$, then $G_A(x) > G_A(y)$.

As we mentioned earlier that we would seek to get an optimum solution which was invariant under the selection of goals, we impose the following condition.

Condition III (Invariance under the selection of goals): If $A = (K,f,p,g)$ and $B = (K,f,p,h)$ are two problems such that g and h belong to $G(K,f,p)$, then, for x and y in K_p, $G_A(x) > G_A(y)$ if and only if $G_B(x) > G_B(y)$.

By condition IV, we want to represent that all objectives are treated impartially. Let π be a permutation of $(1,\dots,n)$ and let d denote a vector those i-th component is d_{π_i}. For a problem $A = (K,f,p,g)$, we put

$$D_A = \left\{ d \in R^n \,\middle|\, d = d_g(x) \text{ for some } x \in K_p \right\} .$$

D_A is said to be symmetric, if $d_\pi \in D_A$ for any $d \in D_A$ and any permutation π. When the decision maker faces a given problem with symmetric D_A, he may naturally evaluate that d and d_π are the same, if he treats the objectives impartially. Then for d and d_π in D_A, if $d = d_g(x)$ and $d_\pi = d_g(y)$, $G_A(x) = G_A(y)$. The decision maker is also considered to pursue a well balanced evaluation of the objectives. Then if there is a $z \in K_p$ for which $d_g(z)$ is a convex combination of d and d_π, it is natural to assume that $G_A(z) \geqq G_A(x)$ ($= G_A(y)$).

Condition IV (Impartiality): Let $A = (K,f,p,g)$ be a problem with symmetric D_A and let π be any permutation of $(1,\dots,n)$. For d and d_π in D_A,

$$G_A(x) = G_A(y),$$

where $d = d_g(x)$ and $d_\pi = d_g(y)$, and

16

$$G_A(z) \geqq G_A(x) \quad (= G_A(y)),$$

where $d_g(z) = \alpha d_g(x) + (1-\alpha)d_g(y) \qquad (0<\alpha<1).$

Let us consider two different problems A and B, and let us suppose that we get the identical graphic representation, if we draw the outcomes, the floors and the goals of each problem. Then we naturally consider that the outcomes of the solutions for two problems are the same. I.e., if $f \in R^n$ gives the outcome of the solution for A, then it also gives the outcome of the solution for B.

By generalizing this consideration slightly, we give the final condition. We define an inclusive copy of a problem. Let A = (K,f,p,g) and B = (K',f',p',g') be two problems. If p' = p and g' = g, and if there exists a mapping θ from K_p into $K'_{p'}$, such that

$$f(x) = f'(\theta(x)) \qquad \text{for all } x \in K_p,$$

then B is said to be an inclusive copy of A.

<u>Condition V (Independence of inclusive copies)</u>: Let A = (K,f,p,g) and B = (K',f',p',g') be two problems, and let B be an inclusive copy of A. Then $G_A(x) = G_B(\theta(x))$ for all $x \in K_p$.

Now we will show that a composition rule satisfying the conditions I, II, III, IV and V exists and unique in the following sense. Let G and G' be two composition rules. They are said to be equivalent if they give the same solution to each problem (K,f,p,g). We will show that all composition rules satisfying the conditions I, II, III, IV and V are equivalent.

Theorem 2.1. Let H be a composition rule which carries a problem A = (K,f,p,g) to the following function

$$H_A(x) = \sum_{i=1}^{n} \log \frac{f_i(x) - p_i}{g_i - p_i}$$

$$= \sum_{i=1}^{n} \log d_{g_i}(x), \qquad (2.1)$$

where $x \in K_p$. Then H satisfies the conditions I, II, III, IV and V.

Proof. If we note that $H(u) = \sum_{i=1}^{n} \log u_i$ is continuous and strictly concave on $u > 0$, then the assertion immediately follows. Q.E.D.

<u>Theorem 2.2.</u> Let G be a composition rule satisfying the conditions I, II, III, IV and V. Then G is equivalent to H.

To prove the theorem, we use three lemmas. The first lemma is an immediate consequence of condition II.

<u>Lemma 2.3.</u> Let us assume condition II and let a problem $A = (K,f,p,g)$ be given. If $d_g(x)$ and $d_g(y)$ satisfy $d_g(x) \geqq d_g(y)$ and $d_g(x) \neq d_g(y)$, then $G_A(x) > G_A(y)$.

<u>Lemma 2.4.</u> Let us assume conditions I, II, and IV. Let $A = (K,f,p,g)$ be a problem with symmetric D_A such that

$$D_A = \left\{ d \in R^n \;\middle|\; d > 0, \; \sum_{i=1}^{n} d_i \leqq n \right\}.$$

Suppose that an $\hat{x} \in K_p$ satisfies $d_{g_i}(\hat{x}) = 1$ for $i = 1,\ldots,n$. Then $[\hat{x}]_A$ is the solution for A.

<u>Proof.</u> By condition I, let $[x^*]_A$ be the solution for A and let $x^* \in [x^*]_A$. We prove that all $d_{g_i}(x^*)$ are identical with 1. Suppose without loss of generality that $d_{g_1}(x^*) \neq d_{g_2}(x^*)$. Let π be a permutation such that

$\pi_1 = 2$, $\pi_2 = 1$ and $\pi_i = i$ for all $i \neq 1, 2$. Let $d^* = d_g(x^*)$. Since D_A is symmetric, $d^*_\pi \in D_A$, where $d^*_\pi = d_g(y)$ for $y \in K_p$. If we put $b = (1/2)d^* + (1/2)d^*_\pi$, then $b \in D_A$. Let $b = d_g(z)$ for $z \in K_p$. By condition IV, we have $G_A(x^*) = G_A(y)$ and $G_A(z) \geqq G_A(x^*)$. By condition I, we have $z \in [x^*]_A$. Then $(1/2)d^*_1 + (1/2)d^*_2 = d^*_1$, so $d^*_1 = d^*_2$. This is a contradiction. Then all $d_{g_i}(x^*)$ are identical. From lemma 2.2, $\sum_{i=1}^{n} d_{g_i}(x^*) = n$, so $d_{g_i}(x^*) = 1$ for all i. Then $x^* \in [\hat{x}]_A$ or $[x^*]_A = [\hat{x}]_A$. Q.E.D.

__Lemma 2.5.__ Let us assume conditions I, II, IV and V. Let $A = (K,f,p,g)$ be a problem such that

$$g = f(\hat{x}) \quad \text{for some } \hat{x} \in E(K,f,p)$$

and

$$\sum_{i=1}^{n} d_{g_i}(x) \leqq n \quad \text{for all } x \in K_p. \qquad (2.2)$$

Then $[\hat{x}]_A$ is the solution for A.

__Proof.__ We construct an inclusive copy of A. Let $m \geqq n$ and let K' be an n-dimensional simplex with the vertecies $x^0, x^1, \ldots, x^n$. We define for each $k = 1, \ldots, n$,

$$f'_k(x^k) = ng_k + (1-n)p_k$$

$$f'_i(x^k) = p_i \qquad \text{for all } i \neq k,$$

and $$f'_i(x^0) = p_i \qquad \text{for all } i.$$

For $x \in K'$, we have a unique expression

$$x = \sum_{k=0}^{n} x(k)x^k,$$

where $x(k)$ is a weight of x^k ($k = 0,\ldots,n$). Then we define

$$f'_i(x) = \sum_{k=0}^{n} x(k)\, f'_i(x^k) \qquad \text{for } i = 1,\ldots,n. \tag{2.3}$$

They are affine functions on K'. We put $p' = p$ and $g' = g$. Here we must prove $g' \in G(K',f',p')$, but for the convenience we give the proof later. We define a required mapping from θ from K_p into $K'_{p'}$. Let $x \in K_p$. From (2.2), we have a unique expression

$$d_g(x) = \sum_{k=0}^{n} \lambda_k(ne^k), \tag{2.4}$$

where $e^0 = 0$ and e^k is a unit vector those k-th component is 1. We note that $d_{g_i}(\hat{x}) = 1$ for $i = 1,\ldots,n$. Then we define

$$\theta(x) = \sum_{k=0}^{n} \lambda_k x^k.$$

Clearly $\theta(x) \in K'$. We have

$$f'_i(\theta(x)) = f'_i\Big(\sum_{k=0}^{n} \lambda_k x^k\Big)$$

$$= \sum_{k=0}^{n} \lambda_k \, f_i'(x^k)$$

$$= n\lambda_i \, g_i \; + \; (1-n\lambda_i)p_i .$$

From (2.4), $d_{g_i}(x) = n\lambda_i$ or $f_i(x) = n\lambda_i g_i + (1-n\lambda_i)p_i$, which implies

$$f_i(\theta(x)) = f_i(x) \qquad (i = 1,\ldots,n),$$

and so $\theta(x) \in K'_{p'}$. Thus $B = (K',f',p',g')$ is an inclusive copy of A, if $g' \in G(K',f',p')$ is known. From (2.3), we get for $x \in K'_{p'}$,

$$d_{g_i'}(x) = nx(i) \qquad (i = 1,\ldots,n), \qquad (2.5)$$

which gives $D_B = \left\{ d \in R^n \;\middle|\; d > 0, \; \sum_{i=1}^{n} d_i \leq n \right\}$. Let $z = \sum_{K=1}^{n}(1/n)x^k$.

Then $\theta(\hat{x}) = z$ by (2.4) and $z \in K'_{p'}$. From (2.5), $d_{g'_i}(z) = 1$ for all i.

Then g' ($= g$) $= f'(z)$, which gives $g' \in G(K',f',p')$, since $z \in E(K',f',p')$ is easily known. Hence, by applying lemma 2.4 to B, z belongs to the solution for B. Then $G_B(z) \geqq G_B(\theta(x))$ for all $x \in K_p$. Then, by condition V, we have $G_A(\hat{x}) \geqq G_A(x)$ for all $x \in K_p$. Hence, by condition I, $[\hat{x}]_A$ is the solution for A. Q.E.D.

22

Proof of theorem 2.2. Let $A = (K,f,p,g)$ be an arbitrary problem and let $[x^*]_A$ be the solution for A with respect to G by condition I. By theorem 2.1, $H_A(x)$ attains its maximum at a unique $[\hat{x}]_A$ subject to $x \in K_p$. Let $\hat{x} \in [\hat{x}]_A$ and let us change a goal from g to h, where $h = f(\hat{x})$. Here note that $\hat{x} \in E(K,f,p)$ and $h \in G(K,f,p)$. Then we have a problem $B = (K,f,p,h)$. By theorem 2.1, $H_B(x)$ attains its maximum at $\hat{x}$ subject to $x \in K_p$, and $d_h(\hat{x}) = e$, where $e_i = 1$ ($i = 1,\dots,n$). We show that

$$\sum_{i=1}^{n} d_{h_i}(x) \leq n \qquad \text{for all } x \in K_p. \tag{2.6}$$

Suppose then that there is an $\bar{x} \in K_p$ such that

$$\sum_{i=1}^{n} d_{h_i}(\bar{x}) > n. \tag{2.7}$$

Let $H(u) = \sum_{i=1}^{n} \log u_i$. Then $H_B(x) = H(d_h(x))$. If we put $z = (1-\varepsilon)\hat{x} + \varepsilon\bar{x}$ with $0 < \varepsilon < 1$, then $z \in K_p$. By the concavity of d_h, we have

$$\begin{aligned} d_h(z) &\geq (1-\varepsilon)d_h(\hat{x}) + \varepsilon d_h(\bar{x}) \\ &= (1-\varepsilon)e + \varepsilon d_h(\bar{x}) \\ &> 0. \end{aligned}$$

By the monotonicity and the differentiability of $H(u)$, we have

$$\begin{aligned} H(d_h(z)) &\geq H(\,(1-\varepsilon)e + \varepsilon d_h(\bar{x})) \\ &= H(\,e + \varepsilon(d_h(\bar{x}) - e)\,) \\ &= H(e) + \nabla H(e)\{\varepsilon(d_h(\bar{x}) - e)\} \\ &\qquad + a(\varepsilon)\|\varepsilon(d_h(\bar{x}) - e)\| \end{aligned}$$

$$(\lim a(\varepsilon) = 0 \quad \text{as } \varepsilon \to 0)$$

$$= H(e) + \varepsilon \left\{ \left(\sum_{i=1}^{n} d_{h_i}(\bar{x}) - n \right) + a(\varepsilon) \left\| d_h(\bar{x}) - e \right\| \right\}$$

From (2.7),

$$H(d_h(z)) > H(e) = H(d_h(\hat{x}))$$

for a sufficiently small ε. Then $H_B(z) > H_B(\hat{x})$, which is impossible. Then we have (2.6).

Hence it follows from lemma 2.5 that $[\hat{x}]_B$ is the solution for B. Then $G_B(\hat{x}) \geqq G_B(x)$ for all $x \in K_p$. Then, by condition III, we have $G_A(\hat{x}) \geqq G_A(x)$ for all $x \in K_p$. This implies that $[x^*]_A = [\hat{x}]_A$ and the theorem is demonstrated.

Theorem 2.1 and theorem 2.2 show that if a problem A is given, our desired composite objective function is $H_A(x)$. It is easy to see that $H_A(x) > H_A(y)$ if and only if $N_A(x) > N_A(y)$, where

$$N_A(x) = \sum_{i=1}^{n} \log \left\{ f_i(x) - p_i \right\}. \qquad (2.8)$$

We know that if we admit the conditions I, II, III, IV and V for a composition rule G, the maximum solution for a problem A with respect to G is always given by maximizing $N_A(x)$. But this does not necessarily imply that G_A and N_A generate the same ordering on K_p, i.e., for x and y in K_p, $G_A(x) > G_A(y)$ if and only if $N_A(x) > N_A(y)$. Thus we need to consider a relative ranking on K_p, there may exist an eligible composite objective function which is different from N_A or its appropriate transformation. But judging from a mathematical regularity, N_A seems to be prominent for practical purposes.

Then if a multi-objective programming problem with K, f and p is given, we seek to maximize

$$\sum_{i=1}^{n} \log \left\{ f_i(x) - p_i \right\}$$

$$\text{subject to} \qquad x \in K \qquad \text{and} \qquad f_i(x) > p_i \quad \text{for } i = 1,\ldots,n. \tag{2.9}$$

This concave programming problem gives a unique solution $[x^*]_A$. If one of f_i is strictly concave, it is easily known that $[x^*]_A$ consists of a unique alternative. As was mentioned, each $f_i(x)$ was measured with its own unit of measurement. Even if we change the scale of unit to another, our compromise solution should be invariant. This is easily known by observing that any linear transformation

$$L(f) = f'$$

$$f' = \alpha_i f_i \ , \quad \alpha_i > 0 \quad (i = 1,\ldots,n)$$

keeps the solution of (2.9) invariant.

3. CONCLUDING REMARKS

We have shown that the Nash concept of fairness is directly applicable to the multi-objective programming theory by deriving a composite objective function of several objectives. Recently J. Bonnardeaux, J. Dolait and J.S. Dyer [2] nicely applied the Nash bargaining game to an actual problem. The Nash concept seems to have a wide applicability beyond the current scope of the bargaining game theory.

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