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Stochastically stable diffeomorphisms and Takens conjecture

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§ Introduction.

Let $\varphi : M \longrightarrow M$ be a homeomorphism of a metric space (M, d) with distance function d. A (double) sequence $\{x_i\}_{i \in Z}$ of points $x_i \in M$ $(i \in Z)$ is called, by definition, a δ-pseudo-orbit of φ iff

$$d(\varphi(x_i), x_{i+1}) \leq \delta$$

for every $i \in Z$, where $\delta > 0$ is a constant (cf. [2]). Given $\varepsilon > 0$, a δ-pseudo-orbit $\{x_i\}$ is called to be ε-traced by a point $y \in M$ iff

$$d(\varphi^i(y), x_i) \leq \varepsilon$$

for every $i \in Z$. We shall call φ stochastically stable, iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit of φ can be ε-traced by some point $y \in M$.

R. Bowen [2] proves that every Anosov diffeomorphism φ of a compact manifold is stochastically stable.

In this note we shall first prove that every topologically stable homeomorphism φ (cf. Def. 1) of a compact manifold (or euclidean space) M is stochastically stable in case $\dim M \geq 3$ (Th. 1). Using this result we give a positive answer to the conjecture of F. Takens in tolerance stability [9] (Th. 2). By virtue of these results it seems to be significant to give necessary and/or sufficient conditions for diffeomorphisms to be stochastically stable and to clarify the relations with other stabilities of diffeomorphisms.

We shall in fact characterize linear automorphisms of R^n (resp. group automorphisms of a torus T^n) to be stochastically stable (Th. 3 and 4). Moreover, we shall see that every isometry of a compact connected Riemannian manifold M ($\dim M \geq 1$) is not stochastically stable.

We shall further show a result due to H. Urakawa which says that if there is a stochastically stable group automorphism φ of a compact connected Lie group G, then G is necessarily a torus.

§1. Definitions and preparatory lemmas.

Let $\varphi : M \longrightarrow M$ be a homeomorphism of a metric space (M, d). We denote by $H(M)$ the group of all homeomorphisms of M.

Definition 1. We call φ topologically stable iff for any $\varepsilon > 0$ there exists $\delta > 0$ with the property that for any $\psi \in H(M)$ with $d(\varphi(x), \psi(x)) < \delta$ for every $x \in M$ there is a continuous map $h : M \longrightarrow M$ such that

$$\text{i)} \quad h \circ \psi = \varphi \circ h,$$

$$\text{ii)} \quad d(h(x), x) < \varepsilon \quad \text{for every } x \in M.$$

Definition 2. A sequence of points $\{x_i\}_{i \in (a,b)}$ $(-\infty \leq a < b \leq +\infty)$ is called a δ-pseudo-orbit of φ iff

$$d(\varphi(x_i), x_{i+1}) \leq \delta$$

for $i \in (a+1, b-2)$. If $a > -\infty$ and $b < \infty$, this sequence will be called a finite δ-pseudo-orbit of φ and if $a = -\infty$ and $b = +\infty$, the sequence will be (sometimes) called an (infinite) δ-pseudo-orbit of φ. $\{x_i\}$ is called to be ε-traced by $x \in M$ iff

$$d(\varphi^i(x), x_i) \leq \varepsilon$$

holds for $i \in (a, b)$.

φ is called stochastically stable iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that any (infinite) δ-pseudo-orbit of φ can be ε-traced by some point $x \in M$. We shall call such φ also a Bowen homeomorphism.

Definition 3. We denote by $\mathrm{Orb}^{\delta}(\varphi)$ the set of all (finite or infinite) δ-pseudo-orbit of φ and $\mathrm{Tr}^{\varepsilon}(\{x_i\}, \varphi) = \mathrm{Tr}(\{x_i\})$ the set of all $y \in M$ such that $\{x_i\}$ is ε-traced by y.

Assumption. In the sequel we assume that every bounded subset of M is relatively compact unless otherwise stated.

We shall now state several lemmas, some of whose proofs will be omitted, since the proofs will be more or less standard.

Lemma 1. Let $h \in H(M)$ be a homeomorphism of M such that h and h^{-1} are both uniformly continuous. Take $\varphi \in H(M)$ and set $\psi = h \circ \varphi \circ h^{-1}$. Then φ is a Bowen homeomorphism if and only if ψ is.

Lemma 2. Let $\varphi \in H(M)$ be stochastically stable. Then, for any integer $k > 0$, φ^k is also stochastically stable.

Lemma 3. Let $\varphi \in H(M)$ be uniformly continuous, and fix an integer $N > 0$. Then for any $\varepsilon > 0$, there is $\delta > 0$ such that if $\{x_i\}_{i=0}^{N} \in \mathrm{Orb}^{\delta}(\varphi)$ then x_0 ε-traces $\{x_i\}_{i=0}^{N}$.

Lemma 4. Let $\varphi \in H(M)$ be uniformly continuous. If φ is a Bowen homeomorphism, then φ^{-1} is.

Lemma 5. Let $\varphi \in H(M)$ be uniformly continuous. If φ^k is a Bowen homeomorphism for some integer $k > 0$, then φ is.

Lemma 6. Let $\varphi \in H(M)$, and $\psi \in H(M')$. The direct product $M \times M'$ is a metric space by the distance function $d((x, y), (x', y')) = \operatorname{Max}\{d(x, x'), d(y, y')\}$ for $x, x' \in M$ and $y, y' \in M'$. Then $\varphi \times \psi$ is a Bowen homeomorphism if and only if φ and ψ are both Bowen.

Lemma 7. Let $\varphi \in H(M)$ and assume that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any integer $k > 0$ and any $\{x_i\}_{i=0}^{k} \in \operatorname{Orb}^{\delta}(\varphi)$ we have $\operatorname{Tr}^{\varepsilon}(\{x_i\}_0^k, \varphi) \neq \emptyset$. Then φ is a Bowen homeomorphism.

Lemma 8. Let φ be a Bowen diffeomorphism of a compact Riemannian manifold M. Then φ is Bowen with respect to any Riemannian metric on M.

Lemma 9. Let M be a differentiable manifold of $\dim M \geq 3$. Let $X_i = \{p_i, q_i\}$ ($i = 1,\ldots,k$) be a subset of M consisting of at most two points p_i and q_i with $d(p_i, q_i) < \delta$. Suppose $X_i \cap X_j = \emptyset$ for $i \neq j$. Then there is a diffeomorphism $\eta : M \longrightarrow M$ such that $d(\eta(x), x) < \delta$ for $x \in M$ and that $\eta(p_i) = q_i$ for $i = 1,2,\ldots, k$.

Lemma 10. Let $\varphi : M \longrightarrow M$ be a homeomorphism of a manifold M with $\dim M \geq 1$ and suppose $M - \operatorname{Fix}(\varphi)$ is dense in M. Take and fix a constant $\delta_1 > 0$ and an integer $k > 0$. Then for any $\{x_i\} \in \operatorname{Orb}^{\delta_1}(\varphi)$ and $\varepsilon_1 > 0$, there is $\{x_i'\} \in \operatorname{Orb}^{3\delta_1}(\varphi)$ such that i) $d(x_i, x_i') < \varepsilon_1$ for $i = 0,1,\ldots, k$ and ii) $X_i = \{\varphi(x_i'), x_{i+1}'\}$ ($i = 0,1,\ldots, k-1$) are disjoint.

Proof. We can assume $\varepsilon_1 < \delta_1$. For this ε_1, there is $\varepsilon_1' > 0$ such that $\varepsilon_1 > \varepsilon_1'$ and that $d(x, y) < \varepsilon_1'$ implies $d(\varphi(x), \varphi(y)) < \varepsilon_1$.

First, we can find $x_i' \in M$ ($i = 0,1,\ldots, k$) such that $x_i' \neq x_j'$ ($i \neq j$)

and that $d(x_i, x_i') < \varepsilon_1'$ $(i = 0,1,\ldots, k)$. Next, we shall show by induction that $X_0,\ldots,X_{k-1}$ are disjoint by taking x_i' suitably. For that, suppose $X_i = \{\varphi(x_i'), x_{i+1}'\}$ $(i = 0,1,\ldots, k-2)$ are disjoint. We shall show that, by changing x_{k-1}' and x_k', if necessary, X_i $(i = 0,1,\ldots, k-1)$ are disjoint.

Consider the point $\varphi(x_{k-1}')$ and suppose $\varphi(x_{k-1}') \in \bigcup_{i=0}^{k-2} X_i$. Then there is a unique $i \leq k - 1$ such that $\varphi(x_{k-1}') = x_i'$, since $x_{k-1}' \neq x_j'$ $(j \leq k-2)$ implies $\varphi(x_{k-1}') \neq \varphi(x_j')$. If $i \leq k - 2$, we can find x_{k-1}'' near x_{k-1}' such that $\varphi(x_{k-1}'') \neq x_i'$. If $i = k - 1$ i.e. $\varphi(x_{k-1}') = x_{k-1}'$, then we can find x_{k-1}'' near x_{k-1}' such that $\varphi(x_{k-1}'') \neq x_{k-1}''$, since $M - \mathrm{Fix}(\varphi)$ is dense and open in M. We denote x_{k-1}'' by x_{k-1}' again. Then we can assume that $x_k' \notin \bigcup_{i=0}^{k-2} X_i$, since $\bigcup_{i=0}^{k-2} X_i$ is a finite set.

Thus we have proved that $X_0, X_1,\ldots, X_{k-1}$ are disjoint.

For $i < 0$ (resp. $i > k$) we define $x_i' = \varphi^{-i}(x_0')$ (resp. $x_i' = \varphi^{i-k}(x_k')$). Then we see that $\{x_i'\} \in \mathrm{Orb}^{3\delta_1}(\varphi)$. For, we have

$$\begin{aligned} d(\varphi(x_i'), x_{i+1}') &\leq d(\varphi(x_i'), \varphi(x_i)) + d(\varphi(x_i), x_{i+1}) + d(x_{i+1}, x_{i+1}') \\ &< \varepsilon_1 + \delta_1 + \varepsilon_1' < 3\delta_1 \end{aligned}$$

for $i = 0,1,\ldots, k-1$. This completes the proof of Lemma 10.

Lemma 11. Let $\varphi \in H(M)$, where M is a differentiable manifold of dimension ≥ 1. Assume φ is topologically stable. Then for any integer $k > 0$, $M - \mathrm{Fix}(\varphi^k)$ is dense in M, where $\mathrm{Fix}(\varphi^k) = \{x \in M \mid \varphi^k(x) = x\}$.

Proof. Induction on k. First, we prove the lemma for $k = 1$.

To prove that $M - \mathrm{Fix}(\varphi)$ is dense in M, we assume that there is an open set $U \neq \emptyset$ such that $U \subset \mathrm{Fix}(\varphi)$. We can suppose that U is a coordinate neighborhood of a point $x_0 \in U$ with coordinate system

$(x_1, \dots, x_n)$. Take $\varepsilon_1 > 0$ such that $Q_{\varepsilon_1} \subset U$, where $Q_{\varepsilon_1} = Q_{\varepsilon_1}(x_0)$ means the cubic neighborhood with center x_0 and of breadth $2\varepsilon_1$. Take $\varepsilon > 0$ such that $4\varepsilon < \varepsilon_1$. For this $\varepsilon > 0$, we can find $\delta > 0$ with the property in Definition 1. Now, take a differentiable function α on M such that $\alpha(x) = 1$ for $x \in Q_{3\varepsilon}$, $\alpha(x) = 0$ for $x \notin Q_{4\varepsilon}$. Define a differentiable vector field Y on M by

$$Y(x) = \begin{cases} \delta_1 \cdot \alpha(x) \quad \left(\frac{\partial}{\partial x_1}\right)_x & x \in Q_{\varepsilon_1} \\ 0 & x \notin Q_{\varepsilon_1} \end{cases}$$

where $\delta_1 > 0$ is a constant. Let $\{\eta_t\}$ be the one-parameter group of diffeomorphisms η_t of M generated by Y and put $\eta = \eta_1$. It is clear that if $\delta_1 < \delta$ is sufficiently small, then we have $d(\eta(x), x) < \delta$ for $x \in M$. Set $\psi = \eta \circ \varphi$, then we have $d(\varphi(x), \psi(x)) < \delta$ for $x \in M$ and hence there is a continuous map $h : M \longrightarrow M$ such that $h \circ \psi = \varphi \circ h$ and $d(h(x), x) < \varepsilon$ for $x \in M$. Since $\alpha = 1$ on $Q_{3\varepsilon}$, we see that there is a sufficiently large integer $k > 0$ such that $\psi^k(x_0) = \eta^k(x_0) \notin Q_{3\varepsilon}$ and hence $h(\psi^k(x_0)) \notin Q_{2\varepsilon}$. On the other hand, since $d(h(x), x) < \varepsilon$ we have $h(x_0) \in Q_\varepsilon \subset U \subset \mathrm{Fix}(\varphi)$, and so we have $h(\psi^k(x_0)) = \varphi^k(h(x_0)) = h(x_0) \in Q_\varepsilon$, which is a contradiction. Thus we have proved the lemma for $k = 1$.

Assume that $k \geq 2$ and that the Lemma is true for any $k' \leq k - 1$. Suppose that $M - \mathrm{Fix}(\varphi^k)$ is not dense. Then there will be a non-empty open set $U \subset \mathrm{Fix}(\varphi^k)$. Since $M - \mathrm{Fix}(\varphi^i)$ is dense in M for $i \leq k-1$ there exists $x_0 \in U$ such that $\varphi^i(x_0) \neq x_0$ for any $i \leq k - 1$. Hence we can assume that U is a coordinate neighborhood of x_0 with coordinate system $(x_1, \dots, x_n)$, $n = \dim M$ and that $\{\varphi^i(U)\}_{i=0}^{k-1}$ is disjoint. Take ε_1 such that $U \supset Q_{\varepsilon_1}(x_0)$, and take $\varepsilon > 0$ with $4\varepsilon < \varepsilon_1$. Since φ is topologically stable there exists a $\delta > 0$ with the property in

Definition 1. For this $\delta > 0$ we can find a diffeomorphism $\eta : M \longrightarrow M$ such that $\eta(U) = U$, $d(\eta(x), x) \leq \delta$ $(x \in M)$, $\eta(x) = x$ $(x \notin U)$ and that $\eta|_{Q_{4\varepsilon}}$ is a parallel translation along the x_1-axis as in the proof of the Lemma for $k = 1$. Define $g \in H(M)$ by

$$g(x) = \begin{cases} \varphi(x) & x \notin \varphi^{k-1}(U) \\ \eta \circ \varphi(x) & x \in \varphi^{k-1}(U). \end{cases}$$

Since $U = \varphi^k(U)$, g is in fact a homeomorphism of M and $d(g, \varphi) \leq \delta$ holds. Therefore, there is a continuous map $h : M \longrightarrow M$ such that

$$h \circ g = \varphi \circ h \quad \text{and} \quad d(h(x), x) < \varepsilon \quad (x \in M)$$

holds. We see easily that $g^k(x) = \eta(x)$ for $x \in U$. Hence we can find a sufficiently large integer $m > 0$ such that $g^{km}(x_0) = \eta^m(x_0) \notin Q_{3\varepsilon}$. On the other hand, we get $h \circ g^{km}(x_0) = \varphi^{km}(h(x_0)) = h(x_0) \in Q_\varepsilon$ since $h(x_0) \in U$ and $d(h(x_0), x_0) < \varepsilon$. Hence we have $g^{km}(x_0) \in Q_{2\varepsilon}$, which is a contradiction. This completes the proof of Lemma 11.

<u>Remark.</u> The author does not know whether the topological stability of φ implies that of φ^k for $k \neq 0$.

§2. Topological and stochastic stabilities, and Takens conjecture.

Theorem 1. Let M be a differentiable (metric) manifold of $\dim M \geq 3$ and assume that there exists $\varepsilon_0 > 0$ such that ε_0-neighborhood $U_{\varepsilon_0}(x)$ of any point $x \in M$ is relatively compact. Let $\varphi : M \longrightarrow M$ be a topologically stable homeomorphism of M. Then φ is stochastically stable. In particular, if M is compact or $M = R^n$ is the euclidean space then the topological stability implies the stochastic stability.

<u>Remark.</u> The author has a proof of Theorem 1 in case $M = S^1$ (the circle). However since the proof is quite different, he will treat it in a future paper.

Proof of Theorem 1. Since φ is topologically stable, for any $\varepsilon > 0$ there is $\delta > 0$ with the property in Definition 1. We can assume $\delta < \mathrm{Min}(\varepsilon, \varepsilon_0)$.

First, we shall prove, for any $\{x_i\} \in \mathrm{Orb}^{\delta/6\pi}(\varphi)$ and any integer $k > 0$, that $\mathrm{Tr}^{2\varepsilon}(\{x_i\}_0^k, \varphi) \neq \emptyset$.

By Lemma 10, 11 we can find $\{x_i'\} \in \mathrm{Orb}^{\delta/2\pi}(\varphi)$ such that $d(x_i, x_i') < \delta$ $(i = 0,\dots, k)$ and that the sets $\{\varphi(x_i'), x_{i+1}'\}$ are disjoint for $i = 0,1,\dots, k-1$. By Lemma 9, there is a $\eta \in H(M)$ such that $d(\eta(x), x) < \delta$ for $x \in M$ and $\eta(\varphi(x_i')) = x_{i+1}'$ for $i = 0,1,\dots, k-1$. Put $\psi = \eta \circ \varphi$, then $d(\varphi(x), \psi(x)) < \delta$. Hence by the property for $\delta > 0$, we can find a continuous map $h : M \longrightarrow M$ such that $h \circ \psi = \varphi \circ h$ and $d(h(x), x) < \varepsilon$ for $x \in M$. Put $y = h(x_0')$. Now we have for $i = 0, 1,\dots, k$,

$$d(\varphi^i(y), x_i) = d(\varphi^i(h(x_0')), x_i) = d(h(\psi^i(x_0')), x_i)$$
$$\leq d(h(x_i'), x_i') + d(x_i', x_i) < \varepsilon + \delta < 2\varepsilon,$$

which shows $y \in \mathrm{Tr}^{2\varepsilon}(\{x_i'\}_0^k, \varphi)$. Thus we have proved that φ satisfies the condition in Lemma 7, which concludes that φ is stochastically stable.

Now we shall recall the notion of extended orbits of a homeomorphism of a compact metric space (cf. [9]).

Let $\varphi : M \longrightarrow M$ be a homeomorphism of a compact metric space (M, d). The set of all non-empty closed subsets of M will be a compact metric space by the distance function $\bar{d}$ defined by

$$\bar{d}(A, B) = \mathrm{Max}\left\{ \max_{b \in B} d(A, b),\ \max_{a \in A} d(a, B) \right\}$$

for $A, B \in C(M)$, where $d(A, b) = \inf_{a \in A} d(a, b)$ (cf. [5]). We denote by $\widetilde{\mathrm{Orb}}^{\delta}(\varphi)$ the set of all $A \in C(M)$, for which there is $\{x_i\} \in \mathrm{Orb}^{\delta}(\varphi)$

such that $A = \mathrm{Cl}\{x_i \mid i \in Z\}$, Cl denoting the closure.

Definition 5. We denote by E_φ the set of all $A \in C(M)$ such that for any $\varepsilon > 0$ there is $A_\varepsilon \in \widetilde{\mathrm{Orb}}^\varepsilon(\varphi)$ with $\bar{d}(A, A_\varepsilon) < \varepsilon$. An element A of E_φ is called an <u>extended orbit</u> of φ.

On the other hand, we define $O_\varphi = \mathrm{Cl}\{O_\varphi(x) \mid x \in M\} \subset C(M)$, where $O_\varphi(x) = \mathrm{Cl}(\mathrm{Orb}_\varphi(x))$ with $\mathrm{Orb}_\varphi(x) = \{\varphi^i(x) \mid i \in Z\}$. We can easily see that E_φ is closed in $C(M)$ and $O_\varphi \subset E_\varphi$ holds for any $\varphi \in H(M)$.

Lemma 12. If $\varphi \in H(M)$ is stochastically stable, then $O_\varphi = E_\varphi$ holds.

Proof omitted.

Now, we shall give an affirmative answer to a conjecture by F. Takens[9].

Theorem 2 (Conjecture of Takens). Let φ be a C^1-diffeomorphism of a compact connected manifold M with $\dim M \geq 1$. Assume that φ is an AS-diffeomorphism, i.e., φ satisfies the Axiom A and the strong transversality condition. Then $O_\varphi = E_\varphi$ holds.

Proof. Consider the direct product $\varphi \times \varphi \times \varphi$, a diffeomorphism of $M \times M \times M$ onto itself. Since φ is an AS-diffeomorphism, we see that $\varphi \times \varphi \times \varphi$ is also AS. By a result of Nitecki[6], $\varphi \times \varphi \times \varphi$ is topologically stable. Hence by Theorem 1 $\varphi \times \varphi \times \varphi$ is a Bowen homeomorphism. Now Lemma 6 says that φ is also a Bowen homeomorphism and so by Lemma 12, $O_\varphi = E_\varphi$ holds.

Definition 4. $\varphi \in H(M)$ is called <u>expansive</u>, iff there exists $\varepsilon_0 > 0$ (called an expansiveness constant of φ) with the property that for any $x, y \in M$ with $x \neq y$, there is $n \in Z$ such that

$$d(\varphi^n(x), \varphi^n(y)) \geq \varepsilon_0.$$

The following Proposition is essentially proved in [2].

Proposition 1. Let M be a metric space such that every bounded set is relatively compact. Let $\varphi : M \longrightarrow M$ be a stochastically stable homeomorphism of M. If φ is expansive, then φ is topologically stable.

§3. Stochastic stability of linear and toral automorphisms.

In this section we shall characterize affine transformations of R^n and toral automorphisms of $T^n = R^n / Z^n$ to be stochastically stable.

Proposition 2. Let $\varphi : R^n \longrightarrow R^n$ be a linear automorphism of R^n. Then φ is stochastically stable if and only if φ is hyperbolic, i.e., if λ is an eigenvalue of φ then $|\lambda| \neq 1$.

Proof. Assume φ is stochastically stable. Consider the complexification $\varphi^C : C^n \longrightarrow C^n$. Identifying C^n with $R^n \times R^n$, we can identify φ^C with $\varphi \times \varphi$. By virtue of Lemma 6 and 8, φ is stochastically stable if and only if φ^C is. Since a linear map is uniformly continuous, it follows from Lemma 1 that φ^C is stochastically stable if and only if every factor of the Jordan canonical form of φ^C is stochastically stable. Now, it suffices to show that if $\psi = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} : C^n \longrightarrow C^n$ (resp. $\psi_0 = \lambda \cdot 1_C$) is stochastically stable, then $|\lambda| \neq 1$. Suppose $|\lambda| = 1$. Set $z_j = j \cdot \lambda^j \cdot \delta$ for $j \in Z$. Since

$$d(\psi_0(z_j), z_{j+1}) = |\psi_0(z_j) - z_{j+1}| = |j \cdot \lambda^{j+1} \delta - (j+1) \lambda^{j+1} \delta| = \delta,$$

we have $\{z_j\} \in \mathrm{Orb}^{\delta}(\psi_0)$. However, since

$$d(\psi_0^n(\xi), z_n) = |\lambda^n \xi - \lambda^n n \delta| = |\xi - n\delta|,$$

there is no ξ such that $d(\psi_0^n(\xi), z_n)$ is bounded for any small $\delta > 0$. In particular, for any $\delta > 0$ we have $Tr^1(\{z_j\}, \psi_0) = \emptyset$. Hence ψ_0 is not stochastically stable.

Similarly, consider the vector $v_j = (0, \ldots, 0, z_j)$ for $j \in Z$, then we see that $\{v_j\} \in \mathrm{Orb}^{\delta}(\psi)$ and that $\mathrm{Tr}^1(\{v_j\}, \psi) = \emptyset$, which means ψ is not stochastically stable. Thus we have proved that ψ is hyperbolic.

Conversely, assume that φ is hyperbolic. Then it is well known that there are subspaces E^s and E^u of R^n and constants $C > 0$, $0 < \lambda < 1$ such that

i) $R^n = E^s \oplus E^u$

ii) $\varphi(E^{\sigma}) = E^{\sigma}$, $\sigma = s, u$,

iii) $\| \varphi^n v \| \leq C \lambda^n \| v \| \qquad v \in E^s$

$\| \varphi^{-n} w \| \leq C \lambda^n \| w \| \qquad w \in E^u$

for $n \geq 0$. Set $\psi = \varphi|_{E^s}$ and $\eta = \varphi|_{E^u}$, then identifying R^n with $E^s \times E^u$ we can identify φ with $\psi \times \eta$. By virtue of Lemma 6, it suffices to show that ψ and η are stochastically stable.

First consider $\eta : E^u \to E^u$ and take $\varepsilon > 0$. Put $\delta = (1-\lambda)\varepsilon / C$ We assert that $\{x_i\} \in \mathrm{Orb}^{\delta}(\eta)$ implies $\mathrm{Tr}^{\varepsilon}(\{x_i\}, \eta) \neq \emptyset$. For $k \in Z$ we set $\alpha_k = x_{k+1} - \eta(x_k) \in E^u$. Then we have $\|\alpha_k\| \leq \delta$ for $k \in Z$. By induction we see that for $k > 0$

$$x_k = \eta^k(x_0) + \eta^{k-1}(\alpha_0) + \eta^{k-2}(\alpha_1) + \ldots + \alpha_{k-1}$$

holds. Put $\xi = \eta^{-1}$. Then we have $\| \xi^k \| \leq C \lambda^k$ for $k > 0$ (cf. iii)). We have also :

$$x_k = \eta^k(x_0 + \xi(\alpha_0) + \xi^2(\alpha_1) + \ldots + \xi^k(\alpha_{k-1})) = \eta^k(x_0 + v_k),$$

where we put $v_k = \xi(\alpha_0) + \xi^2(\alpha_1) + \ldots + \xi^k(\alpha_{k-1})$ for $k > 0$.

We shall show that $\{v_k\}_{k=1}^{\infty}$ is a Cauchy sequence. In fact, for any $p > k > 0$, we have

$$\| v_p - v_k \| = \left\| \sum_{i=k+1}^{p} \xi^i(\alpha_{i-1}) \right\| \leq \sum_{i=k+1}^{p} \| \varphi^{-i}(\alpha_{i-1}) \|$$

$$\leq C \sum_{i=k+1}^{p} \lambda^i \| \alpha_{i-1} \| \leq C \cdot \delta \cdot \lambda^{k+1} / (1-\lambda) \to 0 \ (k \to \infty).$$

Hence there is $\beta \in E^u$ such that $\lim_{k\to\infty} v_k = \beta$. Put $y = x_0 + \beta$. Then we have

$$\eta^k(y) - x_k = \eta^k(x_0 + \beta - x_0 - v_k)$$
$$= \eta^k\left(\sum_{i=k+1}^{\infty} \xi^i(\alpha_{i-1})\right) = \sum_{i=k+1}^{\infty} \xi^{i-k}(\alpha_{i-1}),$$

and hence we have

$$d(\eta^k(y), x_k) = \| \eta^k(y) - x_k \| \le \sum \| \xi^{i-k} \| \delta \le c\delta/(1-\lambda) = \varepsilon .$$

By Lemma 12, we see that η is stochastically stable. Similarly we conclude that ψ^{-1} is stochastically stable. By Lemma 4, ψ is also stochastically stable. Thus we have proved that φ is stochastically stable.

Theorem 3. Let $\varphi : R^n \longrightarrow R^n$ be a linear automorphism. Then the following conditions are equivalent :

1) φ is hyperbolic
2) φ is expansive
3) φ is structurally stable
4) φ is stochastically stable
5) φ is topologically stable.

Proof. Equivalence 1)$\longleftrightarrow$3) was proved by Hartman (see Theorem 2.3[7] for details).

1)$\longleftrightarrow$2) is standard.

1)$\longleftrightarrow$4) is by Proposition 2.

5)$\longrightarrow$4) for $n \geq 2$ is by Theorem 1. For $n = 1$, $\varphi : R^1 \longrightarrow R^1$ is given by $\varphi(x) = \lambda \cdot x$ for some $\lambda \neq 0$. If φ is topologically stable, then $\lambda \neq \pm 1$. For if $\lambda = \pm 1$, then $\varphi^2 = 1_{R^1}$ and $\mathrm{Fix}(\varphi^2) = R^1$, which contradicts Lemma 11. Thus $|\lambda| \neq 1$, which means φ is hyperbolic and so stochastically stable. Finally 4)$\longrightarrow$5), since 4)$\longrightarrow$2) and so we can apply Theorem 2. This completes the proof of Theorem 3.

Proposition 3. Let $f: R^n \longrightarrow R^n$ be a linear automorphism and $\xi \in R^n$ a fixed vector. Define the affine transformation $\varphi: R^n \longrightarrow R^n$ by

$$\varphi(x) = f(x) + \xi$$

for $x \in R^n$. Then φ is stochastically stable if and only if f is.

Proof. Let $\{x_i\} \in \mathrm{Orb}^{\delta}(\varphi)$. Put

$$x_i' = x_i - \left(f^{i-1}(\xi) + f^{i-2}(\xi) + \dots + \xi\right)$$

for $i \in Z$. We see that $\{x_i'\} \in \mathrm{Orb}^{\delta}(f)$. It is easy to verify that $\{x_i\} \rightarrow \{x_i'\}$ is a one-one correspondence between $\mathrm{Orb}^{\delta}(\varphi)$ and $\mathrm{Orb}^{\delta}(f)$ and that $\mathrm{Tr}^{\varepsilon}(\{x_i\}, \varphi) = \mathrm{Tr}^{\varepsilon}(\{x_i'\}, f)$ for every $\varepsilon > 0$. Thus φ is stochastically stable if and only if f is.

Proposition 4. Let M and $\widetilde{M}$ be metric spaces and $\pi : \widetilde{M} \longrightarrow M$ be a locally isometric covering map of $\widetilde{M}$ onto M. Assume that M is compact and that every ε-neighborhood $U_{\varepsilon}(x)$ of $x \in M$ is connected for small $\varepsilon > 0$. Let $f \in H(\widetilde{M})$ and $\varphi \in H(M)$ such that $\pi \circ f = \varphi \circ \pi$. Then, f is stochastically stable if and only if φ is.

Proof omitted.

Lemma 13. Let f be a linear automorphism and $\varphi : T^n \longrightarrow T^n$ be a group automorphism of T^n such that $\pi \circ f = \varphi \circ \pi$, where $\pi : R^n \longrightarrow T^n = R^n/Z^n$ is the projection. Then f is expansive if φ is.

Proof omitted.

Theorem 4. Let $\varphi : T^n \longrightarrow T^n$ be a group automorphism of the torus $T^n = R^n/Z^n$. Then the following conditions are equivalent :

1) φ is an Anosov diffeomorphism,

2) φ is expansive,

3) φ is structurally stable,

4) φ is stochastically stable,

5) φ is topologically stable,

6) φ satisfies Axiom A and the strong transversality condition.

Proof. 1) $\longrightarrow$ 5) is proved by Walters [10].

5) $\longrightarrow$ 4) is proved by Theorem 1 for case $n \geq 2$. In case $n = 1$, if $\varphi : T^1 \longrightarrow T^1$ is a group automorphism $\varphi^2 = 1_{T^1}$ and so φ is not topologically stable by Lemma 11.

To prove 4) $\longrightarrow$ 1), we denote by $f : R^n \longrightarrow R^n$ the linear automorphism covering φ, i.e., $\pi \circ f = \varphi \circ \pi$. Since φ is stochastically stable, f is also so by Proposition 4. Hence by Theorem 3, f is hyperbolic. Then φ is clearly an Anosov diffeomorphism.

1) $\longrightarrow$ 3) is proved by Anosov [1].

3) $\longrightarrow$ 1), since $T_0\varphi$ (the differential of φ at the neutral element 0 of T^n) is hyperbolic by a result of Franks [3], and hence φ is an Anosov diffeomorphism.

1) $\longrightarrow$ 2) is proved also by Anosov [1].

2) $\longrightarrow$ 1), since f is expansive by Lemma 13, and hence f is hyperbolic by Theorem 3 and so φ is Anosov.

1) $\longrightarrow$ 6) is verified by the very definition and a result of Anosov [1].

6) $\longrightarrow$ 3) is proved by Robbin [8].

This completes the proof of Theorem 4.

§4. Isometries of compact Riemannian manifolds.

In this section we shall prove that any isometry of a compact connected Riemannian manifold M with $\dim M \geq 1$ is not stochastically stable.

Lemma 14. Let M be a compact connected Riemannian manifold. Suppose $\varphi \in H(M)$ is an isometry of M. Then, $M \in E_\varphi$.

Proof omitted.

Theorem 5. Let $\varphi : M \longrightarrow M$ be an isometry of a compact connected Riemannian manifold M with $\dim M \geq 1$. Then φ is not stochastically stable.

Proof. Suppose φ is stochastically stable. Since M is compact, the non-wandering set $\Omega(\varphi)$ of φ is not empty. Take and fix a point $p_0 \in \Omega(\varphi)$. For $\varepsilon = \text{diameter}(M)/7$, there exists $\delta > 0$ such that $\varepsilon > \delta$ and that $\{x_i\} \in \text{Orb}^{\delta}(\varphi)$ implies $\text{Tr}^{\varepsilon}(\{x_i\}, \varphi) \neq \emptyset$. Put $U = U_{\delta/2}(p_0)$. Then, since $p_0 \in \Omega(\varphi)$, there is an integer $k > 0$ such that $\varphi^k(U) \cap U \neq \emptyset$. We can assume that $\varphi^i(U) \cap U = \emptyset$ for $i = 1, \ldots, k-1$. Take a point $x_0 \in U$ such that $\varphi^k(x_0) \in U$. Now, set $x_{nk+i} = \varphi^i(x_0)$ for $n \in Z$ and $0 \leq i < k$. We see easily that $\{x_i\}_{i \in Z} \in \text{Orb}^{\delta}(\varphi)$. Hence we can find a point $y \in M$ such that $d(\varphi^i(y), x_i) \leq \varepsilon$ for $i \in Z$. In particular, we have $d(\varphi^{nk}(y), x_{nk}) \leq \varepsilon$ and hence $d(\varphi^{nk}(y), x_0) \leq \varepsilon$ for $n \in Z$. Put $\psi = \varphi^k$ and $y_n = \psi^n(y)$. We have $y_n \in U_{\varepsilon}(x_0)$ for $n \in Z$.

Now, since ψ is an isometry, we have $M \in E_{\psi}$ by Lemma 14. Since ψ is stochastically stable by Lemma 2, we have $E_{\psi} = O_{\psi}$ by Lemma 12. Therefore, $M \in O_{\psi}$ and so there is $z \in M$ such that

$$\bar{d}(O_{\psi}(z), M) < \varepsilon . \tag{4.1}$$

Since $y \in M$, there is $m \in Z$ such that $y \in U_{\varepsilon}(\psi^m(z))$. Since ψ is an isometry we have $\psi^m(z) \in U_{\varepsilon}(y)$, and hence $\psi^n(\psi^m(z)) \in U_{\varepsilon}(\psi^n(y)) \subset U_{2\varepsilon}(x_0)$ and finally we get

$$\text{Orb}_{\psi}(z) \subset U_{2\varepsilon}(x_0). \tag{4.2}$$

Now (4.1) and (4.2) imply $M \subset U_{\varepsilon}(O_{\psi}(z)) \subset U_{3\varepsilon}(x_0)$ and we have $\text{diam}(M) \leq 6\varepsilon$, which is a contradiction.

Proposition 5. Let G be a compact connected Lie group. Suppose that there is a group automorphism $\varphi : G \longrightarrow G$, which is stochastically stable with respect to some Riemannian metric on G. Then, G is a torus.

Proof. Let A (resp. S) be the maximal abelian (resp. semi-simple) normal subgroup of G, and set $Z = A \cap S$. Then we know (cf. [4]) that $G = A \cdot S$ and Z is a finite group. It is well known that $\varphi(A) = A$ and $\varphi(S) = S$. Put $\xi = \varphi|_A$ and $\eta = \varphi|_S$. Since $\pi : A \times S \longrightarrow G$ defined by $\pi(a, x) = a \cdot x$ for $a \in A$, $x \in S$ is a (finite) covering map and since $\pi \circ (\xi \times \eta) = \varphi \circ \pi$, we see, by Proposition 3, that $\xi \times \eta$ and hence η is stochastically stable. Since η is an automorphism of S, η leaves invariant the Killing form β of the Lie group S, which is negative definite and so η is an isometry of the invariant Riemannian metric on S induced by $-\beta$. By Theorem 5, dim M = 0, and hence G = A is a torus.

§5. Final remarks

Remark 1. The author has examples of diffeomorphisms, which are stochastically stable but not structurally stable.

Remark 2. In case $M = S^1$ (the circle), we can prove that a C^2-diffeomorphism $\varphi : S^1 \longrightarrow S^1$ is stochastically stable if and only if there exists an integer $k > 0$ such that φ^k is topologically stable. The author does not know whether we can take $k = 1$ in the above statement.

Remark 3. The author has a characterization for projective transformations to be stochastically stable.

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