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The Center Manifold Theorem and the Structure of a Flow in the Vicinity of an Almost Periodic Motion

by

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I. Introduction. One of the fundamental techniques which is used for the study of the behavior of solutions in the vicinity of a given solution is the method of linearization. Let us briefly describe this technique. Assume that $x' = f(x)$ is a differential equation with C^2-coefficients on R^n. Let $x = \phi(t)$, $-\infty < t < \infty$, be a given solution of this equation. Now form a change of variables $x = \phi(t) + y$. Then y satisfies the differential equation

$$y' = f(\phi(t) + y) - f(\phi(t)) = g(y, t).$$

If we let $A(t)$ denote the linear part of g, i.e., $A(t)$ is the Jacobian matrix of f evaluated along the orbit $\phi(t)$, then the equation for y can be written in the form

(1) $$y' = A(t)y + F(y, t)$$

where the linear part of F vanishes at $y = 0$. The method of linearization can be described as follows:

Study the behavior of solutions of the linear equation $y' = A(t)y$ near $y = 0$, and then show (if possible) that the solutions of the nonlinear equation (1) near $y = 0$ inherit the same behavior.

If the given solution $\phi(t)$ is a fixed point of $x' = f(x)$, i.e., a zero of f, then the matrix A, as well as the function F, are independent of t, and the study of the linear equation $y' = Ay$ is very simple. If it happens that the matrix A has no eigenvalues with real part zero, then the linear equation $y' = Ay$ admits an exponential dichotomy, which in turn, is inherited by the nonlinear equations, cf. [2, 3, 4, 5]. However, if A has some eigenvalues with real part zero, then the linear equation admits a trichotomy. That is, the y-space R^n admits a splitting into three linear subspaces, $R^n = S_- + S_0 + S_+$, each invariant under A, where S_-, S_0 and S_+ are the algebraic sums of the generalized eigenspaces of A that correspond, respectively, to the eigenvalues of A with negative, zero and positive real parts. This splitting is then inherited by the nonlinear equation near $y = 0$. S_- is replaced by the stable manifold, S_+ by the unstable manifold and S_0 by the center manifold, cf. [2, 5, 6].

The next level of difficulty occurs when the given solution $\phi(t)$ is nonconstant and periodic in t with minimal period $\omega > 0$. In this case the matrix $A(t)$ is periodic in t with the same period ω. Likewise the nonlinear term $F(y, t)$ is ω-periodic in t. One way of attacking this problem is by means of the Floquet theory, which asserts that there is a periodic change of variables $u = P(t)y$, where $P(t)$ is nonsingular and periodic in t, such that in terms of the u-variable Eq. (1) becomes

(2) $$u' = Bu + G(u, t),$$

where B is constant and G is periodic in t. Because of the fact that $\phi(t)$ is a nontrivial periodic solution of an autonomous differential equation $x' = f(x)$, the matrix B has $\lambda = 0$ as an eigenvalue. In other words, the invariant subspace S_0 for the linear equation $u' = Bu$ is nontrivial. Therefore the center manifold for the full nonlinear equation (2) has dimension ≥ 1. This is easy to see geometrically if one notes that the center manifold for the nonlinear equation contains the trajectory $\{\phi(\tau): 0 \leq \tau \leq \omega\}$, which is a one-dimensional set S^1.

The problems one faces for a nonperiodic solution $\phi(t)$ are numerous. Difficulties occur even in the case where $\phi(t)$ is almost periodic in t. In this lecture we shall describe a theory of linearization which encompasses the study of the behavior of solutions in the vicinity of a given almost periodic motion. Our theory is more refined then that developed in the above references. Specifically we wish to generalize to nonautonomous nonlinear equations the roles of the generalized eigenspaces of linear autonomous equation $x' = Ax$. This involves studying carefully the dynamical properties of the eigenvalues of the matrix A. More precisely, the real parts of the eigenvalues describe the exponential growth rates of the solutions of $y' = Ay$. This feature does generalize to nonautonomous problems. The theory we present here is based on a spectral theory for nonautonomous linear differential equations developed by Sacker and Sell [10].

Consider the linear differential equation $y' = A(t)y$, $y \in R^n$, with bounded uniformly continuous coefficients. The spectrum $\Sigma(A)$ is defined in Section II, but in the constant coefficient case it is the set $\{Re\ \mu\}$ where μ ranges over the eigenvalues of A. More generally the spectrum is the union of a finite number of compact intervals $\Sigma(A) = \bigcup_{i=1}^{k} [a_i, b_i]$.

Associated with each spectral interval $[a_i, b_i]$ there is a nontrivial linear subspace $V_i(A)$ of R^n called the spectral subspace, which is characterized in terms of the solutions of $y' = A(t)y$ with exponential growth rates in the interval $[a_i, b_i]$. (In the constant coefficient case, $V_i(A)$ is the algebraic sum of the generalized eigenspaces corresponding to eigenvalues λ with $Re\ \lambda = a_i = b_i$.) We show in Theorem 1 that the nonlinear equation

(3) $$y' = A(t)y + F(y, t)$$

inherits the same structure, provided F satisfies certain reasonable assumptions. That is, associated with each spectral interval there exist "branch manifolds" $V_i(A, F)$ such that $V_i(A, F)$ is homeomorphic to $V_i(A)$. Moreover, $V_i(A, F)$ can be characterized in terms of solutions of (3) with exponential growth rates approximately in the interval $[a_i, b_i]$.

Our Theorem 1 includes the center manifold theorem described above and represent the appropriate generalization of this theorem to nonautonomous systems. The existence of a nontrivial center manifold for the nonlinear equation (3) is thus assured whenever $\lambda = 0$ is in the spectrum $\Sigma(A)$ of the related linear equation. If $\lambda = 0$ is not in the spectrum $\Sigma(A)$, then there is a splitting of R^n into stable and unstable manifolds of complimentary dimensions.

Theorem 1 is general enough to apply to the study of the behavior of solutions in the vicinity of an arbitrary <u>bounded</u> solution of $x' = f(x)$. However because of certain technicalities, which are described below, one must exercise care in applying this result. Fortunately all the technical conditions underlying Theorem 1 are satisfied in the case where $\phi(t)$ is an almost periodic solution of $x' = f(x)$, and this leads to a rather interesting application.

Specifically let $\phi(t)$ be an almost periodic solution of the autonomous equation $x' = f(x)$, where f is a C^2-function on R^n. Then the hull $H(\phi) = Cl\{\phi(\tau): \tau \in R\}$ is a space with topological dimension ℓ and $\ell \leq n$. By the Pontryagin-Cartwright Theorem the topological dimension ℓ is the same as the algebraic dimension of the Fourier-Bohr frequency module. We will show in Theorem 3 that for the induced linear equation $y' = A(t)y$ one has $0 \in \Sigma(A)$ and that $\dim V_0(A) \geq \ell$, where $V_0(A)$ denotes the spectral subspace of $y' = A(t)y$ corresponding to the spectral interval $[a_0, b_0]$ containing $\lambda = 0$. This then means that the center manifold for ϕ has dimension $\geq \ell$ and that it contains the hull $H(\phi)$. In the event that the center manifold has dimension equal to ℓ, then the hull $H(\phi)$ is locally homeomorphic to R^ℓ. Since $H(\phi)$ is a compact Abelian group, it follows that $H(\phi)$ is a Lie group and consequently one has $H(\phi)$ is homeomorphic with T^ℓ, the ℓ-dimensional torus.

As we shall see, the dimension of the center manifold can be computed exactly in terms of exponential dichotomies for the shifted linear equation $y' = (A(t) - \lambda I)y$, where λ is real. The almost periodic theory described in the last paragraph has the following variation. Assume that there is a $\mu < 0$ and a $\lambda > 0$ such that both $y' = (A(t) - \mu I)y$ and $y' = (A(t) - \lambda I)y$ admit exponential dichotomies, and let N_μ and N_λ denote the dimensions of the stable manifolds, respectively. Then $N_\lambda - N_\mu \geq \ell$. (This is a reformulation of the fact that $\dim V_0(A) \geq \ell$.) Furthermore if $N_\lambda - N_\mu = \ell$, then $H(\phi)$ is homeomorphic with T^ℓ and the solution $\phi(t)$ is quasi-periodic (Theorem 4).

As mentioned above, the essential features of the spectral theory for nonautonomous linear differential equations will be described in Section II. In Section III we shall state our main result (Theorem 1) which describes how the stable manifolds, the unstable manifolds and the branch manifolds are inherited by the nonlinear equation (3). In Section IV we shall interpet this result in the case that the given solution $\phi(t)$ is almost periodic in t.

The proofs of the results describe in this lecture can be found in [10, 13, 14].

II. Linear Theory. Let M^n denote the collection of all $(n \times n)$-matrix-valued functions $A(t)$ $(t \in R)$ with bounded uniformly continuous real coefficients. We assume that M^n has the topology of uniform convergence on compact sets. For each $A \in M^n$ we define the translate A_τ by $A_\tau(t) = A(\tau + t)$. The mapping $\sigma(A, \tau) = A_\tau$ then determines a flow on M^n. The hull of A is defined by $H(A) = Cl\{A_\tau : \tau \in R\}$ and is an invariant set for the flow σ. Furthermore it follows from the Ascoli-Arzelá Theorem that the hull $H(A)$ is a compact subset of M^n, cf. [11, 12].

For each $A \in M^n$ and $x \in R^n$, we let $\phi(x, A, t)$ denote the solution of the initial value problem $x' = A(t)x$, $x(0) = x$. The function $\phi(x, A, t)$ is linear in x, and the equation

$\Phi(A, t)x = \phi(x, A, t)$ defines a linear operator $\Phi(A, t)$ on R^n. The operator $\Phi(A, t)$ is the fundamental matrix solution of $x' = A(t)x$. The mapping π defined by

$$\pi(x, A, t) = (\phi(x, A, t), \sigma(A, t))$$

is a linear skew-product flow on $X \times M^n$, cf. [9].

Let $|\cdot|$ denote a norm on R^n. Then define a matrix norm $|\cdot|$ by the usual equation $|B| = \sup\{|Bx|: |x| = 1\}$.

Let A be a nonempty subset of M^n. We shall say that P is a <u>projector on</u> A if for each $A \in A$, $P(A)$ is a projection in R^n and the mapping $(A, x) \to P(A)x$ is jointly continuous in $(x, A) \in R^n \times A$. We shall say that π <u>admits an exponential dichotomy over</u> A if there is a projector P on A and constants $K \geq 1$ and $\alpha > 0$ such that

(4) $$|\Phi(A, t)\, P(A)\, \Phi^{-1}(A, s)| \leq Ke^{-\alpha(t-s)}, \quad s \leq t$$

$$|\Phi(A, t)[I - P(A)]\, \Phi^{-1}(A, s)| \leq Ke^{-\alpha(s-t)}, \quad t \leq s$$

for all $A \in A$. In the event that A consists of a single point $\{A\}$, this is the standard notion of an exponential dichotomy for the differential equation $x' = A(t)x$, [3]. It is shown in [9] that if π admits an exponential dichotomy over the single point $\{A_0\}$, then π admits an exponential dichotomy over the hull $H(A_0)$.

Let $A \in M^n$ and consider the linear differential equation

(5) $$x' = A(t)x.$$

If one makes the change of variables $y = e^{-\lambda t}x$, where λ is real, then Eq. (5) becomes the shifted equation

(6) $$y' = (A(t) - \lambda I)y.$$

The new coefficient matrix $(A(t) - \lambda I)$ is an element of M^n. Let $\Phi_\lambda(A, t)$ denote the fundamental matrix solution of (6), then $\Phi_\lambda(A, t) = e^{-\lambda t}\Phi(A, t)$. For $\lambda \in R$ we define the stable and unstable subspaces by

$$S_\lambda(A) = \{x \in R^n: |e^{-\lambda t}\phi(x, A, t)| \to 0 \text{ as } t \to +\infty\}$$
$$U_\lambda(A) = \{x \in R^n: |e^{-\lambda t}\phi(x, A, t)| \to 0 \text{ as } t \to -\infty\}.$$

Let $A \in M^n$. We define the <u>resolvent set</u> $\rho(A)$ as the collection of all $\lambda \in R$ such that Eq. (6) admits an exponential dichotomy. The <u>spectrum</u> $\Sigma(A)$ is the complement of $\rho(A)$.

If $\lambda \in \rho(A)$, then for all $B \in H(A)$ the linear subspaces $S_\lambda(B)$ and $U_\lambda(B)$ are complimentary, i.e. $S_\lambda(B) \cap U_\lambda(B) = \{0\}$ and $R^n = S_\lambda(B) + U_\lambda(B)$. Moreover, the linear skew-product flow

$$\pi_\lambda(x, B, t) = \left(e^{-\lambda t}\phi(x, B, t), \sigma(B, t)\right)$$

admits an exponential dichotomy over $H(A)$. This in turn, implies that the subspaces $S_\lambda(B)$ and $U_\lambda(B)$ vary "continuously" in $B \in H(A)$, since they are, respectively, the range and null space of the projector associated with the exponential dichotomy, cf. [9] for details.

The following Spectral Theorem is proved in [10]:

<u>Theorem A</u>. <u>Let</u> $A \in M^n$ <u>where</u> $n \geq 1$. <u>Then the spectrum</u> $\Sigma(A)$ <u>is the union of</u> k <u>nonoverlapping intervals</u>

$$\Sigma(A) = \bigcup_{i=1}^{k} [a_i, b_i]$$

<u>where</u> $1 \leq k \leq n$. <u>Furthermore associated with each spectral interval</u> $[a_i, b_i]$ <u>and each</u> $B \in H(A)$ <u>there exists an integer</u> n_i <u>(independent of</u> B<u>)</u> <u>and a linear subspace</u> $V_i(B)$ <u>of</u> R^n <u>with</u> $\dim V_i(B) = n_i$ <u>and such that for all</u> $B \in H(A)$ <u>one has</u>

(i) $1 \leq n_i$ <u>and</u> $n_1 + \ldots + n_k = n$,

(ii) $V_i(B) \cap V_j(B) = \{0\}$ <u>if</u> $i \neq j$

(iii) $R^n = V_1(B) + \ldots + V_k(B)$,

(iv) $V_i(B) = S_\lambda(B) \cap U_\mu(B)$ whenever $(\mu, \lambda) \cap \Sigma(A) = [a_i, b_i]$. Moreover the linear subspaces $V_i(B)$ vary continuously in B.

The subspace $V_i(B)$ is called the spectral subspace associated with the spectral interval $[a_i, b_i]$.

For each $\lambda \in \rho(A)$, the flow π_λ admits an exponential dichotomy, which means that there is a projector P (depending on λ) and constants $K \geq 1$ and $\alpha > 0$ so that Ineq. (4) holds with Φ_λ replacing Φ. The constants K and α also depend on λ. In order to build some uniformity into the theory, we define $\rho(A, K, \alpha)$ to be the set of $\lambda \in \rho(A)$ such that there is a projector P on $H(A)$ with the property that

(7) $$|\Phi_\lambda(B, t)P(B)\Phi_\lambda^{-1}(B, s)| \leq Ke^{-\alpha(t-s)}, \quad s \leq t$$
$$|\Phi_\lambda(B, t)[I - P(B)]\Phi_\lambda^{-1}(B, s)| \leq Ke^{-\alpha(s-t)}, \quad t \leq s$$

for all $B \in H(A)$. It is now easy to verify that the following Standing Hypothesis is always satisfied:

Standing Hypothesis: Let $A \in M^n$ and let $\Sigma(A) = \bigcup_{i=1}^k [a_i, b_i]$ denote the spectrum of A. Assume that the intervals $[a_i, b_i]$ are ordered so that

$$a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_k \leq b_k.$$

Then there exists constants $K \geq 1$ and $\alpha > 0$ so that $\rho(A, K, \alpha)$ contains points $\{\lambda_0, \lambda_1, \ldots, \lambda_k\}$ that satisfy

(8) $$\lambda_0 < a_1 \leq b_1 < \lambda_1 < a_2 \leq \ldots < \lambda_{k-1} < a_k \leq b_k < \lambda_k.$$

III. The Invariant Manifolds. In this section we will describe how the structure of the linear equation

$$x' = A(t)x$$

is inherited by the nonlinear equation

(9) $$x' = A(t)x + F(x, t)$$

where $A \in M^n$, $F \in F$ and F is the collection of all functions $F(x, t)$ from $R^n \times R$ to R^n satisfying.

Hypothesis H. F is Lipschitz-continuous in x, $F(0, t) = 0$ for all t and for every $\theta > 0$ there is a $\delta > 0$ such that

$$|F(x, t) - F(y, t)| \leq \theta|x - y|$$

for all x, y in R^n with $|x| \leq \delta$, $|y| \leq \delta$, and for all $t \in R$.

Let $A \in M^n$ be fixed. A local flow can be generated on the spaces $R^n \times H(A) \times F$ as follows: First we assume that F has the topology of uniform convergence on compact sets. Then for $G \in F$ and $B \in H(A)$, we let $\phi(x, B, G, t)$ denote the unique noncontinuable solution of the initial value problem

(10) $$x' = B(t)x + G(x, t), \ x(0) = x.$$

The solution $\phi(x, B, G, t)$ need not be defined for all $t \in R$ because of the so-called finite-escape phenomenon. Nevertheless ϕ depends continuously on (x, B, G, t) and the mapping

$$\pi(x, B, G, \tau) = (\phi(x, B, G, \tau), B_\tau, G_\tau)$$

defines a local flow on $R^n \times H(A) \times F$, where $G_\tau(x, t) = G(x, \tau + t$ cf. [11, 12].

For $F \in F$ we define the hull $H(F)$ as

$$H(F) = Cl\{F_\tau : \tau \in R\}.$$

It follows from the general theory of flows that the hull $H(F)$ is an invariant set in F.

Let $\phi(x, A, F, t)$ be a solution of (9) that is defined for all $t \geq 0$, or $t \leq 0$, where $x \neq 0$ and define the Lyapunov characteristic exponents by

$$\lambda^+(x, A, F) = \lambda^+(x) = \limsup_{t\to+\infty} \frac{1}{t} \log|\phi(x, A, F, t)|,$$

$$\lambda^-(x, A, F) = \lambda^-(x) = \limsup_{t\to-\infty} \frac{1}{t} \log|\phi(x, A, F, t)|,$$

respectively.

It is convenient to study, for certain purposes, the restriction of the local flow π to smaller regions in the x-space. More precisely, let $a > 0$ and define $D_a = \{x \in R^n\colon |x| < a\}$. Then the restriction of the differential equation (9) to D_a defines a local flow on $D_a \times H(A) \times F$. We shall call this flow the induced flow on $D_a \times H(A) \times F$. Notice that a set $\Gamma \subseteq D_a \times H(A) \times F$ is an invariant set in the induced flow whenever one has

$$(x, B, F) \in \Gamma \Longrightarrow \pi(x, B, F, t) \in \Gamma \quad \text{for all} \quad t \in I$$

where I is the maximal interval containing 0 for which

$$|\phi(x, B, F, t)| < a \quad \text{for all} \quad t \in I.$$

We are now prepared to state our main result.

Theorem 1. Assume that

$$x' = A(t)x + F(x, t)$$

is given where $A \in M^n$ and that the Standing Hypothesis is satisfied. Assume further that F satisfies Hypothesis H. Then there is an $a > 0$ with the following properties:

(A) For $\lambda = \lambda_i$ $(i = 0, 1, \ldots, k)$, $B \in H(A)$ and $G \in H(F)$ there exist in R^n Lipschitz-continuous manifolds $S_i(B, G) \cap D_a$ and $U_i(B, G) \cap D_a$ homeomorphic to $S_\lambda(B) \cap D_a$ and $U_\lambda(B) \cap D_a$, respectively. Furthermore the sets

$$\{(x, B, G) \in X \times H(A) \times H(F)\colon x \in S_i(B, G) \cap D_a\}$$

$$\{(x, B, G) \in X \times H(A) \times H(F)\colon x \in U_i(B, G) \cap D_a\}$$

are invariant sets in the induced flow on $D_a \times H(A) \times H(F)$. Moreover, $S_i(B, G) \cap D_a$ contains the set

$\{x \in D_a: |\phi(x,B,G,t)| < a$ for all $t \geq 0$ and $\lambda^+(x,B,G) < \lambda_i\}$,

and $U_i(B, G) \cap D_a$ contains the set

$\{x \in D_a: |\phi(x,B,G,t)| < a$ for all $t \leq 0$ and $\lambda^-(x,B,G) > \lambda_i\}$.

(B) For $\mu = \lambda_{i-1}$ and $\lambda = \lambda_i$ $(i = 1, \ldots, k)$, $B \in H(A)$ and $G \in H(F)$ there exist in R^n Lipschitz-continuous manifolds $W_i(B, G) \cap D_a$ homeomorphic to $V_i(B) \cap D_a$. Furthermore the set

$$\{(x, B, G) \in X \times H(A) \times H(F): x \in W_i(B, G) \cap D_a\}$$

is an invariant set in the induced flow on $D_a \times H(A) \times H(F)$. Moreover, $W_i(B, G) \cap D_a$ contains the set

$\{x \in D_a: |\phi(x,B,G,t)| < a$ for all $t \in R$, $\mu < \lambda^-(x,B,G)$ and $\lambda^+(x,B,G) < \lambda\}$.

In particular if $\mu < 0$ and $\lambda > 0$ then $W_i(B, G) \cap D_a$ contains the set

$\{x \in D_a: |\phi(x,B,G,t)| < a$ for all $t \in R$ and $\lambda^-(x,B,G) = \lambda^+(x,B,G) = 0\}$.

The manifolds $S_i(B, G) \cap D_a$ $(i = 0, 1, \ldots, k)$ are called the stable manifolds for Eq. (10). The manifolds $U_i(B, G) \cap D_a$ $(i = 0, 1, \ldots, k)$ are called the unstable manifolds for Eq. (10). These manifolds have the same dimensions as the corresponding stable and unstable manifolds for the linear equations $x' = B(t)$; or $x' = A(t)x$. The manifolds $W_i(B, G) \cap D_a$ $(i = 1, \ldots, k)$ are called the branch manifolds for Eq. (10). They have the same dimension as the corresponding spectral subspaces for the linear equation $x' = A(t)x$.

Remark 1. Let us now return to the "center manifold". Once again consider the equation

(11) $$x' = A(t)x + F(x, t)$$

where $A \in M^n$ and the Standing Hypothesis is satisfied and where F satisfies Hypothesis H. Assume now that $\lambda = 0$ is in the spectrum $\Sigma(A)$ and let $[a_0, b_0]$ denote the spectral interval containing 0. Let $V_0(A)$ denote the corresponding spectral subspace in R^n and let $n_0 = \dim V_0(A) \geq 1$. Let $W_0(A, F) \cap D_a$ denote the corresponding branch manifold for the nonlinear equation (11). The manifold $W_0(A, F) \cap D_a$ is called the <u>center manifold</u> for (11). It has dimension n_0. Next set $\mu = \lambda_{i-1}$ and $\lambda = \lambda_i$ where $\mu < 0 < \lambda$. Then the stable manifold $S_{i-1}(A, F) \cap D_a$ has dimension ℓ, and the unstable manifold $U_i(A, F) \cap D_a$ has dimension m where $\ell + m + n_0 = n$. Furthermore for any $x \in S_{i-1}(A, F) \cap D_a$ one has

$$|\phi(x, A, F, t)| \leq L|x|e^{-\beta t}, \quad t \geq 0,$$

and for any $x \in U_i(A, F) \cap D_a$ one has

$$|\phi(x, A, F, t)| \leq L|x|e^{\beta t}, \quad t \leq 0$$

where L and β are appropriate positive constants. The three spaces $S_{i-1}(A, F) \cap D_a$, $W_0(A, F) \cap D_a$ and $U_i(A, F) \cap D_a$ show that the trichotomy

$$X = S_\mu(A) + V_0(A) + U_\lambda(A)$$

for the linear equation $x' = A(t)x$ is inherited by the nonlinear equation (11). We also have the following result.

<u>Theorem 2</u>. <u>Let</u> $A \in M^n$ <u>and choose</u> $\mu, \lambda \in \rho(A)$ <u>with</u> $\mu < \lambda$.

(A) <u>If there is a</u> $B \in H(A)$ <u>and an</u> $F \in \mathcal{F}$ <u>with the property that for every</u> $a > 0$ <u>there is a nonzero vector</u> $x \in D_a$ <u>with</u> $|\phi(x, B, F, t)| < a$ <u>for all</u> $t \in R$ <u>and with</u> $\mu < \lambda^-(x, B, F)$ <u>and</u> $\lambda^+(x, B, F) < \lambda$, <u>then the spectrum</u> $\Sigma(A)$ <u>meets</u> (μ, λ), i.e. $\Sigma(A) \cap (\mu, \lambda) \neq \phi$.

(B) <u>Assume that</u> $\mu < 0 < \lambda$. <u>If there is a</u> $B \in H(A)$ <u>and</u>

an $F \in \mathsf{F}$ with the property that for every $a > 0$ there is a nonzero vector $x \in D_a$ such that $|\phi(x, B, F, t)| < a$ for all $t \in R$ and $\lambda^-(x, B, F) = \lambda^+(x, B, F) = 0$, then $0 \in \Sigma(A)$.

IV. The Flow in the Vicinity of an Almost Periodic Motion. Let $x = \phi(t)$ be a given almost periodic solution of the autonomous differential equation $x' = f(x)$ on R^n with C^2-coefficients. Let $H(\phi) = Cl\ \{\phi(\tau)\colon \tau \in R\}$ denote the hull of ϕ. The space $H(\phi)$ is the space of a compact Abelian topological group with a dense subgroup parameterized by the additive group R, cf. [7, 12]. Let ℓ denote the topological dimension of $H(\phi)$. By the Pontryagin-Cartwright Theorem the topological dimension ℓ is the same as the algebraic dimension of the Fourier-Bohr frequency module and $\ell \leq n - 1$, cf. [1, 8].

Let $x = \phi(t) + y$ be a solution of $x' = f(x)$. Then y satisfies the differential equation

$$(12) \qquad y' = A(t)y + F(y, t)$$

where $A(t) = \left.\frac{\partial f}{\partial x}\right|_{x=\phi(t)}$ and $F(y, t) = f(\phi(t) + y) - f(\phi(t)) - A(t)y$. Since $A(t)$ is almost periodic in t one has $A \in \mathsf{M}^n$. Furthermore F satisfies Hypothesis H.

We shall now adopt the notation of Remark 1. In the event that $0 \in \Sigma(A)$ we shall let $[a_0, b_0]$ denote the spectral interval containing 0. Also $V_0(A)$ will denote the spectral subspace in R^n associated with $[a_0, b_0]$ and finally we let $W_0(A, F) \cap D_a$ denote the center manifold for Eq. (12).

Theorem 3. Let $x = \phi(t)$ be an almost periodic solution of $x' = f(x)$ and assume that the topological dimension of the hull $H(\phi)$ is $\ell \geq 1$. Then the following statements are valid:

(A) $0 \in \Sigma(A)$ and $\dim W_0(A, F) \cap D_a = \dim V_0(A) \geq \ell$.

(B) If $\dim V_0(A) = \ell$, then $H(\phi)$ is diffeomorphic with T^ℓ, the ℓ-dimensional torus, and the solution $\phi(t)$ is quasi-periodic, that is, there exists a continuous function $\Psi: R^\ell \to R^n$ such that $\Psi(u_1, \ldots, u_\ell)$ has period 1 in each variable and

$$\phi(t) = \Psi(\alpha_1 t, \ldots, \alpha_\ell t), \ t \in R$$

for appropriate choice of constants $\alpha_1, \ldots, \alpha_\ell$.

The following statement is simply a reformulation of Theorem 3.

Theorem 4. Let $x = \phi(t)$ be an almost periodic solution of $x' = f(x)$ where f is a C^2-function and assume that the topological dimension of the hull $H(\phi)$ is $\ell \geq 1$. Let $A(t) = \left.\frac{\partial f}{\partial x}\right|_{x=\phi(t)}$ denote the linear part of f evaluated along $\phi(t)$. Let $\rho(A)$ denote the collection of $\lambda \in R$ such that the linear equation

$$x' = \big(A(t) - \lambda I\big)x$$

admits an exponential dichotomy. For each $\lambda \in \rho(A)$ let N_λ denote the dimension of the stable manifold for the associated exponential dichotomy. Then the following statements are valid

(A) If $\lambda, \mu \in \rho(A)$ with $\mu < 0 < \lambda$, then $N_\lambda - N_\mu \geq \ell$.

(B) If λ and μ can be chosen in $\rho(A)$ so that $\mu < 0 < \lambda$ and $N_\lambda - N_\mu = \ell$, then $H(\phi)$ is diffeomorphic with T^ℓ, the ℓ-dimensional torus, and $\phi(t)$ is quasi-periodic.

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