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Stability and existence of almost periodic solutions of functional differential equations with infinite retardation

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1. Introduction. Assuming the uniqueness of solutions for the initial value problem, several authors discussed that the existence of a bounded solution with some stability property implies the existence of almost periodic solutions(cf. [3], [11], [14], [16], [17]).

Without the assumption that the solution is unique for the initial value problem, Yoshizawa[18] and Kato[10] have shown the existence of an almost periodic solution of almost periodic systems. They have considered functional differential equations with finite retardation and required the existence of a bounded solution with some stability property. And they have utilized the fact that any solution which takes its initial function in a bounded set $C = C([-h, 0], R^n)$ at t_o remains in a compact subset of C for all $t \geqq t_o + h$. It is known by Hale[5] that if the retardation is infinite, the solution operator never becomes completely continuous but is α - contraction of some order. However, we can find a compact set which contains all of solutions in the hull of the bounded solution. Therefore, by considering only the solutions in hull of the bounded solution, we can introduce new concepts of stabilities, which we called stabilities with respect to hull in [8]. Stabilities with respect to hull are weaker than the usual ones, because these stability properties do not necessarily imply the uniqueness of the bounded solution for the initial value problem. By using the existence of a bounded solution with some kind of stability with respect to hull, we shall discuss the existence of an almost periodic solution of almost periodic systems.

2. Phase space B. The following Banach space has considered by Hale[4] and Hale and Kato[6]. Let $x = (x^1, x^2, \dots, x^n)$ be any vector in R^n and let $|x|$ be any norm of x. Let B be a linear real vector space of functions mapping $(-\infty, 0]$ into R^n with elements designated by $\hat{\phi}$, $\hat{\psi}$, and $\hat{\phi} = \hat{\psi}$ means $\hat{\phi}(t) = \hat{\psi}(t)$ for all $t \leqq 0$. Assume that a semi-norm $|\cdot|_{\hat{B}}$ is given in $\hat{B}$, and assume that $B = \hat{B}/|\cdot|_{\hat{B}}$ is a Banach space with the norm $|\cdot|_B$ which is naturally induced by $|\cdot|_{\hat{B}}$. B consists of equivalence classes ϕ of $\hat{\phi} \in \hat{B}$.

For a $\beta \geqq 0$ and a $\hat{\phi} \in \hat{B}$, let $\hat{\phi}^\beta$ denote the restriction of $\hat{\phi}$ to $(-\infty, -\beta]$, and let $|\cdot|_\beta$ be a semi-norm in B defined by

$$|\phi|_\beta = \inf_{\hat{\eta}\in\hat{B}}\{\inf_{\hat{\psi}\in\hat{B}}\{|\hat{\psi}|_{\hat{B}}: \hat{\psi}^\beta = \hat{\eta}^\beta\}: \eta = \phi\}.$$

For an R^n-valued function $\hat{x}$ defined on a interval $(-\infty, \sigma)$ and for a $t \in (-\infty, \sigma)$, let $\hat{x}_t$ be a function defined on $(-\infty, 0]$ such that

$$\hat{x}_t(\theta) = \hat{x}(t+\theta), \ \theta \leqq 0.$$

Given an $A > 0$ and a $\hat{\phi} \in \hat{B}$, let $F_A(\hat{\phi})$ be the set of all functions $\hat{x}$ defined on $(-\infty, A]$ such that $\hat{x}_0 = \hat{\phi}$ and $\hat{x}(t)$ is continuous on $[0, A]$, and denote

$$F_A = \bigcup\{F_A(\hat{\phi}): \hat{\phi} \in \hat{B}\}.$$

We assume that $\hat{B}$ has the following properties:

(I) If $\hat{x} \in F_A$, $A > 0$, then $\hat{x}_t \in \hat{B}$ and $\hat{x}_t$ is continuous in t on $[0,A]$.

(II) $|\hat{\phi}(0)| \leqq M_1|\hat{\phi}|_{\hat{B}}$ for some $M_1 > 0$.

(III) If a sequence $\{\hat{\phi}_k\}$, $\hat{\phi}_k \in \hat{B}$, is uniformly bounded on $(-\infty, 0]$ with respect to norm $|\cdot|$ and converges to $\hat{\phi}$ uniformly on any compact subset of $(-\infty, 0]$, then $\hat{\phi} \in \hat{B}$ and $|\phi_k - \phi|_B \to 0$ as $k \to \infty$.

By $\hat{\tau}^\beta$, $\beta \geqq 0$, we shall denote an operator on $\hat{B}$ into $\hat{B}^\beta = \{\{\hat{\psi} \in \hat{B}: \hat{\psi}^\beta = \hat{\phi}^\beta\}: \hat{\phi} \in \hat{B}\}$ such that $\hat{\psi} \in \hat{\tau}^\beta\phi$ if and only if $\hat{\psi}(\theta) = \hat{\phi}(\theta+\beta)$ for $\theta \in (-\infty, -\beta]$.

(IV) If $\phi = \psi$ in B, then $|\eta - \xi|_\beta = 0$ for any $\beta \geqq 0$, where $\hat{\eta} \in \hat{\tau}^\beta\hat{\phi}$ and $\hat{\xi} \in \hat{\tau}^\beta\hat{\psi}$.

Under this axiom, it is possible to consider a linear operator $\tau^\beta : B \to B^\beta$, $B^\beta = B/|\cdot|_\beta$, defined by

$$\tau^\beta\phi = \{\psi\}_\beta \text{ for a } \psi \in B \text{ such that } \hat{\psi} \in \hat{\tau}^\beta\hat{\phi},\ \{\psi\}_\beta = \{\phi \in B: |\phi - \psi|_\beta = 0\}.$$

(V) $|\tau^\beta\phi|_\beta \to 0$ as $t \to \infty$.

(VI) There exists a $K > 0$ such that for any ϕ in B and any $\sigma \geqq 0$

$$|\phi|_B \leqq K[\inf_{\hat\psi \varepsilon \hat B} \{ \sup_{-\sigma \leqq \theta \leqq 0} |\hat\psi(\theta)|: \psi = \phi\} + |\phi|_\sigma].$$

(VII) B is separable.

Remark 1. Property (III) is equivalent to the following property: For any $b > 0$ and $\varepsilon > 0$, there exist an $N > 0$ and a $\delta > 0$ such that $\hat B_\varepsilon \supset U_{N,\delta} \cap \chi_b$, where $\hat B_\varepsilon = \{\hat\phi \varepsilon \hat B: |\hat\phi|_{\hat B} < \varepsilon\}$, $U_{N,\delta} = \{\hat\phi \varepsilon \hat B: \hat\phi$ is continuous on $[-N, 0]$ and $\sup_{-N \leqq \theta \leqq 0} |\hat\phi(\theta)| < \delta\}$ and $\chi_b = \{\hat\phi \varepsilon \hat B: \hat\phi$ is continuous on $(-\infty, 0]$ and $\sup_{-\infty < \theta \leqq 0} |\hat\phi(\theta)| \leqq b\}$. By Property (III), if $\{\xi^n\}$ is any sequence which is continuous and uniformly bounded on R, $R = (-\infty, \infty)$, and converges to function $v(t)$ uniformly on R as $n \to \infty$, then $|\xi^n_t - v_t|_B \to 0$ uniformly on R as $n \to \infty$. Therefore, by slightly modifying the arguments used in the proof of Theorem 1 in [7], we can show that if $\xi(t+t_n)$ converges to a function $u(t)$ uniformly on I, $I = [0, \infty)$, as $n \to \infty$, then $|\xi_{t+t_n} - u_t|_B \to 0$ uniformly on I as $n \to \infty$, where $\xi(t)$ is in F_∞ and uniformly continuous and bounded on I and $\{t_n\}$ is a sequence such that $t_n \to \infty$ as $n \to \infty$.

Remark 2. Hale and Kato[6] has pointed out that property (III) implies that all bounded continuous functions mapping $(-\infty, 0]$ into R^n are in $\hat B$.

The following two spaces *C* and *B* have properties (I) ~ (VII) (cf. [2], [6], [7], [12]).

Example 1. The space C consists of all continuous functions mapping $(-\infty, 0]$ into R^n such that $\phi(\theta)e^{\gamma\theta} \to 0$ as $\theta \to -\infty$ with norm $|\phi|_C = \sup_{-\infty<\theta\leqq 0} |\phi(\theta)|e^{\gamma\theta}$, $\gamma > 0$.

Example 2. Let $r \geqq 0$, $p \geqq 1$, and let $g(\theta)$ be a nondecreasing positive function defined on $(-\infty, 0]$ such that $\int_{-\infty}^{0} g(\theta)d\theta < \infty$. The space B consists of all functions ϕ mapping $(-\infty, 0]$ into R^n, which are Lebesgue measurable on $(-\infty, 0]$ and are continuous on $[-r, 0]$ with norm $|\phi|_B = \{(\sup_{-r\leqq\theta\leqq 0} |\phi(\theta)|^p + \int_{-\infty}^{0} |\phi(\theta)|^p g(\theta)d\theta\}^{1/p}$. When $r = 0$, we do not assume the continuity of ϕ at $\theta = 0$.

3. Asymptotically almost periodic solutions of almost periodic systems. Consider an almost periodic system

$$\dot{x}(t) = F(t, x_t), \tag{1}$$

where $F(t, \phi)$ is continuous on $R \times \overline{B_M}$, $\overline{B_M} = \{\phi \in B\colon |\phi|_B \leqq M\}$, and almost periodic in t uniformly for $\phi \in \overline{B_M}$. We assume that there exists an $L > 0$ such that $|F(t, \phi)| \leqq L$ on $R \times \overline{B_M}$. Let $\xi(t)$ be a solution of (1) defined on I, which satisfies $|\xi_t|_B \leqq \beta$, $0 < \beta < M$, for all $t \in I$.

Define S by

$$S = \{\phi_t\colon t \geqq 0,\ \phi \in S^*\},$$

where

$$S^* = \{\phi \in F_\infty: \phi(\theta) = \xi(\theta),\ \theta \in (-\infty, 0],\ |\phi(\theta)| \leqq M_1\beta \text{ for all } \theta \geqq 0$$
$$\text{and } |\phi(\theta) - \phi(\theta')| \leqq L|\theta - \theta'| \text{ for any } \theta, \theta' \geqq 0\}.$$

Then $\bar{S}$ is compact, where $\bar{S}$ is closure of S, and $\xi \in S^*$. For details, see [8]. Let $H(\xi)$, $H(F)$ and $H(\xi, F)$ be the hulls of $\xi(t)$, $F(t, \phi)$ and $(\xi(t), F(t, \phi))$, respectively. $H^+(\xi)$, $H^+(F)$ and $H^+(\xi, F)$ are subsets of $H(\xi)$, $H(F)$ and $H(\xi, F)$ whose elements are $x(t)$, $G(t, \phi)$ and $(x(t), G(t, \phi))$ such that there exists a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$ such that $\xi(t+t_k) \to x(t)$ as $k \to \infty$ uniformly on any compact interval in R and $F(t+t_k, \phi) \to G(t, \phi)$ as $k \to \infty$ uniformly on $R \times \bar{S}$, respectively. Clearly, $x(t)$ is a solution of $\dot{x}(t) = G(t, x_t)$, if $(x(t), G(t, \phi)) \in H(\xi, F)$.

Let $f(t)$ be a continuous function defined on $a \leqq t < \infty$. $f(t)$ is said to be asymptotically almost periodic if it is a sum of a continuous almost periodic function $p(t)$ and a continuous function $q(t)$ defined on $a \leqq t < \infty$ which tends to zero as $t \to \infty$, that is

$$f(t) = p(t) + q(t).$$

It is well known that $f(t)$ is asymptotically almost periodic if and only if for any sequence $\{\tau_k\}$ such that $\tau_k \to \infty$ as $k \to \infty$ there exists a subsequence $\{\tau_{k_j}\}$ for which $f(t+\tau_{k_j})$ converges uniformly on $a \leqq t < \infty$ (cf. [19]).

Proposition 1 (cf. Theorem 1 in [8]). If the solution $\xi(t)$ of (1) is asymptotically almost periodic, then for any $G \in H^+(F)$, there exists a sequence $\{\tau_k\}$ such that $\xi(t+\tau_k)$ tends to an almost periodic solution of the system

$$(2) \qquad \dot{x}(t) = G(t, x_t)$$

uniformly on R as $k \to \infty$.

We shall define $H^+_G(\xi)$ by

$$H^+_G(\xi) = \{x(t): (x(t), G(t, \phi)) \in H^+(\xi, F)\}.$$

Now we shall give definitions of stabilities with respect to hull and some separation conditions.

Definition 1. The solutions in $H^+(\xi)$ are uniformly stable with respect to $H^+(\xi)$ with a common $\delta(\cdot)$ (in short, u.s.$H^+(\xi).\delta(\cdot)$), if for any $\varepsilon > 0$, $t_o \in I$ and $G \in H^+(F)$, $|x_{t_o} - y_{t_o}|_B < \delta(\varepsilon)$ implies $|x_t - y_t|_B < \varepsilon$ for all $t \geqq t_o$, whenever $x(t), y(t) \in H^+_G(\xi)$.

Definition 2. The solutions in $H^+(\xi)$ are weakly uniformly asymptotically stable with respect to $H^+(\xi)$ with a common pair $(\delta_o, \delta(\cdot))$ (in short, w.u.a.s.$H^+(\xi).(\delta_o, \delta(\cdot)))$, if the solutions in $H^+(\xi)$ are u.s. $H^+(\xi).\delta(\cdot)$ and for any $t_o \in I$ and $G \in H^+(F)$, $|x_{t_o} - y_{t_o}|_B < \delta_o$ implies $|x_t - y_t|_B \to 0$ as $t \to \infty$, whenever $x(t), y(t) \in H^+_G(\xi)$.

Definition 3. The solution in $H^+(\xi)$ are globally weakly uniformly asymptotically stable with respect to $H^+(\xi)$ with a common $\delta(\cdot)$ (in short, g.w.u.a.s.$H^+(\xi).\delta(\cdot)$), if the solutions in $H^+(\xi)$ are u.s.$H^+(\xi)$. $\delta(\cdot)$ and for any $G \in H^+(F)$ $|x_t - y_t|_B \to 0$ as $t \to \infty$, whenever $x(t)$, $y(t) \in H^+_G(\xi)$.

Definition 4. The solutions in $H^+(\xi)$ are uniformly asymptotically stable with respect to $H^+(\xi)$ with a common triple $(\delta_o, \delta(\cdot), T(\cdot))$ (in short, u.a.s.$H^+(\xi).(\delta_o, \delta(\cdot), T(\cdot))$), if the solutions in $H^+(\xi)$ are u.s. $H^+(\xi).\delta(\cdot)$ and for any $\varepsilon > 0$, any $t_o \in I$ and $G \in H^+(F)$, $|x_{t_o} - y_{t_o}|_B < \delta o$ implies $|x_t - y_t|_B < \varepsilon$ for $t \geqq t_o + T(\varepsilon)$, whenever $x(t)$, $y(t) \in H^+_G(\xi)$.

Remark 3. For ordinaly differential equations, the concept of weakly uniformly asymptotic stability was given by Sell[17, 18]. Relationships between weakly uniformly asymptotic stability and uniformly asymptotic stability have been discussed in [14, 19]. For a clas of phase spaces for functional differential equations with infinite retardation, Hale and Kato[6] have discussed relationships between weakly uniformly asymptotic stability and uniformly asymptotic stability. Their space has not property (III) in our space but has the following property which is stronger than property (V) in our space; For any $M > 0$ and $\varepsilon > 0$, there exists a $T^* > 0$ such that if $\phi \in B_M$ and $T \geqq T^*$, then $|\tau_T \phi|_T < \varepsilon$.

Definition 5. $H^+(\xi, F)$ is said to satisfy a separation condition if for any $G \in H^+(F)$, $H^+_G(\xi)$ is a finite set and if ϕ and ψ, $\phi, \psi \in H^+_G(\xi)$, are distinct solutions of (2), then there exists a $\lambda(G, \phi, \psi) > 0$ such that $|\phi_t - \psi_t|_B \geqq \lambda(G, \phi, \psi)$ for all $t \in R$.

Remark 4. The separation condition on $H^+(\xi, F)$ is weaker condition that Amerio's condition[1].

Proposition 2.(cf. Theorem 5 in [8]). Suppose that $H^+(\xi, F)$ satisfies the separation condition. Then $\xi(t)$ is an asymptotically almost periodic of (1).

Proposition 3. The following three propositions are equivalent:

(i) $H^+(\xi, F)$ satisfies the separation condition.

(ii) The solutions in $H^+(\xi)$ are w.u.a.s.$H^+(\xi)$.$(\delta_o, \delta(\cdot))$.

(iii) The solutions in $H^+(\xi)$ are u.a.s.$H^+(\xi)$.$(\delta_o, \delta(\cdot), T(\cdot))$.

Finally, we shall discuss the existence of an almost periodic solution of a linear almost periodic system. Let the space C and B be the same ones given in Examples 1 and 2 in Section 2 and $|\cdot|$ be the Euclidean norm.

Proposition 4. Suppose that $A(t, \phi)$ is continuous in $(t, \phi) \in R \times C$ $(R \times B)$, linear in ϕ and almost periodic in t uniformly for ϕ and that the null solution of the system

(3) $$\dot{x}(t) = A(t, x_t)$$

is uniformly stable.

Then for any almost periodic function f(t), the system

(4) $$\dot{x}(t) = A(t, x_t) + f(t)$$

has an almost periodic solution, whenever it has a bounded solution on I.

We shall prove only the case where the phase space is *C*, because the case where the phase space is *B* can be shown by the completely same way by replacing $|\phi|_* = \{\int_{-\infty}^{0} |\phi(\theta)|^2 e^{2\gamma\theta} d\theta\}^{1/2}$ by

$$|\phi|_* = \begin{cases} \{|\phi(0)|^2 + \int_{-\infty}^{0} |\phi(\theta)|^2 g(\theta) d\theta\}^{1/2}, & \text{if } r = 0, \\ \{\int_{-r}^{0} |\phi(\theta)|^2 d\theta + \int_{-\infty}^{0} |\phi(\theta)|^2 g(\theta) d\theta\}^{1/2}, & \text{if } r > 0, \end{cases}$$

in the following proof.

Proof of Proposition 4. Since $A(t, \phi)$ is a linear almost periodic function and f(t) is almost periodic, there exists a solution p(t) of (4) which is defined and bounded on R (shortly, R-bounded) and satisfies $\sup_{t\in R} |p_t|_* = \Lambda(A+f)$, where

$$\Lambda(A+f) = \inf\{\sup_{t\varepsilon R} |x_t|_*: x(t) \text{ is an R-bounded solution of } (4)\}.$$

For detailes, see [9].

We shall show that the solutions in $H^+(p)$ are g.w.u.a.s.$H^+(p).\delta(\cdot)$. Then system (4) has an almost periodic solution by Proposition 1, 2 and 3.

Since p(t) is R-bounded, there exists a $\beta > 0$ such that $|p_t| \leqq \beta$ for all $t \varepsilon R$. Let $B(t, \phi) + h(t)$ be in $H^+(A+f)$ and assume that q^1 and q^2 are in $H^+_{B+h}(p)$ and $|q^1_{t_o} - q^2_{t_o}|_C < \delta(\varepsilon/4)$ for some $t_o \varepsilon R$ and ε, $0 < \varepsilon < \beta$, where $\delta(\cdot)$ is the one given for uniform stability of the null solution of (3). Put $z(t) = (q^1(t) - q^2(t))/2$, then z(t) is the solution of

$$(5) \qquad \dot{x}(t) = B(t, x_t).$$

Since every solution of (5) is unique for initial value problem (cf. see Theorem 2.2 in [6]), we have

$$|z_t|_C < \varepsilon/2 \quad \text{for all } t \geqq t_o,$$

which implies that the solutions in $H^+(p)$ are u.s.$H^+(p).\delta(\cdot)$.

Hence $|z_t|_C \to 0$ as $t \to \infty$ or there exists a $\alpha > 0$ such that $|z_t|_C \geqq \alpha$ for all $t \varepsilon R$. Since $|\cdot|$ is the Euclidean norm, we have

$$\{|q^1_t|^2_* + |q^2_t|^2_*\}/2 = |y_t|^2_* + |z_t|^2_*,$$

where $y(t) = \{q^1(t) + q^2(t)\}/2$, which implies $\inf_{t\varepsilon R} |z_t|_* = 0$, because $q^1(t)$ and $q^2(t)$ are solutions which satisfy

$$\sup_{t\varepsilon R} |q^1|_* = \sup_{t\varepsilon R} |q^2|_* = \Lambda(A+f) = \Lambda(B+h).$$

We can show that $\inf_{t\varepsilon R} |z_t|_* = 0$ implies $\inf_{t\varepsilon R} |z_t|_{\mathcal{C}} = 0$ (see, [9]). Therefore $|q^1{}_t - q^2{}_t|_{\mathcal{C}} \to 0$ as $t \to \infty$. Thus the solutions in $H^+(p)$ are g.w.u.a.s.$H^+(p).\delta(\cdot)$. This completes the proof.

Remark 5. For ordinary differential equations, Nakajima[13] has shown that if the conditions in Proposition 4 hold, then system (4) satisfies Favard's separation condition.

References

[1] L.Amerio, Soluzioni quasi-periodiche, o limitate, di sistemi differenziali non lineari quasi-periodici, o limitati, Ann. Mat. Pura. Appl., 39(1955), 97-119.

[2] B.D.Coleman and V.J.Mizel, On the stability of solutions of functional differential equations, Arch. Rational Mech. Anal., 30(1968), 174-196.

[3] L.G.Deysach and G.R.Sell, On the existence of almost periodic motions, Michigan Math. J., 12(1965), 87-95.

[4] J.K.Hale, Dynamical systems and stability, J. Math. Anal. Appl., 26(1969), 39-59.

[5] _______ , Functional differential equations with infinite delays, J. Math. Anal. Appl., 48(1974), 276-283.

[6] J.K.Hale and J.Kato, Phase space for retarded equations with infinite delay, to appear.

[7] Y.Hino, Asymptotic behavior of solutions of some functional differential equations, Tôhoku Math. J., 22(1970), 98-108.

[8] _____ , Stability and existence of almost periodic solutions of some functional differential equations, Tôhoku Math. J., 28(1976), 389-409.

[9] _____ , Favard's separation theorem in functional differential equations with infinite retardations, to appear in Tôhoku Math. J.

[10] J.Kato, Uniformly asymptotic stability and total stability, Tôhoku Math. J., 22(1970), 254-269.

[11] R.K.Miller, Almost periodic differential equations as dynamical systems with applications to the existence of a. p. solutions, J. Differential Eqs., 1(1965), 337-345.

[12] T.Naito, On autonomous linear functional differential equations with infinite retardations, J. Differential Eqs., 21(1976), 297-315.

[13] F.Nakajima, Separation conditions and stability properties in almost periodic systems, Tôhoku Math. J., 26(1974), 305-314.

[14] G.Seifert, Almost periodic solutions for almost periodic systems of ordinary differential equations, J. Differential Eqs., 2(1966), 305-319.

[15] ________ , Almost periodic solutions and asymptotic stability, J. Math. Anal. Appl., 21(1968), 136-149.

[16] G.R.Sell, Periodic solutions and asymptotic stability, J.Differential Eqs., 2(1966), 143-157.

[17] ________ , Nonautonomous differential equations and topological dynamics. I, II, Trans. Amer. Math. Soc., 127(1967), 241-262, 263-283.

[18] T.Yoshizawa, Asymptotically almost periodic solutions of an almost periodic system, Funkcial. Ekvac., 12(1969), 23-40.

[19] ________ , Stability theory and the existence of periodic solutions and almost periodic solutions, New York, Springer-Verlag, 1975.