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DYER-LASHOF OPERATIONS FOR CERTAIN INFINITE LOOP SPACES

by

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§1. Results (with notation in §2.)

1. Let $X = BU \times Z$. Consider $H_*(X;Z_p) = Z_p[a_0, a_0^{-1}, a_1, \cdots]$,

$\deg a_i = 2i$, for an arbitrary odd prime p.

$$Q^m(a_k) = \sum C^{k,m}_{n_1,\cdots,n_p} a_{n_1} a_{n_2} \cdots a_{n_p} ,$$

where $n_1 + \cdots + n_p = k + m(p-1)$, $n_1 \geq n_2 \geq \cdots \geq n_p$.

The coefficients $C^{k,m}_{n_1,\cdots,n_p}$ are defined as the (unique) solution of the system of equations:

$$\sum_{\nu=0}^{[k/p]} \binom{k-\nu(p-1)}{\nu} C^{k-\nu(p-1),m+\nu}_{n_1,\cdots,n_p} = (-1)^m B^{k,m}_{n_1,\cdots,n_p}, \quad \cdots (1.1)$$

where

$$B^{k,m}_{n_1,\cdots,n_p} = \sum_{\sigma \in \mathfrak{S}_p/H} A^{k,m}_{n_{\sigma(1)},\cdots,n_{\sigma(p)}},$$

(H is the stabilizer of $\{n_1,\cdots,n_p\}$)

$A^{k,m}_{n_1,\cdots,n_p}$ = the coefficient of $X^{m(p-1)}$ in the polynomial

$$(1+X)^{n_1}(1+2X)^{n_2}\cdots(1+(p-1)X)^{n_{p-1}}.$$

The equation can be solved in some special cases. Set

$$k = k'(p - 1) - \alpha, \quad 0 \leq \alpha < p - 1.$$

i) In case $k' \leq p$: $C^{k,m}_{n_1,\cdots,n_p} = \sum_{j=1}^{k'} (-1)^{m+k'-j} \binom{k'+\alpha-1}{k'-j} B^{j(p-1)-\alpha,m+k'-j}_{n_1,\cdots,n_p}$.

ii) In case $k' = p + 1$: $C^{k,m}_{n_1,\cdots,n_p} = \sum_{j=2}^{k'}$ (same as above).

iii) In case there is a number n such that $\sum_{i=1}^{p} (n_i - n) < p - 1$, $n_i \geq n$,

$$(-1)^m B^{k,m}_{n_1,\cdots,n_p} = \#(\mathfrak{S}_p/H)\binom{n}{m},$$

and hence $C^{k,m}_{n_1,\cdots,n_p} = \#(\mathfrak{S}_p/H)\binom{n-k-1}{n-m}$, applying Th. 25.3 of Adem.

iv) In case there is a number n such that $n_1 = n_2 = \cdots = n_{p-1} = n$, $n_p = n - 1$, $C^{k,m}_{n_1,\cdots,n_p} = \binom{n-k-2}{n-m}$.

As for the remaining generator a_0^{-1}:

$$Q^m(a_0^{-1}) = \sum (-1)^{\lambda} \binom{\lambda}{\lambda_1,\cdots,\lambda_m} (Q^1(a_0))^{\lambda_1} \cdots (Q^m(a_0))^{\lambda_m} a_0^{-p(\lambda+1)},$$

where $\sum i\lambda_i = m$, $\lambda = \sum \lambda_i$.

2. Let $X = K(\pi,n)$: the Eilenberg-MacLane space for an arbitrary finitely generated abelian group π, and $n \geq 1$.

For any prime p, the Dyer-Lashof operations on $H_*(K(\pi,n);Z_p)$ are trivial, except that $Q^0(1) = 1$.

§2. Proofs of 1. : $X = BU \times Z$.

In [1], S. Priddy has computed the Dyer-Lashof operations on $H_*(BU \times Z;Z_2)$. We shall adapt his methods to the mod p case for an arbitrary odd prime p, and obtain some results.

For the necessary background literature on infinite loop spaces and Dyer-Lashof operations, we refer to J. P. May's papers [2], [4].

It is known that

$$H_*(BU \times Z) = Z_p[a_0, a_0^{-1}, a_1, a_2, \cdots]$$

and that the generator a_i comes from the generator e_{2i} of $H_*(BZ_p)$, (Z_p is regarded as the subgroup of U_1, with generator $a = e^{2\pi i/p}$) which is the dual of $y_2^{\,i} \in H^*(BZ_p) = \Lambda(y_1) \otimes Z_p[y_2]$.

According to [1] the diagram

$$\begin{array}{ccccc}
B(Z_p \wr Z_p) & \longrightarrow & B(\mathfrak{S}_p \wr U_1) & \xrightarrow{Bj} & BU_p \\
\| & & \| & & \big\downarrow \\
WZ_p \underset{Z_p}{\times} (BZ_p)^p & \longrightarrow & W\mathfrak{S}_p \underset{\mathfrak{S}_p}{\times} (BU_1)^p & & \\
 & & \big\downarrow & & \\
 & & W\mathfrak{S}_p \underset{\mathfrak{S}_p}{\times} (BU \times Z)^p & \xrightarrow[\theta]{} & BU \times Z
\end{array}$$

is homotopy commutative, where j is the inclusion of the wreath product and θ is the Dyer-Lashof map.

Consider the following commutative diagram of group homomorphisms

$$
\begin{array}{ccccccc}
Z_p \times Z_p & \xrightarrow{1\times\Delta^p} & Z_p \int Z_p & \hookrightarrow & \mathfrak{S}_p \int U_1 & \xhookrightarrow{j} & U_p \\
\downarrow{\scriptstyle 1\times\Delta^p} & & & & & & \downarrow{\scriptstyle P(\ )P^{-1}} \\
Z_p \times (\underbrace{Z_p \times \cdots \times Z_p}_{p}) & \xrightarrow{\varphi} & \underbrace{Z_p \times \cdots \times Z_p}_{p} & \hookrightarrow & \underbrace{U_1 \times \cdots \times U_1}_{p} & \xrightarrow{m} & U_p
\end{array}
$$

where

$$\varphi(a; b_1, \cdots, b_p) = (ab_1, a^2 b_2, \cdots, a^p b_p)$$

(group multiplications of the cyclic group Z_p)

m = juxtaposition of matrices

$$P = \frac{1}{\sqrt{p}} \left(a^{ij} \right)_{\substack{1 \leq i \leq p \\ 1 \leq j \leq p}}, \quad (a = e^{2\pi i/p}),$$

(note that $P^{-1} = \bar{P}$).

We now apply $H_* B(\ )$ to the diagram, and compute the images of the element $e_{2m(p-1)} \otimes e_{2k} \in H_*(BZ_p \times BZ_p)$:

$$e_{2m(p-1)} \otimes e_{2k} \overset{B(1\times\Delta^p)_*}{\underset{(1)}{\longmapsto}} \sum_{\nu} (-1)^{\nu+k} \binom{k-\nu(p-1)}{\nu} e_{2(m+p\nu-k)(p-1)} \otimes e_{2k-2\nu(p-1)}^{\ p}$$

$$\sum_{\nu} (-1)^{\nu+k} \binom{k-\nu(p-1)}{\nu} e_{2(m+p\nu-k)(p-1)} \otimes e_{2k-2\nu(p-1)}^{\ p} \overset{Bj_*}{\underset{(2)}{\longmapsto}} (-1)^m \sum_{\nu} \binom{k-\nu(p-1)}{\nu} Q^{m+\nu}(a_{k-\nu(p-1)})$$

$$e_{2m(p-1)} \otimes e_{2k} \overset{B(1\times\Delta^p)_*}{\underset{(3)}{\longmapsto}} \sum_{i_1+\cdots+i_p=k} e_{2m(p-1)} \otimes e_{2i_1} \otimes \cdots \otimes e_{2i_p}$$

$$\sum_{i_1+\cdots+i_p=k} e_{2m(p-1)} \otimes e_{2i_1} \otimes \cdots \otimes e_{2i_p} \overset{m_*\varphi_*}{\underset{(4)}{\longmapsto}} \sum_{i_1+\cdots+i_p=k} \ \sum_{j_1+\cdots+j_{p-1}=m(p-1)} 2^{j_2} \cdots (p-1)^{j_{p-1}} \binom{j_1+i_1}{i_1} \cdots \binom{j_{p-1}+i_{p-1}}{i_{p-1}} a_{j_1+i_1} \cdots a_{j_{p-1}+i_{p-1}} a_{i_p}$$

$$(-1)^m \sum_{\nu} \binom{k-\nu(p-1)}{\nu} Q^{m+\nu}(a_{k-\nu(p-1)}) \overset{\text{id.}}{\longmapsto} \sum_{i_1+\cdots+i_p=k} \ \sum_{j_1+\cdots+j_{p-1}=m(p-1)} 2^{j_2} \cdots (p-1)^{j_{p-1}} \binom{j_1+i_1}{i_1} \cdots \binom{j_{p-1}+i_{p-1}}{i_{p-1}} a_{j_1+i_1} \cdots a_{j_{p-1}+i_{p-1}} a_{i_p}$$

(proofs): (1) is referred to Lemma 4.6 of [3].

(2) is obtained from the definition of Q^m. (see [3].)

(3) and (4) are proved by investigating the duality. In fact,

$\langle y_1^{\varepsilon_1} y_2^{i_1} \otimes y_1^{\varepsilon_2} y_2^{i_2}, \Delta_*(e_{2k})\rangle = \langle y_1^{\varepsilon_1+\varepsilon_2} y_2^{i_1+i_2}, e_{2k}\rangle$

$= 1$ iff $\varepsilon_1 = \varepsilon_2 = 0$, $i_1 + i_2 = k$,

and hence $\Delta_*(e_{2k}) = \sum_{i_1+i_2=k} e_{2i_1} \otimes e_{2i_2}$;

$\langle y_2^{k_1+k_2}, \mu_*(e_{2k_1} \otimes e_{2k_2})\rangle$

$= \langle (y_2\otimes 1 + \lambda y_1\otimes y_1 + 1\otimes y_2)^{k_1+k_2}, e_{2k_1} \otimes e_{2k_2}\rangle = \binom{k_1+k_2}{k_1}$,

and hence $\mu_*(e_{2k_1} \otimes e_{2k_2}) = \binom{k_1+k_2}{k_1} e_{2k_1+2k_2}$.

Note that $\varphi = (\varphi_1 \times \cdots \times \varphi_p)\circ T \circ (\Delta^p \times 1)$,

where T is the shuffling map and $\varphi_\nu = \mu\circ(s_\nu \times 1)$ with $s_\nu = \mu^\nu \circ \Delta^\nu : a \mapsto a^\nu$.

We see that

$$\begin{aligned} s_{\nu *}(e_{2j}) &= (\mu^\nu)_* \sum_{k_1+\cdots+k_\nu=j} e_{2k_1}\otimes\cdots\otimes e_{2k_\nu} \\ &= \sum_{k_1+\cdots+k_\nu=j} \binom{k_1+\cdots+k_\nu}{k_1,\ldots,k_\nu} e_{2j} \\ &= (\underbrace{1+\cdots+1}_{\nu})^j e_{2j} = \nu^j e_{2j}, \end{aligned}$$

and hence $\varphi_*(e_{2n}\otimes e_{2i_1}\otimes\cdots\otimes e_{2i_p})$

$$= \sum_{j_1+\cdots+j_p=n} 1^{j_1} 2^{j_2} \cdots p^{j_p} \binom{j_1+i_1}{i_1} \cdots \binom{j_p+i_p}{i_p} e_{2j_1+2i_1}\otimes\cdots\otimes e_{2j_p+2i_p}.$$

From the preceding diagram we have

$$\sum_{\nu} \binom{k-\nu(p-1)}{\nu} Q^{m+\nu}(a_{k-\nu(p-1)}) = (-1)^m \sum B^{k,m}_{n_1,\cdots,n_p} a_{n_1} \cdots a_{n_p},$$

and (1.1) follows.

Note that the solution of the equation (1.1) is determined uniquely, since the left hand side is of the form $C^{k,m}$ + (k: lower), and the right hand side is known.

(1.1) can be solved only in some special cases.

First, we get i) and ii), using

<u>Lemma</u>. If $i < p$ and $0 \leqslant \alpha < p-1$, then

$$\binom{j(p-1)-\alpha}{i} = (-1)^i \, {}_{i+1}H_{j+\alpha-1} = (-1)^i \binom{i+j+\alpha-1}{j+\alpha-1} \quad \text{mod } p.$$

(this is proved by considering the power series $(1+X)^{jp-(j+\alpha)}$)

and the fact that

$$c_k = \sum_{j=1}^{k} \binom{k+\alpha-1}{k-j} a_j \quad \text{is the solution of} \quad \sum_{i=0}^{k-1} (-1)^i \binom{k+\alpha-1}{i} c_{k-i} = a_k.$$

Secondly, in view of $(1+X)\cdots(1+(p-1)X) = 1 - X^{p-1}$ mod p, $A^{k,m}$ are expressed by some binomial coefficients in cases iii) and iv), and in these cases $C^{k,m}$ are computed by using some formulas of binomial coefficients. Th. 25.3 of Adem is referred to [1].

Computation of $Q^m(a_0^{-1})$ is same as in [1]. In fact, we get from the Cartan formula in $H_*(QS^0)$,

$$Q^n([-1]) = \sum_{\Sigma i\lambda_i=n} (-1)^{\lambda} \binom{\lambda}{\lambda_1,\ldots,\lambda_n} (Q^1[1])^{\lambda_1} \cdots (Q^n[1])^{\lambda_n} [-1]^{p(\lambda+1)}$$

in $H_*(QS^0)$, and the result follows.

Note that

$$Q^m(a_0) = (-1)^m \sum_{n_1+\cdots+n_{p-1}=m(p-1)} 2^{n_2} \cdots (p-1)^{n_{p-1}} a_{n_1} \cdots a_{n_{p-1}} a_0,$$

by i).

§3. Proof of 2. : $X = K(\pi,n)$.

Remark 3.1. The Dyer-Lashof map

$$\theta: \ WZ_p \underset{Z_p}{\times} (K(\pi,n))^p \longrightarrow K(\pi,n)$$

always represents an element of $H^n(WZ_p \underset{Z_p}{\times} (K(\pi,n))^p;\pi)$.

On the other hand, the properties of θ:

i) $\theta\big|_{(WZ_p)^0 \underset{Z_p}{\times} X^p}: \ X^p \longrightarrow X$ is equal to the p-ple product map of the H-space X (see [6]),

ii) $\theta_*(e_m \otimes \underbrace{1 \otimes \cdots \otimes 1}_{p}) = 0$ in $H_m(X;Z_p)$, for any $m \geq 1$ (see [2]),

show that

$$\theta_*: \ H_i(WZ_p \underset{Z_p}{\times} (K(\pi,n))^p;Z) \longrightarrow H_i(K(\pi,n);Z)$$

is uniquely determined for $i = n$.

Hence the universal coefficient theorem shows that θ is unique as an element of $H^n(WZ_p \underset{Z_p}{\times} (K(\pi,n))^p;\pi)$.

Since θ is unique, the Dyer-Lashof operations on the homology of $K(\pi+\pi',n) = K(\pi,n) \times K(\pi',n)$ can be regarded as the tensor product of those on the direct summands $H_*(K(\pi,n))$, $H_*(K(\pi',n))$.

So it suffices to consider the Dyer-Lashof operations on

$H_*(K(Z_{p^h},n);Z_p)$ and $H_*(K(Z,n);Z_p)$ (p: prime, $h,n \geqslant 1$).

<u>Lemma 3.2</u>. Let $H^* = \Lambda(x_\alpha;\alpha\in A) \otimes Z_p[y_\beta;\beta\in B]$ be an algebra over Z_p, of locally finite type, and with product map Δ^*. Then in its dual coalgebra (H_*,Δ_*), an element is primitive (i.e., $\Delta_*(z) = z\otimes 1 + 1\otimes z$) if and only if it is a linear combination of elements $\{(x_\alpha)^*, (y_\beta)^*\}$.

<u>Proof</u>. Let $s\binom{i_1}{\alpha_1}\binom{i_2}{\alpha_2}\cdots\binom{j_1}{\beta_1}\binom{j_2}{\beta_2}\cdots$ be the dual basis of $x_{\alpha_1}^{i_1} x_{\alpha_2}^{i_2}\cdots y_{\beta_1}^{j_1} y_{\beta_2}^{j_2}\cdots$. Then

$$\Delta_*\left(s\binom{i_1}{\alpha_1}\cdots\binom{j_1}{\beta_1}\cdots\right) = \sum_{\substack{i_1'+i_1''=i_1 \\ \vdots \\ j_1'+j_1''=j_1 \\ \vdots}} s\binom{i_1'}{\alpha_1}\cdots\binom{j_1'}{\beta_1}\cdots \otimes s\binom{i_1''}{\alpha_1}\cdots\binom{j_1''}{\beta_1}\cdots,$$

and the primitive elements are determined as above.

We shall use the Cartan formula for the coproduct Δ_*:

$$\Delta_* Q^r(x) = \sum_i \sum Q^i(x') \otimes Q^{r-i}(x'') \qquad (\text{if } \Delta_*(x) = \sum x' \otimes x'') \cdots (3.3)$$

and the Nishida relations:

$$Sq_*^s \circ Q^r = \sum_i \binom{r-s}{s-2i} Q^{r-s+i} \circ Sq_*^i \qquad (p = 2) \quad \cdots\cdots (3.4)$$

$$\left.\begin{aligned} \mathcal{P}_*^s \circ Q^r &= \sum_i (-1)^{i+s} \binom{(r-s)(p-1)}{s-pi} Q^{r-s+i} \circ \mathcal{P}_*^i \\ \mathcal{P}_*^s \circ \beta_* \circ Q^r &= \sum_i (-1)^{i+s} \binom{(r-s)(p-1)-1}{s-pi} \beta_* \circ Q^{r-s+i} \circ \mathcal{P}_*^i \\ &\quad + \sum_i (-1)^{i+s} \binom{(r-s)(p-1)-1}{s-pi-1} Q^{r-s+i} \circ \mathcal{P}_*^i \circ \beta_* \end{aligned}\right\} (p: \text{odd}) \cdots (3.5)$$

(see [7]).

As for the structure of the cohomology of $K(\pi,n)$, we refer to [8] and [9]. In any case, Lemma 3.2 always applies to the cohomology group of our consideration.

Proposition 3.6. In $H_*(K(Z_{2^h},n);Z_2)$, $Q^r(x) = 0$, except $Q^0(1) = 1$.

Proof. We shall prove this by induction on the degree.

Take a homogeneous element x. By the induction hypothesis,

$$\|y\| < \|x\| \Rightarrow Q^r(y) = 0 \text{ for any } r, \text{ except that } Q^0(1) = 1.$$

Then the Cartan formula (3.3) implies that $\Delta_* Q^r(x) = Q^r(x) \otimes 1 + 1 \otimes Q^r(x)$, i.e., $Q^r(x)$ is primitive.

Hence by Lemma 3.2, it suffices to show that $\langle Sq_h^I(\iota_n), Q^r(x)\rangle = 0$, where $d(I) + n = r + \|x\|$. (see [8].)

Write $I = (i_1,\cdots,i_t)$. If $i_t > 1$, put $I' = I$ and $v = \iota_n$, then $Sq_h^I = Sq^{I'}$. If $i_t = 1$, put $I' = (i_1,\cdots,i_{t-1})$ and $v = \beta_h(\tilde{\iota}_n)$, then $Sq_h^I(\iota_n) = Sq^{I'}(v)$. ($\tilde{\iota}_n$ is the fundamental class in $H^n(K(Z_{2^h},n);Z_{2^h})$ and β_h is the Bockstein homomorphism.)

Note that $\langle Sq_h^I(\iota_n), Q^r(x)\rangle = \langle v, Sq_*^{I'}\circ Q^r(x)\rangle$.

Now we use the Nishida relation (3.4). By the induction hypothesis, $Q^{r-s+i}(Sq_*^i(x)) = 0$ if $i > 0$. Hence

$$Sq_*^{I'}\circ Q^r(x) = \binom{r-i_1}{i_1}\binom{r-i_1-i_2}{i_2}\cdots\binom{r-d(I')}{i_t \text{ or } i_{t-1}}Q^{r-d(I')}(x).$$

Recall $r-d(I') = n-\|x\|+(d(I)-d(I')) \leq n+1-\|x\|$.

Since $\|x\| \geq n$ and $Q^r(x) = 0$ if $r < \|x\|$,

$$Sq_*^{I'}\circ Q^r(x) = 0 \quad \text{unless} \quad \|x\| = n = 1,\ r-d(I') = i_t = 1.$$

Hence it suffices to show that

$$\langle \beta_h(\tilde{\iota}_1), Q^1(u_1)\rangle = 0. \qquad (u_1 \text{ is the fundamental homology class.})$$

In fact, $\langle \beta_h(\tilde{\iota}_1), Q^1(u_1)\rangle = \langle \beta_h(\tilde{\iota}_1), u_1^2\rangle = \langle \mu^*(\beta_h(\tilde{\iota}_1)), u_1\otimes u_1\rangle$

$$= \langle (\beta_h\otimes\rho + \rho\otimes\beta_h)\circ\mu^*(\tilde{\iota}_1), u_1\otimes u_1\rangle = \langle \beta_h(\tilde{\iota}_1)\otimes 1 + 1\otimes\beta_h(\tilde{\iota}_1), u_1\otimes u_1\rangle$$

$= 0$, by virtue of Lemma 3.7 below.

Thus we have shown $Q^r(x) = 0$, and the proposition is proved by induction.

Lemma 3.7. The diagram

$$\begin{array}{ccc}
H^1(X;Z_{2^h}) & \xrightarrow{\beta_h} & H^2(X;Z_2) \\
\mu^* \downarrow & & \downarrow \mu^* \\
H^1(X\times X;Z_{2^h}) & \xrightarrow{\beta_h} & H^2(X\times X;Z_2) \\
\kappa \uparrow & & \uparrow \kappa \\
\sum_{i_1+i_2=1} H^{i_1}(X;Z_{2^h})\otimes H^{i_2}(X;Z_{2^h}) & \xrightarrow[\beta_h\otimes\rho+\rho\otimes\beta_h]{} & \sum_{j_1+j_2=2} H^{j_1}(X;Z_2)\otimes H^{j_2}(X;Z_2)
\end{array}$$

is commutative, where ρ = reduction mod 2, β_h = Bockstein homomorphism, κ = the cross product.

Proof. This follows from the commutativity of the following diagrams:

$$\begin{array}{ccccc}
C^*(X;Z_2)\otimes C^*(X;Z_{2^{h+1}}) & \xrightarrow{1\otimes\rho} & C^*(X;Z_2)\otimes C^*(X;Z_2) & \xrightarrow[\kappa]{} & C^*(X;Z_2) \\
i_*\otimes 1 \downarrow & & & & \downarrow i_* \\
C^*(X;Z_{2^{h+1}})\otimes C^*(X;Z_{2^{h+1}}) & & \xrightarrow[\kappa]{} & & C^*(X;Z_{2^{h+1}})
\end{array}$$

$$\begin{array}{ccc}
C^*(X;Z_{2^{h+1}})\otimes C^*(X;Z_{2^{h+1}}) & \xrightarrow[\kappa]{} & C^*(X;Z_{2^{h+1}}) \\
\delta\otimes 1 + 1\otimes\delta \uparrow & & \uparrow \delta \\
C^*(X;Z_{2^{h+1}})\otimes C^*(X;Z_{2^{h+1}}) & \xrightarrow[\kappa]{} & C^*(X;Z_{2^{h+1}}) \\
j_*\otimes j_* \downarrow & & \downarrow j_* \\
C^*(X;Z_{2^h})\otimes C^*(X;Z_{2^h}) & \xrightarrow[\kappa]{} & C^*(X;Z_{2^h})
\end{array} .$$

Proposition 3.8. In $H_*(K(Z_{p^h},n);Z_p)$, p: odd, $Q^r(x) = 0$, except $Q^0(1) = 1$.

Proof. This is entirely same as 3.6. In fact: it suffices to prove $\langle \mathcal{P}^I_h(\iota_n), Q^r(x)\rangle = 0$, with $d(I) + n = 2r(p-1) + \|x\|$.

Write $I = (\varepsilon_0, i_1, \varepsilon_1, \cdots, i_t, \varepsilon_t)$ and put $I' = I - (\varepsilon_t)$.

The Nishida relation (3.5) and the induction hypothesis show that

$$\mathcal{P}_*^{I'} \circ Q^r(x) = \lambda \beta_*^{\varepsilon_{t-1}} \circ \cdots \circ \beta_*^{\varepsilon_0} \circ Q^{r-i_1-\cdots-i_t}(x), \qquad \lambda \in Z_p.$$

Note that $r-i_1-\cdots-i_t = \frac{n-\|x\|+\Sigma\varepsilon_j}{2(p-1)}$. If $\varepsilon_0+\cdots+\varepsilon_{t-1} > 1$, then $\beta_*^{\varepsilon_{t-1}} \circ \cdots \circ \beta_*^{\varepsilon_0} = 0$. If $\varepsilon_0+\cdots+\varepsilon_{t-1} \leq 1$, then $r-i_1-\cdots-i_t \leq 2/2(p-1)$, and hence ≤ 0, thus $Q^{r-i_1-\cdots-i_t}(x) = 0$.

Thus $\langle \mathcal{P}_h^I(\iota_n), Q^r(x) \rangle$ is always shown to be zero.

Proposition 3.9. In $H_*(K(Z,n);Z_p)$, p: any prime, $Q^r(x) = 0$, except $Q^0(1) = 1$.

Proof. This is now obvious, by the proof of 3.6 and 3.8.

Now we complete the proof 2., combining 3.1 and 3.6 ~ 3.9.

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