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On geometric reductions of homology 3-spheres of genus two

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1. Introduction. Our main concern is the study of homology 3-spheres obtained by Heegaard splittings of genus two. It is showed that Heegaard splittings of genus two are closely related to symmetric planer graphs with four vertices (see Lemma 4). Such planer graphs are said to be Whitehead graphs (or simply W-graphs) for the splittings. Futhermore we can establish procedures of simplifying homology 3-spheres with Heegaard splittings of genus two such that a W-graph of it is type(2) or type(3) and that a presentation of the fundamental group associated with a W-graph of it is Π_1-reducible (see Theorem 1 and Theorem 2).

All spaces and maps considered here are polyhedral. S^n is a n-sphere and D^n is a n-disk. Let $M \subseteqq W$ be manifolds; the interior and boundary of M are denoted int(M), ∂M, respectively; M is properly embedded if $M \cap \partial W = \partial M$; N(M,W) is a regular neighborhood of M in W.

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2. Whitehead graphs of 3-manifolds of genus two.

Let G_i(i=1,2,3) be one of the following planer graphs with the maltiple degree a, a', b, b', c, d of edges;

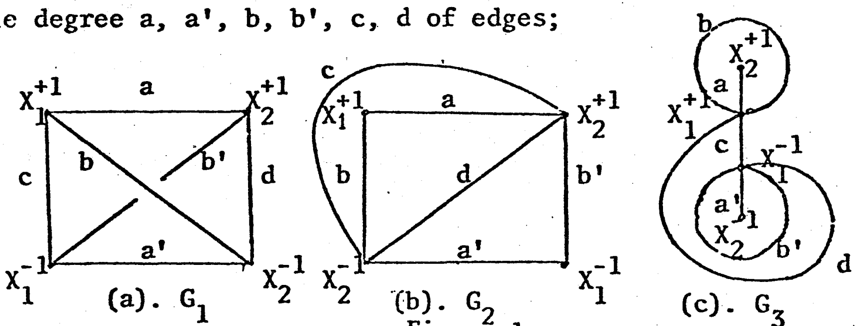

Figure 1.

The graph G_i(i=1,2,3) is said to be symmetric if a = a' and b = b'.

Let M be a closed orientable 3-manifold and $(W_1,W_2; h)$ a Heegaard splitting of genus two for the manifold M. Such the manifold M is said to be a 3-manifold of genus two. Then there is a (matching) homeomorphism $h\colon \partial W_2 \longrightarrow \partial W_1$ of boundaries of solid tori W_1, W_2 of genus two and the manifold M is the identification space $W_1 \underset{h}{\cup} W_2$ by the homeomorphism h. A solid torus of genus two is the result of attaching two disjoint " 1-handles" $D^2 \times [-1,1]$ to a 3-ball B^3 by sewing the parts $D^2 \times \{\pm 1\}$ to 2×2 disjoint 2-disks on ∂B^3 in such a way that the result is an orientable 3-manifold with boundary. In particular, the two properly embedded 2-disks $D^2 \times \{0\}$ in the solid torus are said to be a meridian disk pair for it, and the boundaries of them are a meridian pair.

Let $\{D_{i1}, D_{i2}\}$ be a meridian disk pair of W_i (i=1,2). Then we have;

Lemma 1. The manifold $M = W_1 \underset{h}{\cup} W_2$ is determined up to homeomorphism by the collection of circles v_1 and v_2 on ∂W_1 such that $v_k = h(\partial D_{2k})$ (k=1,2).

Proof. By the definition, $W_2 - \{N(D_{21}, W_2) \cup N(D_{22}, W_2)\}$ is a subset of W_2 such that the closure of it is a 3-ball and so the manifold M is uniquely determined up by the collection.

Now we describe the construction of the Whitehead graph G(h) corresponding to the Heegaard splitting $(W_1, W_2;\ h)$. We cut ∂W_1 along the circles ∂D_{11}, ∂D_{12}. As a result, we obtain a 2-sphere S^2 with four holes; X_1^{+1}, X_2^{+1}, X_1^{-1}, X_2^{-1}. Under this operation, the circles v_1 and v_2 are cut up (we may assume that $v_k \cap (\partial D_{11} \cup \partial D_{12}) \neq \phi$ for k=1,2. If otherwise, the splitting $(W_1, W_2;\ h)$ is equivalent to the Heegaard splitting of genus two for $(S^1 \times S^2)$ # M' or $S^3 \times M'$ where M' is a 3-manifold of genus one and # is a connected sum (see Waldhausen [3]).), and they turn into a collection of segments joining (in some order or another) the holes in the sphere S^2. Let us suppose that these holes, X_1^{+1}, X_2^{+1}, X_1^{-1}, X_2^{-1} are the vertices, and the segments of the circles the edges, of a graph. So we obtain a W-graph G(h) realized on the sphere S^2. Similarly we can obtain a W-graph $G(h^{-1})$ which is called the conjugate W-graph of the graph G(h). Note that W-graphs are not simple, that is, multi-graphs, and are associated with the meridian disk pairs $\{D_{11},D_{12}\}$, $\{D_{21},D_{22}\}$. Then we have;

Lemma 2. Let G(h) be an arbitrary W-graph of a Heegaard splitting $(W_1,W_2;\ h)$. Then two cases happen; (1). If G(h) is disconnected, then the Heegaard splitting is splitted into the connected sum of Heegaard splittings of genus one. (2). If G(h) is connected, then it is isomorphic to one of the three symmetric graphs G_1, G_2, G_3 above defined.

Proof. Let $\{D_{i1},D_{i2}\}$ be a meridian disk pair of W_i (i=1,2). At first, we cancel trivial loop-edges of G(h). Let E^o be a trivial loop-edge, that is, E^o bounds a 2-disk in the 2-sphere S^2 with four holes X_1^{+1}, X_2^{+1}, X_1^{-1}, X_2^{-1} such that the interior of the disk does not contain any other edges. Then E^o is cancelled by an isotopy in ∂W_1 and so we may assume that G(h) is a graph without any trivial loop-edges. Hence two cases happen; Case(1). G(h) is a planer graph with four vertices and disconnected; Then there is a properly embedded 2-disk D^2 in W_1 such that $\partial D^2 \cap (\partial D_{11} \cup \partial D_{12}) = \phi$, $\partial D^2 \cap G(h) = \phi$ and ∂D^2 is not homotopic to zero in ∂W_1. Suppose that ∂D^2 separate ∂W_1 into two components (if otherwise, the circles $h(\partial D_{21})$, $h(\partial D_{22})$ are contained in a torus $S^1 \times S^1$ with one hole, which is contained in ∂W_1, and they are mutually parallel in ∂W_1 but it is impossible.) Then by $\partial D^2 \cap G(h) = \phi$, ∂D^2 bounds a 2-disk in W_2 and so the splitting $(W_1,W_2;\ h)$ is splitted into a connected sum of Heegaard splittings of genus one. Case(2). G(h) is a connected planer graph with four vertices; The solid torus W_1 is obtained by sewing X_i^{+1} (i=1,2) to X_i^{-1} (i=1,2) and so this operation induces the symmetry of the graph G(h). Hence G(h) is isomorphic to one of the three graphs G_1, G_2, G_3 with symmetry. The proof is complete.

Hereafter all W-graphs considered in this paper are connected, if otherwise specified, and a W-graph is said to be type(i) if it is isomorphic to the graph G_i (i=1,2,3). Futhermore we define the complexity $C\{G(h)\}$ of a W-graph $G(h)$ to be the total sum of degree of all edges in $G(h)$, that is, $C\{G(h)\} = 2a + 2b + c + d$. Then we have;

Lemma 3. Let $G(h)$ be an arbitrary W-graph of a Heegaard splitting $(W_1,W_2;\ h)$ of genus two and $G(h^{-1})$ the conjugate W-graph of $G(h)$. Then $C\{G(h)\} = C\{G(h^{-1})\}$.

Proof. The proof is directly from the definition of $G(h^{-1})$.

3. Presentations for $\Pi_1(M)$ associated with W-graphs.

Let M be a 3-manifold of genus two with a Heegaard splitting $(W_1,W_2;\ h)$ and $\{D_{i1},D_{i2}\}$ a meridian disk pair of W_i and $G(h)$ a W-graph associated with $\{D_{i1},D_{i2}\}$. Futhermore let $v_i = h(\partial D_{1i}\}$ and $w_j = h^{-1}(\partial D_{2j}\}$ (i=1,2; j=1,2). By Lemma 1, the fundamental group $\Pi_1(M)$ is determined by (oriented) circles v_1, v_2 or w_1, w_2; Choose a base point $z \in \partial W_1$(or ∂W_2) for $\Pi_1(W_1)$ (or $\Pi_1(W_2)$) and $\Pi_1(M)$. Futhermore choose the canonical generators A, B associated with $\{D_{11},D_{12}\}$ for the free group $\Pi_1(W_1)$ (or B, C associated with $\{D_{21},D_{22}\}$ for the free group $\Pi_1(W_2)$), that is, A (or B) is a homotopy class in $\Pi_1(W_1)$ represented by an (oriented) circle which transversely intersects D_{11}(or D_{12}) at only one point and is disjoint from D_{21}(or D_{11}), and C (or D) is a homotopy class in $\Pi_1(W_2)$ represented by an (oriented) circle which transversely intersects D_{21} (or D_{22}) at only one point and is disjoint from D_{22}(or D_{21}). Then we have;

Lemma 4. $\Pi_1(M) = \{A,\ B;\ v_1'(A,B) = v_2'(A,B) = 1\}$

$$= \{C,\ D;\ w_1'(C,D) = w_2'(C,D) = 1\}$$

where $v_i'(A,B)$ and $w_j'(C,D)$ (i=1,2;j=1,2) are determined by circles v_i and w_j respectively when orientations of circles v_i, w_j are fixed and oriented arcs, which join the base point z to the point in the circle v_i or w_j, are sellected.

Note that the oriented arcs, which induce $v_1'(A,B)$ or $v_2'(A,B)$ (or, $w_1'(C,D)$ or $w_2'(C,D)$), can be sellected in such a way that they are disjoint from $D_{11} \cup D_{12}$ (or $D_{21} \cup D_{22}$) and so $v_i'(A,B)$ (or $w_j'(C,D)$) is characterized by a double sequence of (A, ε_k) or (B, ε_k) (or, (C, ε_k) or (D, ε_k)) such that ε_k is the intersection number between v_i and $D_{11} \cup D_{12}$ (or w_j and $D_{21} \cup D_{22}$) and the sequence preserves the order induced from the orientation of v_i (or w_j) given in Lemma 4.

Let $\{\alpha, \beta; v_1'(\alpha,\beta) = v_2'(\alpha,\beta) = 1\}$, $\{\tilde{\alpha}, \tilde{\beta}; v_1''(\tilde{\alpha},\tilde{\beta}) = v_2''(\tilde{\alpha},\tilde{\beta}) = 1\}$ be two presentations for $\Pi_1(M)$ associated with $\{D_{i1}, D_{i2}\}$ $(i=1,2)$ given by Lemma 4. Then they are said to be simple equivalent if $\alpha = \tilde{\alpha} = A$ and $\beta = \tilde{\beta} = B$, or $\alpha = \tilde{\alpha} = C$ and $\beta = \tilde{\beta} = D$.

The presentation $\{\alpha, \beta; v_1'(\alpha,\beta) = v_2'(\alpha,\beta) = 1\}$ is Π_1-reducible iff there is a presentation $\{\alpha, \beta; \tilde{v}_1(\alpha,\beta) = \tilde{v}_2(\alpha,\beta) = 1\}$, which is simple equivalent to it, such that in the class of words $\tilde{v}_1(\alpha,\beta)$, $\tilde{v}_2(\alpha,\beta)$ the one is contained in the other as a subword.

Let $P(\alpha,\beta)$ be a coefficient matrix of two linear equations $\bar{v}_1(\alpha,\beta)$, $\bar{v}_2(\alpha,\beta)$ which are obtained from the abelianizations of $v_1'(\alpha,\beta)$, $v_2'(\alpha,\beta)$. Then we have;

Lemma 5. The manifold M is a homology 3-sphere iff the determinant of the 2×2 matrix $P(\alpha,\beta)$ is ± 1.

Proof. Let $H_1(M)$ be the first homology group of M. Then we have that $H_1(M) = \{\alpha,\beta; \bar{v}_1(\alpha,\beta) = \bar{v}_2(\alpha,\beta) = 0\}$ and so the lemma is valid.

4. Geometrically reducible.

Let G(h) be a W-graph associated with a meridian disk pair $\{D_{i1}, D_{i2}\}$ of a Heegaard splitting $(W_1, W_2;\ h)$. Then the W-graph G(h) is geomtrically reducible iff there is a W-graph $G'(\tilde{h})$ of the splitting such that $C\{G'(\tilde{h})\} < C\{G(h)\}$ and $\tilde{h}$ is either of h or h^{-1}. Then we have;

Lemma 6. Let G(h) be a W-graph of a Heegaard splitting $(W_1, W_2;\ h)$.

If the W-graph $G(h)$ is type(3), then it is geometrically reducible.

Proof. Let $G(h)$ be a W-graph of type(3) associated with a meridian disk pair $\{D_{i1},D_{i2}\}$ and S^2 the 2-sphere with four holes X_1^{+1}, X_2^{+1}, X_1^{-1}, X_2^{-1} and we may assume that X_i^{+1} , X_i^{+1} are obtained from cutting ∂W_1 along the circle ∂D_{1i} (i=1,2). Then there is a properly embedded 2-disk D_{13} in W_1 such that D_{13} is disjoint from $D_{11} \cup D_{12}$ and all of edges (X_1^{+1},X_1^{+1}), (X_1^{-1},X_1^{-1}), that is loops, (X_1^{+1},X_2^{+1}), (X_1^{-1},X_2^{-1})(see Figure 1(c)) and that ∂D_{13} is not homologous to zero in ∂W_1 and transversely intersects each of edges (X_1^{+1},X_1^{-1}) at only one point. Thus there is a W-graph $G'(h)$ associated with a meridian disk pair $\{D_{12},D_{13}\}$ such that $2a + c + d = C\{G'(h)\} < C\{G(h)\} = 2a + 2b + c + d$ (see Figure 1(c)). The proof is complete.

Futhermore we have;

Proposition 1. If the W-graph $G(h)$ is type(2), then it is geometrically reducible or there is a W-graph $G'(h)$ of type(1) with $C\{G'(h)\} = C\{G(h)\}$, presupposed that the splitting gives a homology 3-sphere.

Proof. Similarly in Lemma 5, there is a properly embedded 2-disk D_{13} in W_1 such that D_{13} is disjoint from $D_{11} \cup D_{12}$ and all of edges (X_1^{+1},X_2^{+1}), (X_1^{-1},X_2^{-1})(see Figure 1(b)) and that ∂D_{13} is not homologous to zero in ∂W_1 and transversely intersects each of edges (X_1^{+1},X_2^{-1}), (X_2^{+1},X_1^{-1}), (X_2^{+1},X_2^{-1}) (note that these edges consist two kinds of edges) at only one point. We may assume that $b \leqq a$ by the symmetry of $G(h)$. Then two cases happen; Case(1) $b < a$. In this case, there is a W-graph $G'(h)$ associated with $\{D_{11},D_{13}\}$ such that $C\{G'(h)\}(= a + b + 2b + c + d) < C\{G(h)\}(= 2a + 2b + c + d)$. Hence the W-graph $G(h)$ is geometrically reducible. Case(2) $a = b$. Let $G(h(\partial D_{2i}))$ be a subgraph of $G(h)$ induced from $h(\partial D_{2i})$(i=1,2). Then $C\{G(h)\} = C\{G(h(\partial D_{21}))\} + C\{G(h(\partial D_{22}))\}$ and let $C\{G(h(\partial D_{2i}))\} = 2a_i+ 2b_i+ c_i+ d_i$ (i=1,2) such that $a = a_1+ a_2$, $b =$

$b_1 + b_2$, $c = c_1 + c_2$, $d = d_1 + d_2$. Suppose that $C\{G(h(\partial D_{21}))\} \leqq C\{G(h(\partial D_{22}))\}$. Here if $a_1 = b_1$, then it follows that $a_2 = b_2$ and each of circles $h(\partial D_{21})$, $h(\partial D_{22})$ transversely intersects ∂D_{11} at even points. But it is impossible by Lemma 5. Thus we may assume that $b_1 < a_1$. Similarly in Case(1), there is a W-graph $G'(h)$ associated with $\{D_{11}, D_{13}\}$ such that $C\{G'(h)\}(= 4a + c + d) = C\{G(h)\}(= 4a + c + d)$ and that $C\{G'(h(\partial D_{21}))\}(= a_1 + b_1 + 2b_1 + c_1 + d_1) < C\{G(h(\partial D_{21}))\}(= 2a_1 + 2b_1 + c_1 + d_1) \leqq C\{G(h(\partial D_{22}))\} < C\{G'(h(\partial D_{22}))\}(= a_2 + b_2 + 2b_2 + c_2 + d_2)$. Let the W-graph $G'(h)$ be type(1) and then the lemma is valid. We may assume that $G'(h)$ is not type(1). To apply the above argument to the graph $G'(h)$, a W-graph $G''(h)$ associated with a meridian disk pair in W_1 is obtained such that $C\{G''(h)\} = C\{G'(h)\}$, $C\{G''(h(\partial D_{21}))\} < C\{G'(h(\partial D_{21}))\}$ and $C\{G'(h(\partial D_{22}))\} < C\{G''(h(\partial D_{22}))\}$. Let the W-graph $G''(h)$ be type(1) and if otherwise, the argument goes on and after finite steps a W-graph $\bar{G}(h)$ of type(1) with $C\{\bar{G}(h)\} = C\{G(h)\}$ is obtained because of $C\{G(h)\}$ being finite. The proof is complete.

Let M be a 3-manifold of genus two and G any W-graph. Then we have;

Theorem 1. The manifold M has a W-graph G' of type(1) such that $C\{G'\} \leqq C\{G\}$, if it is a homology 3-sphere.

Proof. It follows directly from Lemma 2, Lemma 6, and Proposition 1.

5. Homology 3-spheres to be Π_1-reducible.

Let $\{A, B;\ v_1'(A,B) = v_2'(A,B) = 1\}$ be an arbitrary presentation for $\Pi_1(M)$ associated with a meridian disk pair $\{D_{i1}, D_{i2}\}(i=1,2)$ (or a W-graph $G(h)$) given by Lemma 4. Then we have;

Theorem 2. If the manifold M is a homology 3-sphere and the presentation $\{A, B;\ v_1'(A,B) = v_2'(A,B) = 1\}$ is Π_1-reducible, then the W-graph $G(h)$ is geometrically reducible.

Proof. The case that $G(h)$ is type(3) is trivial by Lemma 6. It is not known whether the change of W-graphs from type(2) to type(1) in

Proposition 1 preserves the Π_1-reducibility or not, and so the case that $G(h)$ is type(2) has to be proved but the proof is similar with the one in the case that $G(h)$ is type(1). Hence we may assume that $G(h)$ is type(1) and $C\{G(h)\} = 2a + 2b + c + d$ (see Figure 1(a)). Then by Lemma 5, which of c or d is non-zero and so suppose that $c \neq 0$ and $0 < b \leqq a$ by the symmetry of $G(h)$. Let S^2 be the 2-sphere with four holes $X_1^{+1}, X_2^{+1}, X_1^{-1}, X_2^{-1}$ obtained from cutting ∂W_1 along ∂D_{11}, ∂D_{12} and W_1 is obtained from sewing X_i^{+1} to X_i^{-1} by a homeomorphism d_i $(i=1,2)$ of disks. Let $\{A, B; v_1'(A,B) = v_2'(A,B) = 1\}$ be a Π_1-reducible and so suppose that, in the class of words $v_1'(A,B)$, $v_2'(A,B)$, the one is contained in the other as a subword. We may assume that $v_1'(A,B)$ is contained in $v_2'(A,B)$. By trivial observations from Lemma 4, the words $v_1'(A,B)$, $v_2'(A,B)$ are induced from edge sequences in S^2 and let them be $\{\Sigma_\alpha\}$, $\{\Sigma_\beta'\}$, respectively. Then it follows from the last assumption that $\Sigma_1 \equiv \Sigma_1', \ldots, \Sigma_\gamma \equiv \Sigma_\gamma'$ ($\gamma = C\{G(h(\partial D_{21}))\} - 1$, and the symbol $\equiv$ means that $\Sigma_i \in \{\Sigma_\alpha\}$ is parallel to $\Sigma_i' \in \{\Sigma_\beta'\}$, that is; there are m nodes $x_1^{+1}, \ldots, x_m^{+1}$ in ∂X_1^{+1}, m nodes $x_1^{-1}, \ldots, x_m^{-1}$, in ∂X_1^{-1} (m= a+b+c), n nodes $x_{1+m+c}^{+1}, \ldots, x_{n+m+c}^{+1}$ in ∂X_2^{+}, n nodes $x_{1+m+c}^{-1}, \ldots, x_{n+m+c}^{-1}$ in ∂X_2^{-1} (n= a+b+d, c ; a pisitive integer such that $C\{G(h(\partial D_{22}))\} < c$), and then Σ_i is parallel to Σ_i' iff followings hold ; (1) $\Sigma_i = (x_{\alpha(i,1)}^{\varepsilon(i,1)} x_{\alpha(i,2)}^{\varepsilon(i,2)})$ and $\Sigma_i' = (x_{\beta(i,1)}^{\varepsilon(i,1)}, x_{\beta(i,2)}^{\varepsilon(i,2)}) (x_{\alpha(i,j)}^{\varepsilon(i,j)}, x_{\beta(i,j)}^{\varepsilon(i,j)}$; nodes in $\partial X_1^{+1} \cup \partial X_1^{-1} \cup \partial X_2^{+1} \cup \partial X_2^{-1})$ (2) $(c - \alpha(i,1))(c - \beta(i,1)) > 0$ and $(c - \alpha(i,2))(c - \beta(i,2)) > 0$.). Let $\{(\Sigma_i, \Sigma_i')\}_{i=1}^{\gamma}$ be a double sequence of parallel edges and futhermore $[\Sigma_i, \Sigma_i']$ the collection of edges such that it contains Σ_i, Σ_i' and all edges which are parallel to Σ_i between Σ_i and Σ_i'. Let $[\Sigma_i, \Sigma_i']$ be $\{(x_{\alpha(i,1)+k}^{\varepsilon(i,1)}, x_{\alpha(i,2)+k}^{\varepsilon(i,2)})\}_{k=o}^{c_i}$ $(c_i = |\alpha(i,1) - \beta(i,1)| + 1)$. Then two cases happen; Case(1). $c_1 \doteq \quad = c_\gamma = \delta$ and for all i $(i=0,.., \gamma-1)$ nodes $x_{\alpha(i,2)}^{\varepsilon(i,2)}, \ldots, x_{\alpha(i,2)+\delta}^{\varepsilon(i,2)} (= x_{\beta(i,2)}^{\varepsilon(i,2)})$ are identified with nodes $x_{\alpha(i+1,1)}^{\varepsilon(i+1,1)}, \ldots, x_{\alpha(i+1,1)+\delta}^{\varepsilon(i+1,1)} (= x_{\beta(i+1,1)}^{\varepsilon(i+1,1)})$ by the homeomorphism d or d^{-1} respectively. Case(2). The case that Case(1) does not hold.

Case(1). In this case, for all i (i=1,..,γ) $(x^{\varepsilon(i,1)}_{\alpha(i,1)+1}, x^{\varepsilon(i,2)}_{\alpha(i,2)+1}) \in \{\Sigma'_\beta\}$. Then there is a properly embedded 2-disk D_{23} in W_2 such that it is disjoint from $D_{21} \cup D_{22}$ and $D_{21} \cup D_{22} \cup D_{23}$ separates W_2 into two components and $C\{G(h(\partial D_{23}))\} = C\{G(h(\partial D_{22}))\} - C\{G(h(\partial D_{21}))\}$; Suppose that edges $x^{\varepsilon(1,1)}_{\alpha(1,1)}$, $x^{\varepsilon(1,1)}_{\beta(1,1)}$ are contained in ∂X^{+1}_1 or ∂X^{-1}_1 or ∂X^{+1}_2 or ∂X^{-1}_2 and so in ∂X^{+1}_1. Then there is a arc q which joins $x^{\varepsilon(1,1)}_{\alpha(1,1)}$ to $x^{\varepsilon(1,1)}_{\beta(1,1)}$ in $\mathrm{int}(N(\partial X^{+1}_1, S^2))$ and is disjoint from $\partial D_{11} \cup \partial D_{12} \cup h(\partial D_{21}) \cup h(\partial D_{22})$. Let S^1 be a circle in ∂W_1 obtained from the connected sum of circles $h(\partial D_{21})$, $h(\partial D_{22})$ along the arc q. Since the closure of $W_2 - N(D_{21}, W_2) \cup N(D_{22}, W_2)$ is a 3-cell, the circle $h^{-1}(S^1)$ bounds a 2-disk D^2 in W_2. Thus let D_{23} be the 2-disk D^2. Hence there is a W-graph G'(h) associated with a meridian disk pair $\{D_{11}, D_{12}, D_{21}, D_{23}\}$ such that $C\{G'(h)\} < C\{G(h)\}$, and so the lemma is valid.

Case(2). For some k (k=1,..,γ), nodes $x^{\varepsilon(k,2)}_{\alpha(k,2)}, \ldots, x^{\varepsilon(k,2)}_{\alpha(k,2)+c_k}$ are identified with the collection of edges which is obtained from removing nodes $x^{\varepsilon(k+1,1)}_{\alpha(k+1,1)+1}, \ldots, x^{\varepsilon(k+1,1)}_{\alpha(k+1,1)+(c_k-1)}$ from the collection which consists of all nodes in ∂X which contains $x^{\varepsilon(k+1,1)}_{\alpha(k+1,1)}$ $(X = X^{+1}_1, X^{-1}_1, X^{+1}_2, X^{-1}_2)$. The case happens only if followings hold; (1): $\varepsilon(k,1) \times \varepsilon(k,2) > 0$ and $\varepsilon(k+1,1) \times \varepsilon(k+1,2) > 0$ and $\varepsilon(k,1) \times \varepsilon(k+1,1) < 0$ (note that $b \leq a$).

Let's prove the last statement. The case except the one which the edge $(x^{\varepsilon(k,1)}_{\alpha(k,1)}, x^{\varepsilon(k,2)}_{\alpha(k,2)})$ is parallel to the edge $(x^{\varepsilon(k+1,1)}_{\alpha(k+1,1)}, x^{\varepsilon(k+1,2)}_{\alpha(k+1,2)})$ is trivial. And so let them be parallel and let $x^{\varepsilon(k,1)}_{\alpha(k,1)}, x^{\varepsilon(k+1,1)}_{\alpha(k+1,1)} \in \partial X^{+1}_1$ and $x^{\varepsilon(k,2)}_{\alpha(k,2)}, x^{\varepsilon(k+1,2)}_{\alpha(k+1,2)} \in \partial X^{-1}_1$. But this case is also trivial from the observation through Figure 2. Hence the condition (1) holds and then it follows from the one that $b + d < a$ or $b + c < a$. Then there is a W-graph G'(h) with $C\{G'(h)\} < C\{G(h)\}$ (see the proof of the Case(1) in

Proposition 1). The proof of the lemma is complete.

Corollary 1. If the manifold M has a group presentation {A ,B; $A^p \cdot B^q = 1$, $A^s \cdot B^t = 1$, p,q,s,t:non-zero integers} associated with a W-graph and $\Pi_1(M)$ is trivial, then it is a 3-sphere.

Proof. By Lemma 5, the determinant of the matrix P(A,B) is ± 1. We may assume that $0 < p < s$ and $0 < q < t$. Thus we can apply Theorem 2 to the case, and so the lemma is valid.

Note that Corollary 1 is also true in the case when the presentation in Corollary 1 is {A,B; $A^p \cdot B^q \cdot A^s \cdot B^t = 1$, $v'(A,B) = 1$, p,q,s,t:non-zero integers, v'(A,B):an arbitrary relation}(see [1]).

Finally we propose a conjecture associated with Poincare conjecture. Let M be a 3-manifold of genus two, G an arbitrary W-graph of it, $\bar{G}$ the conjugate of G. Then we set;

Conjecture(A): If $\Pi_1(M)$ is trivial, then an arbitrary presentation of $\Pi_1(M)$ associated with G is Π_1-reducible or the one associated with $\bar{G}$ has a reduced part(such as: $A \cdot A^{-1}$, and see [2])(see Algorithm(A) in [2]).

Let G_1 be the symmetric planer graph in Figure 1(a) with parameters a, b, c, d respect of edges. Then we can construct homology 3-spheres through the graph G_1 by the converse operation of Lemma 2 using Lemma 5. The construction, if each of the parameters vary, cover all homology 3-spheres by Theorem 1 and practically give the method to make up homotopy 3-spheres by computer and was carried out on Facom 230-45s over the graph G_1 with limited a, b, c, d. The result of the trial computation are convincing evidence for the truth of Conjecture(A).

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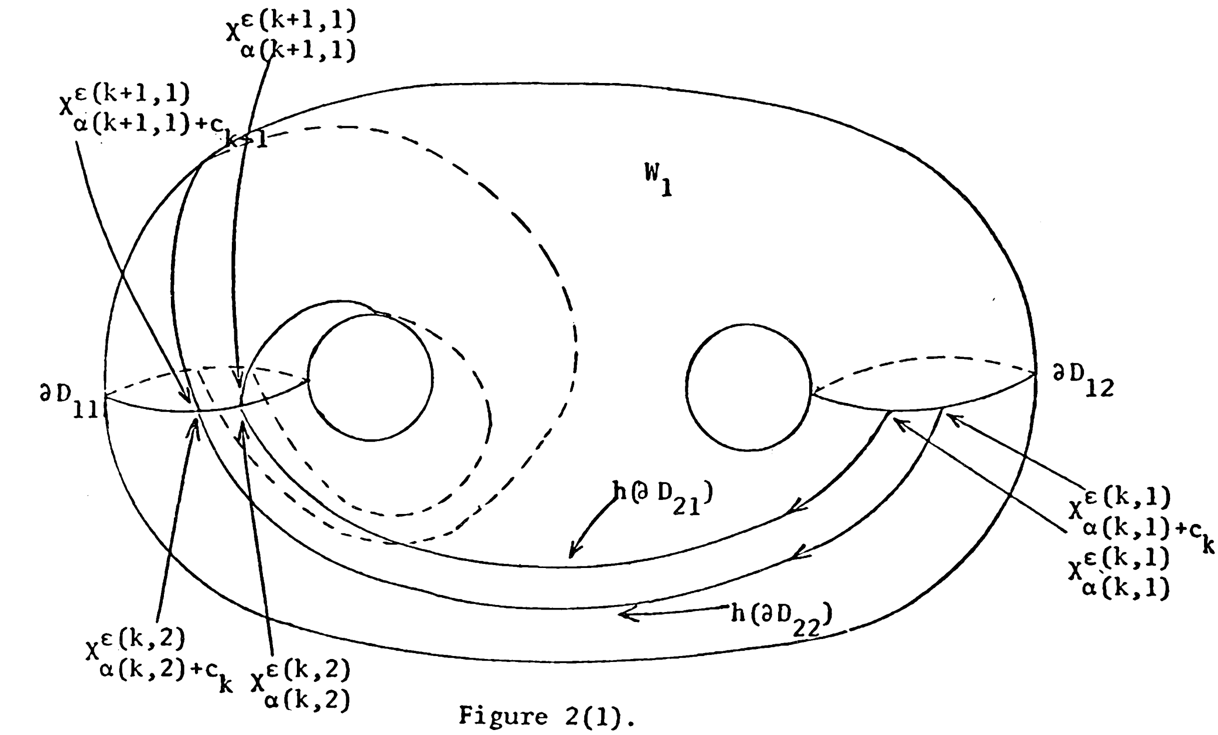

Figure 2(1).

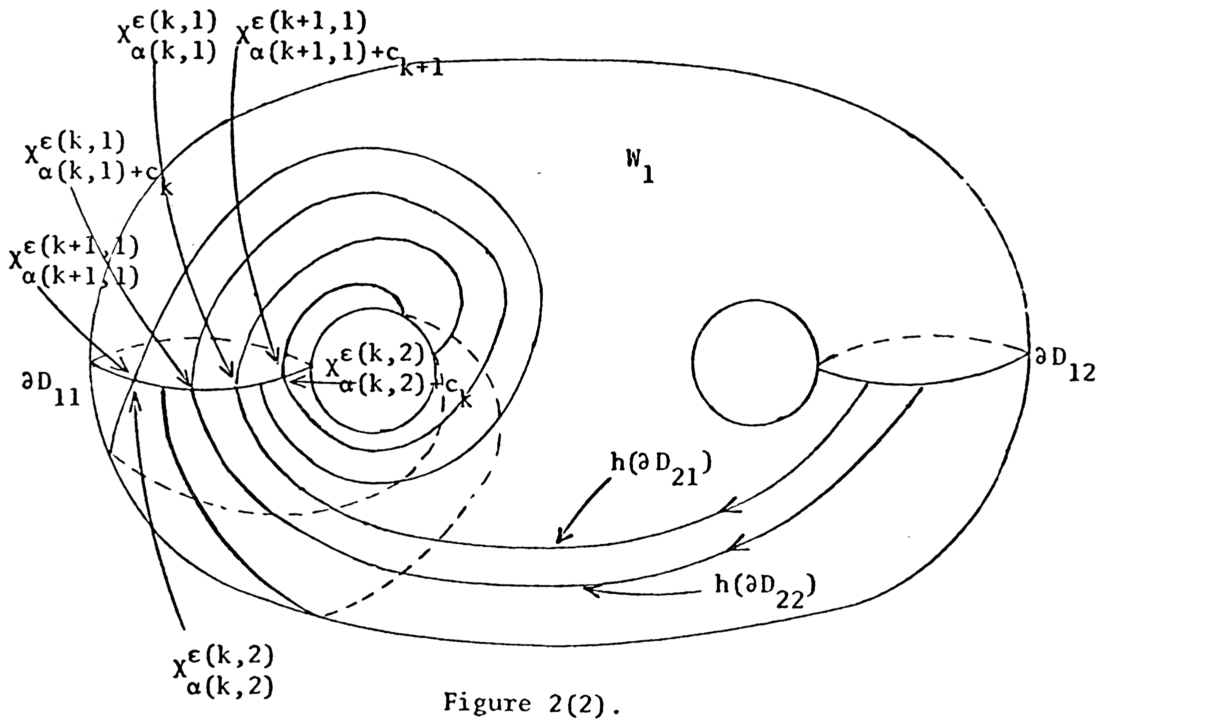

Figure 2(2).