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The generalized least-square estimate of autoregressive coefficients

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The purpose of this note is to derive the asymptotic distribution of the generalized least-square estimate of autoregressive coefficients. Let $\{\varepsilon_t : t = 0, \pm 1, \pm 2, \cdots\}$ be a real-valued stationary Gaussian process with mean 0 and suppose that the spectrum of the process is absolutely continuous with respt to the Lebesgue measure and has the spectral density $f_\varepsilon(\omega)$ $(-\pi \le \omega < \pi)$. Let $\{X_t\}$ be a stationary Gaussian process generated by the equation $X_t - \sum_{j=1}^{p} \alpha_j X_{t-j} = \varepsilon_t$, where the coefficients α_j are such that the zeroes of $z^p - \sum_{j=1}^{p} \alpha_j z^{p-j}$ are all inside the unit circle. Denote by f_X the spectral density of the process $\{X_t\}$ and denote by $f_{X\varepsilon}$ the cospectral density of the bivariate process $\{(X_t, \varepsilon_t)\}$; then it evidently holds that $f_X(\omega) = f_\varepsilon(\omega) / |1 - \sum \alpha_j e^{i\omega j}|^2$ a.e., and $f_{X\varepsilon}(\omega) = (1 - \sum \alpha_j e^{i\omega j}) f_X(\omega)$. For later use, write the covariances $E(X_t X_{t-s})$, $E(X_t \varepsilon_{t-s})$ and $E(\varepsilon_t \varepsilon_{t-s})$ respect

-ively as $\gamma_X(s)$, $\gamma_{X\varepsilon}(s)$ and $\gamma_\varepsilon(s)$.

If the spectral density f_ε is known, an estimate of the α_j can be obtained, based on observations $X_{1-p}, \cdots, X_1, \cdots, X_N$, as the value which minimizes the weighted square integral given as

$$\int_{-\pi}^{\pi} \Big| \sum_{t=1}^{N} \Big(X_t - \sum_{j=1}^{p} \alpha_j X_{t-j} \Big) e^{i\omega t} \Big|^2 / f_\varepsilon(\omega) \, d\omega.$$

Call the estimate $\hat{\alpha}_{j,N}$ thus obtainted the generalized least-square estimate of the α_j. Suppose that the inverse of $f_\varepsilon(\omega)$ is integrable and let $D(k)$ be the k-th Fourier coefficient of $1/f_\varepsilon(\omega)$; namely, $D(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1/f_\varepsilon(\omega)\, d\omega$. Let Γ_N be the p by p matrix whose (k,j) element is $\sum_{t=1}^{N}\sum_{s=1}^{N} X_{t-k} X_{s-j} D(t-s)$ and let γ_N be the p-vector whose k-th component is $\sum_{t=1}^{N}\sum_{s=1}^{N} X_{t-k} X_s D(t-s)$. Denote by α without suffix the p-vector whose k-th component is α_k. Define $\hat{\alpha}_N$ in the same way; that is, the j-th element of $\hat{\alpha}_N$ is $\hat{\alpha}_{j,N}$. Then it holds that $\Gamma_N \hat{\alpha}_N = \gamma_N$ and, if α^0 is the true value of α, $\Gamma_N(\hat{\alpha}_N - \alpha^0) = \eta_N$ where η_N is the p-vector with $\sum\sum X_{t-j}\varepsilon_s D(t-s)$ in the jth element. Let Ω be the p by p matrix whose (k,j) element is $\frac{1}{2\pi}\int_{-\pi}^{\pi} |1 - \sum \alpha_\ell e^{i\omega\ell}|^{-2} e^{i\omega(k-j)} d\omega$; then the following theorem establishes that the generalized least-square estimate $\hat{\alpha}_N$ is asymp-totically normally distributed with covariance matrix Ω^{-1}; namely, the asymptotic covariance of the generalized least-square estimate is the same with that of the least-square estimate applied to the ordinary autoregressive process (that is, $\{\varepsilon_t\}$ is an independent process).

Theorem. Assume that

i) f_ε and $1/f_\varepsilon$ are square-integrable and $1/f_\varepsilon$ has a bounded derivative with respect to ω,

ii) $\sum_{j=0}^{\infty} |\gamma_\varepsilon(j)| < \infty$, and $\sum_{j=0}^{\infty} |D(j)| < \infty$

iii) The process $\{\varepsilon_t\}$ is uniform mixing; namely, if $B(t \leq p)$ and $B(t \geq g)$ are the Borel fields determined by $\{\varepsilon_t ; t \leq p\}$ and $\{\varepsilon_t ; t \geq g\}$ respectively, there exists a sequence of positive numbers g_n such that $g_n \to 0$ as $|n| \to \infty$ and $|P_r(E \cap F) - P_r(E) P_r(F)| < g_{g-p}$, where $E \in B(t \leq p)$ and $F \in B(t \geq g)$. Then under assumptions i), ii), iii), $\sqrt{N}(\hat{\alpha}_N - \alpha^0)$ is asymptotically normally distri-buted with zero mean vector and with covariance matrix Ω^{-1}.

Proof.

Define $F_{k,N}(\ell)$ as $F_{k,N}(\ell) = \sum_{m=1}^{N-\ell} X_{m+\ell-k}\,\varepsilon_m$ for $\ell \geq 0$; $F_{k,N}(\ell) = \sum_{m=|\ell|}^{N} X_{m+\ell-k}\,\varepsilon_m$ for $\ell < 0$. Moreover let

$\xi_N(k) = \sum_{\ell=-N+1}^{N-1} \{F_{k,N}(\ell) - E(F_{k,N}(\ell))\} D(\ell) / \sqrt{N}$,

$\xi_{N,L}(k) = \sum_{\ell=-L}^{L} \{F_{k,N}(\ell) - E(F_{k,N}(\ell))\} D(\ell) / \sqrt{N}$ and $\xi^*_{N,L}(k) = \xi_N(k) - \xi_{N,L}(k)$. Observe first of all for a fixed positive integer that by assumptions i) and iii) the statistics $\{F_{k,N}(\ell) - E(F_{k,N}(\ell))\}/\sqrt{N}$

$(\ell = 0, \pm 1, \cdots, \pm L\ ;\ k = 1, \cdots, p)$ are asymptotically jointly normally distributed with covariances

$$C_{\ell,m}(k,j) = \lim_{N\to\infty} E\,\{(F_{k,N}(\ell) - E(F_{k,N}(\ell)))(F_{j,N}(m) - E(F_{j,N}(m)))\}/N$$
$$= \sum_{u=-\infty}^{\infty} \{ r_X(u)\, r_\varepsilon(u+\ell+m-k-j) + r_{X\varepsilon}(u+\ell-k)\, r_{\varepsilon X}(u-m+j)\}$$

where the last expression above is finite by assumption ii [cf. Hannan (1970), p 209 and 228]. Accordingly, $\xi_{N,L}(k)$, $k = 1, 2, \cdots, p$, are asymptotically jointly normally distributed with mean 0 and with covariance matrix whose (k,j) element $C_L(k,j)$ is given as $C_L(k,j) = \sum_{\ell=-L}^{L} \sum_{m=-L}^{L} D(\ell)\, D(m)\, C_{\ell,m}(k,j)$. Now,

$$\lim_{L\to\infty} C_L(k,j) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} D(\ell) D(m) \{ \sum_{u=-\infty}^{\infty} r_X(u)\, r_\varepsilon(u+\ell-m-k+j)$$
$$+ r_{X\varepsilon}(u+m-j)\, r_{\varepsilon X}(u-\ell+k)\}$$

where the right-hand side converges absolutely. Repeated applications of the Parceval equality lead to the equation

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} D(\ell) D(m) \sum_{u=-\infty}^{\infty} r_X(u)\, r_\varepsilon(u+\ell-m-k+j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_X(\omega)}{f_\varepsilon(\omega)} e^{i(k-j)\omega}\, d\omega,$$

whereas, since $\int_{-\pi}^{\pi} e^{i\ell\omega} f_{X\varepsilon} f_\varepsilon^{-1}\, d\omega = 0$ for $\ell < 0$,

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} D(\ell)\, D(m)\, r_{X\varepsilon}(u+m-j)\, r_{\varepsilon X}(u-\ell+k)$$
$$= \sum_{u=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(u+k)} \frac{f_{X\varepsilon}(\omega)}{f_\varepsilon(\omega)}\, d\omega\ \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(u-j)} \frac{f_{X\varepsilon}(\omega)}{f_\varepsilon(\omega)}\, d\omega = 0.$$

Therefore, it follows that

$$\lim_{L\to\infty} C_L(k,j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_X(\omega)\, f_\varepsilon^{-1}(\omega)\, e^{i(k-j)\omega}$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|1-\sum_{\ell=1}^{p}\alpha_\ell e^{i\omega\ell}\right|^{-2} e^{i(k-j)\omega}\,d\omega.$$

Secondly, an upper bound of the absolute mean of $\xi^*_{N,L}(k)$ is evaluated as follows:

$$E|\xi^*_{N,L}(k)| \leqslant \sum_{L+1\leqslant|\ell|\leqslant N-1} |D(\ell)| \left[E\{F_{k,N}(\ell) - E(F_{k,N}(\ell))\}^2/N\right]^{\frac{1}{2}},$$

whereas, for a certain positive number S, it holds uniformly that

$$E\{F_{k,N}(\ell) - E(F_{k,N}(\ell))\}^2/N < S.$$

Therefore $E|\xi^*_{N,L}(k)| \leqslant 2S\sum_{\ell=L+1}^{\infty}|D(\ell)|$ uniformly in N and k. By use of Chebychev's inequality for the first-order absolute moment, it follows from Assumption ii) that there exists a L_0 such that for $L > L_0$ $P_r\{|\xi^*_{L,N}| > \delta\} < \varepsilon$ for all N $(\geqslant L)$. Then the limit theorem given by T.W. Anderson (1971) says that the asymptotic distribution of $\xi_N = \xi_{L,N} + \xi^*_{L,N}$ is multivariate normal distribution with zero mean vector and with covariance matrix Ω, where ξ_N, $\xi_{L,N}$ and $\xi^*_{L,N}$ are p-vectors whose k-th elements are $\xi_N(k)$, $\xi_{L,N}(k)$ and $\xi^*_{L,N}(k)$ respectively.

The convergence of $\frac{1}{N}\Gamma_N$ to Ω is shown as this. Observe that $E\frac{1}{N}\sum_{t=1}^{N}\sum_{s=1}^{N} X_{t-k}X_{s-j}\,D(t-s) = \sum_{\ell=-N+1}^{N-1}\left(1-\frac{|\ell|}{N}\right)\gamma(\ell+k-j)\,D(\ell)$.
Then,

$$\lim_{N\to\infty} E\frac{1}{N}\sum_{t=1}^{N}\sum_{s=1}^{N} X_{t-k}X_{s-j}\,D(t-s) = \sum_{\ell=-\infty}^{\infty}\gamma_X(\ell+k-j)\,D(\ell)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_X(\omega)\, f_\varepsilon(\omega)^{-1} e^{i(k-j)\omega}\, d\omega .$$

On the other hands, it is straightforward to see that

$$\frac{1}{N} \sum_{t=1}^{N} \sum_{s=1}^{N} \{ X_{t-k} X_{s-j} - E(X_{t-k} X_{s-j}) \} D(t-s)$$

converges in mean-square to 0. Thus $\frac{1}{N}\Gamma_N$ converges in probability to Ω.

Finally the convergence of $\frac{1}{\sqrt{N}} E\{ \sum_{\ell=-N+1}^{N-1} F_{k,N}(\ell) D(\ell) \}$ to 0 is demonstrated as follows. By the application of the Grenander – Rosenblatt theorem (1953, pp 543–544) after its slight modification

$$\frac{1}{\sqrt{N}} \left| E\left(\sum F_{k,N}(\ell) D(\ell) \right) - \sum_{-\infty}^{\infty} \gamma_{X\varepsilon}(k-\ell) D(\ell) \right| = O\left(\frac{\log^5 N}{\sqrt{N}} \right) .$$

On the other hand,

$$\sum_{-\infty}^{\infty} \gamma_{X\varepsilon}(k-\ell)\, D(\ell) = 0$$

since $\displaystyle\sum_{-\infty}^{\infty} \gamma_{X\varepsilon}(\ell-k)\, D(\ell) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\omega} \frac{1 - \Sigma \alpha_j e^{ij\omega}}{|1 - \Sigma \alpha_j e^{ij\omega}|^2}\, d\omega = 0.$

Consequently, $\frac{1}{\sqrt{N}} \Sigma (F_{k,N}(\ell) - E(F_{k,N}(\ell))) D(\ell)$ is asymptotically distributed in the same way as $\frac{1}{\sqrt{N}} \Sigma F_{k,N}(\ell) D(\ell)$. To summerize, $\sqrt{N}(\hat{\alpha}_N - \alpha_0)$ is asymptotically distributed as $\left(\frac{1}{N}\Gamma_N\right)^{-1} \xi_N$, while $\frac{1}{N}\Gamma_N \to \Omega$ in probability and ξ_N is asymptotically normal with zero mean vector and with covariance Ω. Thus the proof is complete.

References

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