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PARAMETER ESTIMATION OF MARKOV RANDOM FIELDS.

東大 理学部 H. Künsch

0. Introduction: A random field is a stochastic process with a multidimensional parameter, usually interpreted as space. There are many examples of random fields in practical applications, for instance annual rainfall in an area, the size of a crop at different plots, or in the case of transmission of photos the optical amplitude. The paper [6] contains a survey of applications of random fields in various disciplines.

We consider here the case of a random field $X(t)$, $t \in \mathbb{Z}^\nu$, $\nu = 1, 2, \ldots$ with $EX(t) \equiv 0$ and which is homogeneous, i.e. all finite dimensional distributions are translation invariant (A generalization to a field with some trend is possible). Whittle [7] and Larimore [2] both considered autoregressive models, which seem to us a too narrow class; we take here the broader class of Markov models. Two methods for the estimation of the parameters of such a model are proposed, and in both cases consistency and asymptotic normality are proved, even if X is not Gaussian.

1. Type of models; a) Markovian models

Choose a finite set $L \subset \mathbb{Z}^\nu$ with $L = -L$ and $0 \notin L$, the set of 'points near 0'. We define then the (L-) boundary of a $D \subset \mathbb{Z}^\nu$ by $\partial D := \{t \notin D \mid \exists s \in D,\ t-s \in L\}$.

Definition: A random field $X(t)$, $t \in \mathbb{Z}^\nu$, has the Markov property w.r. to a set D if $\forall$ $\sigma\{X(t), t \in D\}$ - measurable r. v. U

$$E(U / X(t),\ t \notin D) = E(U / X(t),\ t \in \partial D) \qquad (1)$$

Remark: If $\nu = 1$ and $L = \{-p, \ldots -1, 1, \ldots p\}$, then for $D = \{t, t+1, \ldots\}$ the above definition is the usual definition of a p-th order Markov process. It can also be proved, that a p-th order Markov process has the Markov property w.r. to all $D \subset \mathbb{Z}^1$. However, for $\nu \geq 2$ the line transect $X(t_1, 0, \dots 0)$ of a Markov field is in general not a Markov process.

If X is homogeneous, Gaussian, $EX = 0$ then the Markov property w.r. to $\{t\}$ is simply $E(X(t) \mid X(s),\ s \neq t) = \sum_{k \in L} a(k)\, X(t-k)$ (2) with some coefficients $a(k)$. (2) is equivalent to $X(t) = \sum_{k \in L} a(k)\, X(t-k) + Y(t)$ (3) with $Y(t)$ independent of $X(s)$, $s \neq t$. As our models, we take solutions of (2), resp. (3).

A simple calculation shows, that

$$E\, Y(t) Y(s) = c^2 \begin{cases} 1 & \text{if } t = s \\ -a(t-s) & \text{if } t-s \in L \\ 0 & \text{elsewhere.} \end{cases}$$

i.e. the error terms in (3) are correlated between themselves. Also, we must have $a(k) = a(-k)$ (therefore $L = -L$).

<u>Theorem 1:</u> There exists a Gaussian homogeneous solution of (2) with $E\,Y(t)^2 = c^2 > 0$ iff $a(k) = a(-k)$, $P(x) := 1 - \sum a(k) e^{ikx} \geq 0$, $\int_{[-\pi,\pi]^\nu} \frac{1}{P(x)}\, dx < \infty$

If we ask X to be purely non deterministic, then X is uniquely given by

$$E\,X(t)\,X(s) = R(t-s) := \frac{c^2}{(2\pi)^\nu} \int \frac{e^{i(t-s)x}}{P(x)}\,dx \qquad (4)$$

<u>Proof</u>: see Rozanov [5]

X can be built up with i.i.d. r.v.:

<u>Theorem 2</u>: If the covariance of X is given by (4), then X satisfies the moving average equation

$$X(t) = \sum_{k\in \mathbb{Z}^\nu} g(t-k)\,U(k)\,, \quad \text{with} \qquad (5)$$

$$E\,U(t)\,U(s) = c^2\,\delta_{t,s} \qquad g(s) = \frac{1}{(2\pi)^\nu}\int e^{isx}\,P(x)^{-\frac{1}{2}}\,dx$$

<u>Proof</u>: Put $U(k) := \int e^{ikx}\,P(x)^{\frac{1}{2}}\,dZ(x)$, where dZ is the random measure with $E|dZ(x)|^2 = \frac{c^2}{(2\pi)^\nu}\,P(x)^{-1}dx$.

q.e.d.

If U is not normal, the field X defined by (5) is in general only 'Markov in the weak sense' i.e. $\hat{E}\,(X(t)\mid X(s),\ s\neq t) = \sum_{k\in L} a(k)\,X(t-k)$ where $\hat{E}$ is the orthogonal projection on the closed linear hull. But since it is **not** known, if (3) has a non Gaussian solution, we take nevertheless X defined by (5) as our model.

If $X(t)$ has been observed for $t\in D$ and if $X(t)$ satisfies (2) with known $a(k)$, then the formula for best linear extrapolation of $X(t)$, $t \notin D$, is well known: If D^c is finite, see Rozanov [4], chapter II. 10, if D is a halfspace, see Pitt [3], proposition 9.1, and if D is finite and $\sum|a(k)| < 1$, see Williams [8]

b) Autoregressive models

Definition: X is called an autoregressive field if $X(t) = \sum_{k \in K} b(k) X(t-k) + U(t)$ (6)

with i.i.d. $U(t)$ and K finite, $\not\ni 0$.

From part 1a) it is clear, that then $U(t)$ must be correlated with some $X(s)$, $s \neq t$.

Theorem 3: There exists a purely non deterministic solution of (6) iff

$$Q(x)^{-1} := \left[1 - \sum_{k \in K} b(k) e^{-ikx}\right]^{-1} \in L_2\left([-\pi,\pi]^\nu, dx\right).$$

The covariance is $EX(t)X(s) = c^2 \int \frac{e^{i(t-s)x}}{|Q(x)|^2} dx$

Proof: Purely non deterministic implies, that the spectral measure is abs. continuous (Rozanov [5]). Then $E\,U(t)U(s) = \int e^{i(t-s)x} |Q(x)|^2 f(x) dx$

So, since the U's are i.i.d. $|Q(x)|^2 f(x) = \text{const.}$

q.e.d.

Therefore **every** autoregressive field is Markov, and the contrary is true, iff $P(x) = \text{const}\,|Q(x)|^2$. For $\nu = 1$, every $P(x)$ has such a decomposition (see Yaglom [9], p. 121), but for $\nu \geq 2$, this is no longer the case; for instance the simplest $P(x) = 1 - a\cos x_1 - b\cos x_2$, $a \neq 0, b \neq 0$ has no such decomposition: If we take $Q(x) = 1 - \alpha e^{ix_1} - \beta e^{ix_2} - \gamma e^{-ix_1} - \delta e^{-ix_2}$, then in $|Q|^2$ appear terms of order 2, which are not all $= 0$ except if $\beta = \gamma = \delta = 0$. For more complicated Q's a similar thing is true.

2. Estimation of parameters

Suppose $X(t)$ is observed in a set D. We want to estimate the $a(k)$'s with the help of $X(t)$, $t \in D$. L is supposed to be known. Since we must have $a(k) = a(-k)$, we choose an M such that $M \cap (-M) = \emptyset$, $M \cup (-M) = L$ and estimate $a(k)$, $k \in M$. For simplicity, we take $D = \{1, \dots T\}^\nu$.

The properties of the estimators, we go to propose ~~in the following~~ later, are based on the properties of the following r.v.

$C(k) := \frac{1}{T^\nu} \sum_{t \in D} X(t) X(t+k) \quad (X(s) := 0 \ \forall s \notin D)$

<u>Lemma 1</u>: If X is given by (5) with $E\,U(t)^4 = 3c^4 + \kappa_4 < \infty$, then $C(k) \xrightarrow[T\to\infty]{P} R(k)$. If moreover $P(x) \neq 0 \ \forall x$, then $T^{\nu/2}[C(k) - R(k)]$ are asymptotically joint normal with mean 0 and covariance $c^4 A_{k,\ell} + \frac{\kappa_4}{c^4} R(k) R(\ell)$, where $A_{k,\ell} := \frac{2}{(2\pi)^\nu} \int \frac{\cos kx \cos \ell x}{P(x)^2} dx$

<u>Proof</u>: Consistency follows easily, because the field $X(t)X(t+k)$, $t \in \mathbb{Z}^\nu$ has a spectral density. Asymptotic normality is a straightforward generalization of Anderson [1], section 8.4.2.

q.e.d.

a) <u>Least square estimators</u>: We take as estimators for $a(k)$ those values $\hat{a}(k)$, which minimize $\sum_{t \in D} [X(t) - \sum_{k \in M} a(k) \{X(t+k) + X(t-k)\}]^2$. By differentiation and division by $4 \cdot T^\nu$:

$$\sum_{k \in M} [C(k+n) + C(k-n)]\, \hat{a}(k) = C(n) \qquad n \in M \qquad (7)$$

As estimation for c^2 we take

$$\widehat{c^2} := \frac{1}{T^\nu} \sum_{t \in D} [X(t) - \sum_{k \in M} \hat{a}(k) \{X(t+k) + X(t-k)\}]^2 \qquad (8)$$

or by (7) $\quad \widehat{c^2} = C(0) - 2 \sum_{k \in M} \hat{a}(k)\, C(k) \qquad (8')$

Put $S_{k,n} := R(k+n) + R(k-n)$. Then

Lemma 2: $(S_{k,n})_{k,n\in M}$ is not singular

Proof: $\sum_n S_{k,n}\gamma_n = 0 \quad \forall k \in M \iff$

$\int e^{ikx} \sum_n \gamma_n \cos nx \frac{dx}{P(x)} = 0 \quad \forall k \in L \iff \sum_n \gamma_n \cos nx = 0$

in $L_2\left(\frac{dx}{P(x)}\right) \iff \gamma_n = 0 \quad \forall n$ q.e.d.

Theorem 4: If X satisfies (5) with $EU(t)^4 < \infty$ then $\hat{a}(k)$, $\hat{c}^2$ are consistent. If moreover $P(x) \neq 0 \ \forall x$, then $T^{\nu/2}[\hat{a}(k) - a(k)]$ are asymptotically joint normal with mean 0 and covariance $c^4 \cdot (S_{k,n})^{-2}$

Proof (3) implies $R(n) - \sum_{k\in M} a(k) S_{k,n} = c^2 \delta_{n,0}$ so consistency follows from lemmas 1 and 2. For asymptotic normality, observe that

$$T^{\nu/2}[\hat{a}(k) - a(k)] \sim \sum_n (S^{-1})_{k,n} T^{\nu/2}\left[C(n) - \sum_\ell a(\ell)\{C(\ell+n) + C(\ell-n)\}\right]$$

$$\sim \sum_n (S^{-1})_{k,n} T^{\nu/2}\left[C(n) - R(n) - \sum_\ell a(\ell)\{C(\ell+n) - R(\ell+n) + C(\ell-n) - R(\ell-n)\}\right],$$

and the result follows from lemma 1 by an easy calculation q.e.d.

b) Maximum likelihood estimators

Using a result from Whittle [7], p. 440-442, we get that

$$\frac{1}{T^\nu} \log \text{likelihood} \approx \text{Const.} - \frac{1}{2}\log c^2 + \frac{1}{2}\frac{1}{(2\pi)^\nu}\int \log\left(1 - 2\sum a(k)\cos kx\right)dx$$

$$- \frac{1}{c^2}\left[C(0) - 2\sum_k a(k)C(k)\right] ;$$

This gives after differentiation the following equations for the maximum likelihood estimators:

$$\frac{\widetilde{c}^2}{(2\pi)^\nu}\int \frac{\cos kx}{1-2\sum_\ell \widetilde{a}(\ell)\cos \ell x}\,dx = C(k), \quad k\in M \qquad (9)$$

$$\widetilde{c}^2 = C(0) - 2\sum_k \widetilde{a}(k)C(k), \qquad (10)$$

Or equivalently to (9) and (10):

$$\frac{\widetilde{c}^2}{(2\pi)^\nu}\int \frac{\cos kx}{1-2\sum \widetilde{a}(\ell)\cos \ell x}\,dx = C(k),\ k\in M\cup\{0\} \quad (11)$$

i.e. we have to fit in $\widetilde{c}^2$, $\widetilde{a}(k)$ in such a way, that for $k\in M\cup\{0\}$ the covariances of the fitted model are equal to $C(k)$. The least square method on the other hand uses also $C(\ell)$, $\ell\notin M\cup\{0\}$ for the estimation.

Theorem 5 If X satisfies (5) with $EU(t)^4<\infty$ and $P(x)\neq 0\ \forall x$, then $\widetilde{a}(k)$ and $\widetilde{c}^2$ are consistent and $T^{\nu/2}[\widetilde{a}(k)-a(k)]$ are asymptotically joint normal with mean 0 and covariances $\left(A_{k,\ell}-2\frac{R(k)R(\ell)}{c^4}\right)^{-1}$ ($A_{k,\ell}$ as in lemma 1).

Proof: $\varphi:(a_k)_{k\in M}, c^2 \longrightarrow \frac{c^2}{(2\pi)^\nu}\int\frac{\cos kx}{1-2\sum a_\ell \cos \ell x}\,dx$, $k\in M\cup\{0\}$ is a differentiable map defined on $\{a_k \mid 1-\sum a_k\cos kx>0\ \forall x\}\times\mathbb{R}_+$ with functional matrix

$\left. \left(\underbrace{c^2 A_{k,\ell}}_{\ell \in M} \middle| \frac{R_k}{c^2} \right) \right\}_{k \in M \cup \{0\}}$ By a similar argument as in lemma 2, we can show, that it is not singular. Together with lemma 1, this implies consistency.

If we develop (9) in a Taylor series, we get: $C(k) - R(k) = \sum_\ell [\tilde{a}(\ell) - a(\ell)]\, c^2 A_{k,\ell} +$

$+ (\hat{c}^2 - c^2)\, \frac{R(k)}{c^2}$ + remainder, $\quad k \in M$

Out of (10): $\hat{c}^2 - c^2 = C(0) - R(0) - 2 \sum_\ell R(\ell)[\tilde{a}(\ell) - a(\ell)]$

$- 2 \sum_\ell a(\ell)\, [C(\ell) - R(\ell)]$ + remainder

Combining these two results:

$T^{1/2}\, [C(k) - R(k) - \frac{R(k)}{c^2} \cdot Z] = \sum_\ell (c^2 A_{k,\ell} - 2\, \frac{R(k) R(\ell)}{c^2})\, T^{1/2} [\tilde{a}(\ell) - a(\ell)]$

+ remainder, where $Z := C(0) - R(0) - 2 \sum a(\ell) [C(\ell) - R(\ell)]$.

The remainder goes to 0 as $T \to \infty$, and by lemma 1, the left hand side is asymptotically joint normal with covariance $c^4 A_{k,\ell} - 2 R(k) R(\ell)$. As in lemma 2, we can show, that this matrix is not singular.

q.e.d

c) Comparison of the 2 methods In the case $\nu = 1$, $L = \{-1, 1\}$, we get for the asymptotic variance of $T^{1/2}(\hat{a} - a)$: $\frac{(1-\beta^2)^2}{(1+\beta^2)^4}$ and for

$T^{1/2}(\tilde{a}-a): \frac{(1-\beta^2)^3}{(1+\beta^2)^4}$, where $a = \frac{\beta}{1+\beta^2}$. So, for $\beta = 0,5 \Leftrightarrow a = 0,4$, we need in the case of the least square method $\frac{1}{3}$ more observations for the same precision, but equation (7) is easier to solve then (11). We haven't had time and occasion to solve numerical examples. We think, it might be a good method to calculate first $\hat{a}(k)$, $\hat{c}^2$ and use these values as starting values for the solution of (11) with Newton's method.

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Present address:

H. Künsch

Hügelstraße 11

8002 Zürich Switzerland