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# A Construction of Approximately Finite-Dimensional non-ITPFI Factors

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The following is a report on some joint work with A. Connes which was carried out in Paris in January, 1976.

A von Neumann algebra is said to be approximately finite-dimensional if it is of the form

$$M = \{ M_1 \subset M_2 \subset \quad \cdots \quad \subset M_n \subset \quad \cdots \quad \}''$$

where each $M_n$ is a finite-dimensional matrix algebra. A factor is said to be ITPFI if it is of the form

$$M = \bigotimes_{n=1}^{\infty} (M_n, \omega_n)$$

where each $M_n$ is a type I factor (and $\omega_n$ is a state on $M_n$). The existence of factors which are approximately finite-dimensional but not ITPFI is an interesting problem. The first construction of such factors was given by Krieger [8]. However in [8] it is only proved that the factors are not "weakly equivalent" to any ITPFI factor. The first proof that these factors are not ITPFI was given by Connes [3]. Alternatively one could now use Krieger's theorem [9] that unitary equivalence implies "weak equivalence" to complete the argument. However Krieger's construction is rather involved, and the arguments of both Krieger [8] and Connes [3] were quite delicate. We give here a new construction for which, in the context of the flow of weights, the argument is rather elementary.

Sec. 1 reviews the relevant aspects of the flow of weights [4], and gives some terminology. Sec. 2 contains the technical lemmas. In Sec. 3 we discuss the examples.

1. Preliminary Material

Let M be a factor, Aut M the group of all automorphisms of M with the topology of pointwise convergence in the predual, and Int M the subgroup of inner automorphisms of M. The flow of weights of M is an ergodic action of $R_+^*$ on some measure space $(X_M, \mu_M)$. The construction of [4] gives not the measure space, but the measure algebra whose elements are unitary equivalence classes $[\phi]$ of integrable weights $\phi$ of infinite type. The flow is then defined by $F_M(\lambda)\ [\phi] = [\lambda\phi]$. Let $\alpha \in$ Aut M. The equation Mod $\alpha\ [\phi] = [\phi\circ\alpha]$ defines a Borel (and hence continuous) homomorphism from the polish group Aut M into the polish group of automorphisms of the measure space $(X_M,\mu_M)$. Clearly $\alpha \in$ Int M implies that Mod $\alpha = 1$. If M is a factor of type $III_0$ then the flow of weights $F_{M\otimes M}(\lambda)$ for $M\otimes M$ is given by the action of $F_M(\lambda)\otimes 1$ on the measure algebra of the $F_M(\lambda)\otimes F_M(\lambda^{-1})$ invariant sets on $X_M \times X_M$.

All Borel spaces considered in this paper are standard (i.e. Borel isomorphic to a Borel subset of the unit interval). A transformation S on a measure space $(X,\mu)$ is called non-singular if it is invertible and both S and $S^{-1}$ are $\mu$-measurable. Given a non-singular S, the orbit of x under S is the set

$$O_S(x) = \{S^j x: j \in Z\}.$$

The full group of S is the set [S] of all non-singular transformations T such that for a.e. x, $Tx \in O_S(x)$. A set $W \subset X$ such that $\mu(S^jW \cap S^kW) = 0$ for all $j \neq k$ is called a wandering set for S. S is said to be dissipative if there is a wandering set W such that $X = \bigcup_{j=-\infty}^{\infty} S^jW$.

## 2. The Technical Lemmas

Let M be a von Neumann algebra, x,y M. The automorphism $\sigma$ of $M \otimes M$ defined by the equation $\sigma(x \otimes y) = y \otimes x$ is called the Sakai flip.

Lemma 2.1: Let M be an ITPFI factor, $\sigma$ the Sakai flip on $M \otimes M$. Then $\sigma \in \overline{\text{Int}}(M \otimes M)$.

Proof: Let $M = \bigotimes_{n=1}^{\infty}(M_n, \omega_n)$ be given on $\bigotimes(H_n, \Omega_n)$ where $\omega_n(x) = (x\Omega_n, \Omega_n)$. Then $M \otimes M = \bigotimes_n (M_n \otimes M_n, \omega_n \otimes \omega_n)$ acts on $K = \bigotimes_n (H_n \otimes H_n, \Omega_n \otimes \Omega_n)$. Let $\psi \in (M \otimes M)_*$, $\varepsilon > 0$.

We can assume that $\bigotimes_n (\Omega_n \otimes \Omega_n)$ is a separating vector for $M \otimes M$ (see lemma 3.15 of [2]). Hence there is a vector $\Psi \in K$ such that

$\psi(x) = (x\Psi, \Psi)$. By lemma 3.1 of [1] there exists $m < \infty$ and $\Psi(m) \in \bigotimes_{n=1}^{\infty}(H_n \otimes H_n)$, $\|\Psi(m)\| = 1$, such that $\|\Psi - \Psi_\varepsilon\| < \varepsilon$ where

$$\Psi_\varepsilon = \Psi(m) \otimes \left( \bigotimes_{n=m+1}^{\infty} (\Omega_n \otimes \Omega_n) \right).$$

Let $\psi_\varepsilon$ be the state defined by $\Psi_\varepsilon$, and let $\sigma_m$ be the Sakai flip on $\bigotimes_{n=1}^{m} (M_n \otimes M_n)$. Then $\sigma\psi_\varepsilon = (\sigma_m \otimes 1)\psi_\varepsilon$. Hence

$$\|(\sigma - \sigma_m \otimes 1)\psi\| < 2\varepsilon.$$

Since $\sigma_m$ is inner, it follows that $\sigma \in \overline{\text{Int}}(M \otimes M)$. QED.

Lemma 2.2: Let R,S be non-singular transformations on the standard measure space $(X,\mu)$. If S is dissipative and R leaves invariant (modulo $\mu$) all S-invariant measurable sets, then $R \in [S]$.

Proof: We first note that if $(E,\nu)$ is a countably separated measure space and $f:E \quad E$ satisfies $f(B) = (B)$ (modulo $\nu$) for all measurable $B \quad E$, then $f(x) = x$ (a.e. $\nu$). Namely let $(B_n)_{n\ N}$ separate points in E. Then

$$\{x: f(x) \neq x\} \subset \bigcup_n \{B_n \setminus f(B_n)\}$$

which is a set of measure zero.

Now let W be a wandering set for S such that $X = \bigcup_{k=-\infty}^{\infty} S^k W$. Let $P_k$ be the projection of X onto $S^k W$ defined by $P_k x = y$ if $x = S^j y$ for some j such that $y \in S^k W$. Let A be any measurable subset of $S^k W$. Then $\bigcup_{p=-\infty}^{\infty} S^p A$ is S-invariant and it follows that $(P_k R P_k)A = A$ (modulo $\mu$). Now clearly $R \in [S]$ if and only if $P_k R P_k(x) = x$ (a.e.) for all k. QED.

The following theorem uses the base and ceiling function construction of a flow. For this purpose it is more convenient to have the flow as an action of R rather than $R_+^*$. Hence we shall use $\mathfrak{F}_M(\lambda) = F_M(e^\lambda)$.

Theorem 2.3: Let M be a factor of type $III_0$ whose flow of weights can be built under a constant ceiling function with a base transformation T such that $T \times T^{-1}$ is dissipative. Then the Sakai flip $\sigma \notin \overline{\mathrm{Int}}(M \otimes M)$ and hence M is not ITPFI.

Proof: Clearly Mod $\sigma$ acts on $X_M \times X_M = (B \times I \times (B \times I)$ by $\sigma(x,s,y,t) = (y,t,x,s)$. Let E be any $T \times T^{-1}$ invariant set in $B \times B$, $\sigma_B$ the flip on $B \times B$. Then $E \times I \times I$ is an $\mathfrak{F}_M(\lambda) \otimes \mathfrak{F}_M(-\lambda)$ invariant set in $X_M \times X_M$. Now assume Mod $\sigma = 1$. Then $\sigma_B$ must preserve E, hence $\sigma_B \in [T \times T^{-1}]$ by the preceding lemma. But this implies that for a.e. $(x,y) \in B \times B$ there exists an integer $n(x,y)$ such that

$$\sigma_B(x,y) = (y,x) = (T^{n(x,y)}x, T^{-n(x,y)}y),$$

i.e. $y \in O_T(x)$. But $O_T(x)$ is countable. QED.

## 3. The Examples

It remains to demonstrate the existence of approximately finite-dimensional factors of type $III_0$ satisfying the conditions of Theorem 2.3. For this we first need the existence of ergodic transformations T such that $T \times T^{-1}$ is dissipative. It is a classical result in ergodic theory that such transformations exist [6]. As a specific example, one can use the Markov shift obtained from a two-dimensional random walk.(These transformations preserve an infinite measure.) The existence now follows from the fact that any flow arises as the flow of weights of some approximately finite-dimensional factor [4, 9]. (The proof of this in the general case is not so easy. However for measure preserving flows the argument is not difficult (see for example [7]).)

We remark that $\sigma \in \overline{\text{Int}}(M \otimes M)$ is not a sufficient condition for M to be ITPFI. Namely let M be an approximately finite-dimensional factor whose flow can be built under a constant ceiling function with a base transformation T which preserves a finite measure. If T is a Bernoulli shift then M is not ITPFI [5]. But then $T \quad T^{-1}$ is ergodic, and it follows easily that Mod $\sigma = 1$. Hence $\sigma \in \overline{\mathrm{Int}}$ (M  M) [4]. In fact if T is any ergodic transformation preserving a finite measure, it follows from the proof of part (2) of lemma 1 of [7] that Mod $\sigma = 1$ (see also [10]).


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