

AUGMENTED TEICHMÜLLER SPACES AND
PERIOD REPRODUCING DIFFERENTIALS

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Introduction.

It is of interest to investigate how some characteristic quantities attached to Riemann surfaces vary under quasiconformal deformations of the surfaces. Such sort of studies have been made by L. Ahlfors [4], L. Bers [5], and so on.

In the present paper we consider first the Teichmüller space T_g of marked closed Riemann surfaces of genus g , and show in § 1 some theorems about the continuity on T_g for the holomorphic differentials with fixed A-periods and the period reproducing differentials. These results can be extended over the Teichmüller spaces of certain classes of open Riemann surfaces.

In § 3 and 4 we examine specifically the case of squeezing deformations with respect to a non-dividing simple closed curve c . For this purpose we consider the augmented Teichmüller space \hat{T}_g and its subset ${}_c\hat{T}_g$ determined by c . We introduce a topology into ${}_c\hat{T}_g$, which we call the fine topology (cf. § 4 for the precise definition). Then it is proved that the fine topology is finer than the conformal topology introduced by W. Abikoff [2]. The period reproducing differentials with a suitable normalization

vary continuously on \hat{T}_g with respect to the fine topology.

The proofs are almost sketchy or omitted, and the details will appear elsewhere.

§ 1. The continuity on T_g for holomorphic differentials.

1. Let R^* be a closed Riemann surface of genus $g \geq 2$ and $\Pi = \{ A_i, B_i \}_{i=1}^g$ be the standard set of generators of the fundamental group of R^* with the single relation $\prod_{i=1}^g A_i \cdot B_i \cdot A_i^{-1} \cdot B_i^{-1} = 1$. We denote by T_g the Teichmüller space with the base point $\bar{R}^* = (R^*, \Pi)$, which is equipped with the usual Teichmüller topology. On each point of T_g a canonical homology basis is induced by Π and is denoted again by $\{ A_i, B_i \}_{i=1}^g$. The 1-cycles and (free) homotopy classes etc. given on R^* induce also on every point of T_g the corresponding ones, which will be denoted by the same notations.

Now let $\bar{R}_0 \in T_g$ be fixed and take a holomorphic abelian differential $\theta_{\bar{R}_0}$ on R_0 . Then on every point \bar{R} of T_g there exists the unique holomorphic differential on R , say $\theta_{\bar{R}}$, which has the same A-periods as $\theta_{\bar{R}_0}$, and we have the following

Theorem 1. Let $f_{\bar{R}}$ be the Teichmüller mapping of \bar{R}_0 to \bar{R} , and $K_{\bar{R}} = \frac{1 + k_{\bar{R}}}{1 - k_{\bar{R}}}$ be the maximal dilatation of $f_{\bar{R}}$. Then we have

$$(1) \quad \|\theta_{\bar{R}} \circ f_{\bar{R}} - \theta_{\bar{R}_0}\|_{R_0} \leq \frac{2k_{\bar{R}}}{1 - k_{\bar{R}}} \|\theta_{\bar{R}_0}\|_{R_0}.$$

Hence, $\theta_{\bar{R}} \circ f_{\bar{R}}$ converges to $\theta_{\bar{R}_0}$ in the Dirichlet norm if \bar{R} converges to \bar{R}_0 in T_g .

Proof. Noting that $\theta_{\bar{R}} \circ f_{\bar{R}}$ is a closed differential on R_0 with the finite Dirichlet norm and $\omega = \theta_{\bar{R}} \circ f_{\bar{R}} - \theta_{\bar{R}_0}$ has vanishing A-periods, we have $(\omega, \omega^*) = 0$ by the bilinear relation, from which we can derive the desired inequality (1). (Cf. [4] and [12].) q.e.d.

Let c denote a non-dividing simple closed curve on (surfaces of) T_g , and $\theta_{c, \bar{R}}$ be the holomorphic reproducing differential for c on $\bar{R} \in T_g$. Namely, it is characterized by the relation

$$(2) \quad \text{Im} \int_{\gamma} \theta_{c, \bar{R}} = c \times \gamma$$

for every 1-cycle γ on R , equivalently by $\int_c \omega = (\omega, \text{Re } \theta_{c, \bar{R}})$ for every harmonic differential ω with the finite Dirichlet norm on R . Then similarly as Theorem 1 we have

Theorem 1'. Under the same assumption as in Theorem 1, we have

$$(3) \quad \|\theta_{c, \bar{R}} \circ f_{\bar{R}} - \theta_{c, \bar{R}_0}\|_{R_0} \leq \frac{2k_{\bar{R}}}{1 - k_{\bar{R}}} \|\theta_{c, \bar{R}_0}\|_{R_0}$$

Hence, $\theta_{c, \bar{R}} \circ f_{\bar{R}}$ converges to θ_{c, \bar{R}_0} in the Dirichlet norm if \bar{R} converges to \bar{R}_0 in T_g .

Remark. As is seen from the proof, the mapping $f_{\bar{R}}$ of \bar{R}_0 to \bar{R} need not be the Teichmüller mapping, but it suffices to be an appropriately smooth quasiconformal mapping (with a suitable change of $k_{\bar{R}}$).

2. As the universal covering surface of any $\bar{R} \in T_g$, we take the unit disk $B = \{ |z| < 1 \}$. Then $f_{\bar{R}}$ can be lifted to a quasi-conformal self-mapping of B . It is extended continuously onto $\{ |z| = 1 \}$ and is uniquely determined by the normalization fixing three points 1, i , and -1 .

We write this lift as $f_{\bar{R}}$ again. The differentials $\theta_{\bar{R}}$ and $\theta_{c, \bar{R}}$ can also be lifted to holomorphic differentials, say $a_{\bar{R}}(z)dz$ and $a_{c, \bar{R}}(z)dz$, over B respectively.

Theorem 2. If \bar{R} converges to \bar{R}_0 in T_g , then $a_{\bar{R}}(z)$ and $a_{c, \bar{R}}(z)$ converges uniformly on every compact set in B to $a_{\bar{R}_0}(z)$ and $a_{c, \bar{R}_0}(z)$ respectively.

Proof. The assertion follows from Theorem 1 and 1' as in [12]. Also see [4]. q.e.d.

Now since R is compact, the quadratic differential $\theta_{c, \bar{R}}^2$ has closed trajectories ([3], [11]). Let $L(\theta_{c, \bar{R}})$ be the admissible (i.e. homotopically independent) curve system determined by $\theta_{c, \bar{R}}^2$, and set

$$S_c = \{ \bar{R} \in T_g : L(\theta_{c, \bar{R}}) \text{ contains } c. \} .$$

Then we have the following result.

Theorem 3. If $\bar{R} \in T_g$ is sufficiently near \bar{R}_0 , then $L(\theta_{c, \bar{R}_0})$ is contained in $L(\theta_{c, \bar{R}})$. Hence S_c is an open set in T_g .

Proof. For a holomorphic differential θ on R , a trajectory arc of θ^2 is the curve along which $\text{Im } \theta = 0$. By means of this fact, (2), and Theorem 2, we can prove this theorem. q.e.d.

Remark. Roughly speaking, the decomposition of R by critical trajectories of $\theta_{C, \bar{R}}^2$ into doubly connected domains also varies continuously on T_g . (For the details see [12].)

3. We are able to extend the above results to the Teichmüller space $T(R)$ for an open Riemann surface R of finite or infinite genus. Here, since the existence of the Teichmüller mappings is not known, we need to take another standard (suitably smooth) quasiconformal mappings for convergent sequences in $T(R)$.

Now to extend Theorem 1 and 1' we need further the existence theorem of the holomorphic differentials with the finite Dirichlet norm which satisfies the given period condition, and also the (generalized) bilinear relation on R . Under these consideration we can extend Theorem 1 for any R belonging to the class O'' (, cf. [7]), and Theorem 1' for any R belonging to the class O_{HD} .

A similar continuity for the holomorphic reproducing differentials also holds for a wider class of open Riemann surfaces if the behavior of those differentials are appropriately restricted.

All the details will appear in [8].

§ 2. The open set S_c in T_g .

In the sequel we shall consider again the Teichmüller space T_g with genus g (≥ 2). Fix a non-dividing simple closed curve c . Here without loss of generality we may assume that c is freely homotopic to A_1 .

A homeomorphism of S_c . Let $T_{g-1,2}$ be the Teichmüller space of marked Riemann surfaces of type $(g-1,2)$. We shall construct a mapping F_1 from S_c into $T_{g-1,2}$ as follows: Let $\bar{R} \in S_c$ and $W_{\bar{R}}$ be the characteristic ring domain of $\theta_{c,\bar{R}}^2$ for c (, that is, the union of all closed trajectories of $\theta_{c,\bar{R}}^2$ which are freely homotopic to c). This $W_{\bar{R}}$ can be mapped conformally onto a ring domain $\{1 < |z| < r^2\}$. Let $C_{\bar{R}}$ be the closed trajectory in $W_{\bar{R}}$ of $\theta_{c,\bar{R}}^2$ corresponding to the circle $\{|z| = r\}$. Then $R - C_{\bar{R}}$ becomes a bordered Riemann surface with two contours. Adding two regions D_1 and D_2 corresponding to $\{0 < |z| < r\}$ and $\{r < |z| < +\infty\}$ along each contours of $R - C_{\bar{R}}$ respectively, we get a Riemann surface R' of type $(g-1,2)$.

Now we may assume that $\{A_i, B_i\}_{i=1}^g$ of \bar{R} lie in $R - C_{\bar{R}}$ except for B_1 , and that $B_1 \cap C_{\bar{R}}$ consists of a single point. As the generators of the fundamental group $\pi_1(R', p)$ of R' , we choose loops $\{A'_i, B'_i, C_1, C_2\}_{i=1}^{g-1}$ so that

- (i) $A'_i = A_{i+1}$, and $B'_i = B_{i+1}$ ($i=1, \dots, g-1$),
- (ii) C_1 and C_2 are closed curves belonging to $\pi_1(R', p)$

which run along B_1 in $R - C_{\bar{R}}$ (considered as a subregion of R')

from the base point p , around the punctures in D_1 and D_2 , respectively, and back to p along B_1 in $R - C_{\bar{R}}$.

(iii) They satisfy the single relation

$$\prod_{i=1}^{g-1} [A_i' \cdot B_i' \cdot A_i'^{-1} \cdot B_i'^{-1}] \cdot C_1 \cdot C_2 = 1.$$

With the marking induced by these generators we get a point \bar{R}' of $T_{g-1,2}$, and defining F_1 by $F_1(\bar{R}) = \bar{R}'$ for every $\bar{R} \in S_c$, we have a mapping from S_c into $T_{g-1,2}$.

Next we define a mapping F_2 from S_c into the upper half plane $U = \{ \text{Im } z > 0 \}$ so that for every $\bar{R} \in S_c$

$$\begin{aligned} \text{Re } F_2(\bar{R}) &= \frac{2}{\| \theta_{c, \bar{R}} \|^2} \cdot \text{Re} \int_{B_1} \theta_{c, \bar{R}}, \quad \text{and} \\ \text{Im } F_2(\bar{R}) &= m_{\bar{R}}, \end{aligned}$$

where $m_{\bar{R}}$ denotes the modulus of $W_{\bar{R}}$.

Thus we have a mapping $F = (F_1, F_2)$ from S_c into the product space $T_{g-1,2} \times U$.

Theorem 4. The mapping F is a homeomorphism of S_c onto $T_{g-1,2} \times U$. In particular S_c is simply connected.

Proof is omitted (cf. [12]).

Remark. Generally S_c does not coincide with the whole space T_g . However we can see that $S_c = T_{1,1}$ for every non-dividing simple closed curve c on $T_{1,1}$.

§ 3. The augmented Teichmüller spaces.

As for the boundaries of the Teichmüller spaces many new investigations have been made since a series of studies by Bers [5], Maskit [10], and Abikoff [1]. In view of studying the limits in deformations of Riemann surfaces, here we shall consider as the boundary of T_g the set of marked closed Riemann surfaces with nodes of (arithmetic genus g) defined by Bers [6], which correspond to regular b -groups (cf. [1]). We denote by \hat{T}_g the set obtained from T_g by adding such points, and call it the augmented Teichmüller space for genus g .

Now let R be a Riemann surface with nodes, and $N(R)$ be the set of nodes of R . Put $R' = R - N(R)$. For two marked Riemann surfaces \bar{R}_1 and \bar{R}_2 (possibly with nodes), a deformation $\langle \bar{R}_1, \bar{R}_2, f \rangle$ is, by definition, a continuous surjection f from R_1 to R_2 which preserves the marking, and satisfies the following conditions ;

- (i) $f^{-1}|_{R_2'}$ (the restriction of f^{-1} on R_2') is a homeomorphism into R_1 , and
- (ii) for any node p in $N(R_2)$, $f^{-1}(p)$ is either a node of R_1 or a simple closed curve on R_1 .

In terms of deformations we can say that the augmented Teichmüller space \hat{T}_g is the set of marked closed Riemann surfaces \bar{R} (possibly with nodes) for which there exists a deformation $\langle \bar{R}^*, \bar{R}, f \rangle$ from the base point \bar{R}^* of T_g .

Definition. Following Abikoff [2] we introduce on \hat{T}_g the conformal topology as follows: First for every $\bar{R} \in \hat{T}_g$ we set

$$D(\bar{R}) = \{ \bar{S} \in \hat{T}_g : \text{For two deformations } \langle \bar{R}^*, \bar{R}, f \rangle \\ \text{and } \langle \bar{R}^*, \bar{S}, f_1 \rangle, \text{ there is a de-} \\ \text{formation } \langle \bar{S}, \bar{R}, f_2 \rangle \text{ such that} \\ f = f_2 \circ f_1. \}.$$

Next given a neighbourhood K of $N(R)$ in R and a positive ε , we define a (K, ε) -conformal neighbourhood $N_{K, \varepsilon}$ of \bar{R} in \hat{T}_g by the set

$$\{ \bar{S} \in D(\bar{R}) : \text{There is a deformation } \langle \bar{S}, \bar{R}, f \rangle \\ \text{such that } f^{-1}|_{(R-K)} \text{ is a } (1+\varepsilon)\text{-quasi-} \\ \text{conformal mapping into } S. \}.$$

Taking the system of $N_{K, \varepsilon}$ for arbitrary K and ε as above as a fundamental neighbourhood system of \bar{R} , we have a topology on \hat{T}_g , which we call the conformal topology on \hat{T}_g .

The conformal topology restricted on T_g is equivalent with the usual Teichmüller topology, and it satisfies the first countability axiom.

§ 4. The fine topology.

1. Let c be a non-dividing simple closed curve on T_g , which is fixed once for all. And set

$$\partial_c T_g = \{ \bar{R} \in \hat{T}_g : N(R) \text{ consists of one point } p \\ \text{and for the deformation } \langle \bar{R}^*, \bar{R}, \\ f \rangle, f^{-1}(p) \text{ is freely homotopic} \\ \text{to } c. \},$$

and

$$\hat{T}_g = T_g \cup \partial_c T_g.$$

Obviously $D(\bar{R}) = \hat{T}_g$ for every $\bar{R} \in \partial_c T_g$. Also note that this boundary space $\partial_c T_g$ for c is naturally identified with $T_{g-1,2}$. We denote this identifying mapping from $\partial_c T_g$ onto $T_{g-1,2}$ by J . Then J induces a topology on $\partial_c T_g$ from the Teichmüller topology on $T_{g-1,2}$, which we call also the Teichmüller topology.

Remark. The conformal topology restricted on $\partial_c T_g$ is equivalent with the Teichmüller topology.

Now by using the homeomorphism F defined in § 2, we can introduce a new topology on \hat{T}_g .

Definition. Let $\hat{U} = U \cup \{\infty\}$. The fundamental neighbourhood system of ∞ is defined by $\{ \hat{U}_n \}_{n=1}^{\infty}$, where

$$\hat{U}_n = \{ z \in \hat{U} : z = \infty, \text{ or } \text{Im } z > n. \}.$$

For every $\bar{R} \in \partial_c T_g$ define

$$F(\bar{R}) = (J(\bar{R}), \infty),$$

and we can extend F so that it gives a bijection from \hat{S}_c to

$T_{g-1,2} \times \hat{U}$, where $\hat{S}_c = S_c \cup \partial_c T_g$. Then we can introduce a topology on ${}_c \hat{T}_g$ so that this extended mapping F gives a homeomorphism from \hat{S}_c onto $T_{g-1,2} \times \hat{U}$ and it is equivalent with the Teichmüller topology on T_g (cf. Theorem 4). We call this topology the fine topology on ${}_c \hat{T}_g$.

In other words, we may define this topology by taking, as a fundamental neighbourhood system of each $\bar{R} \in \partial_c T_g$, the system $\{ V_n \}_{n=1}^\infty$, where

$$V_n = \{ \bar{S} \in \hat{S}_c : d(F_1(\bar{S}), F_1(\bar{R})) < \frac{1}{n}, \text{ and} \\ \bar{S} \in \partial_c T_g \text{ (, that is, } F_2(\bar{S}) = \infty \\ \text{or } \text{Im } F_2(\bar{S}) > \log n. \}.$$

Here $d(,)$ denotes the Teichmüller distance on $T_{g-1,2}$.

Theorem 5. The fine topology is finer than the conformal topology restricted on ${}_c \hat{T}_g$.

Proof. From the definition and the above remark, it suffices to prove that a sequence $\{ \bar{R}_n \}_{n=1}^\infty$ in T_g converging to $\bar{R}_0 \in \partial_c T_g$ in the sense of the fine topology converges to \bar{R}_0 also in the sense of the conformal topology. But for such a sequence $\{ \bar{R}_n \}$ we can easily find sequences $\{ c_n \}$ and $\{ f_n \}$ satisfying the condition (C) below, hence the assertion follows from the next lemma.

q.e.d.

Lemma. A sequence $\{ \bar{R}_n \}$ in T_g converges to $\bar{R}_0 \in \partial_c T_g$ in the sense of the conformal topology if and only if for every n there

exists an M -quasiconformal mapping f_n from $R_n - c_n$ into R_0 (, where c_n is a simple closed curve on R_n freely homotopic to c ,) which preserves the marking and satisfies the following condition (C) :

(C) For any neighbourhood K of $N(R_0)$ and positive ϵ , there is an N such that for every $n \geq N$ we have

$$(i) \quad f_n(R_n - c_n) \supset R_0 - K, \quad \text{and}$$

$$(ii) \quad f_n^{-1}|_{(R_0 - K)} \text{ is } (1+\epsilon)\text{-quasiconformal.}$$

Here M is a positive constant independent of n .

Proof. Using the extension theorem on quasiconformal mappings ([9] Theorem II -8-1) we can show the assertion. q.e.d.

Remark. We can see that the conformal topology and the fine topology are equivalent to each other also when they are restricted on the foliage

$$\hat{U}_{\bar{R}'} = \{ \bar{R} \in \hat{S}_c : F_1(\bar{R}) = \bar{R}' \}$$

for each $\bar{R}' \in T_{g-1,2}$. Also note that these topologies restricted on such a foliage corresponds to the Schiffer's variations by attaching a handle.

2. Let p_1 and p_2 be the punctures of $\bar{R}' = J(\bar{R}) \in T_{g-1,2}$ corresponding to C_1 and C_2 respectively for each $\bar{R} \in \partial_c T_g$, and $\phi_{\bar{R}'}$ be the elementary differential of the third kind of \bar{R}' with singularities p_1 and p_2 (cf. [12]). We define

$$\theta_{\bar{R}} = \frac{2 \cdot \theta_{c, \bar{R}}}{\|\theta_{c, \bar{R}}\|^2} \quad \text{for every } \bar{R} \in T_g, \text{ and}$$

$$\theta_{\bar{R}} = \frac{i}{2\pi} \cdot \phi_{\bar{R}}, \quad (\bar{R}' = J(\bar{R})) \quad \text{for every } \bar{R} \in \partial_c T_g.$$

Then we have the following

Theorem 6. Suppose that a sequence $\{\bar{R}_n\}_{n=1}^\infty$ in \hat{T}_g converges to $\bar{R}_0 \in \hat{T}_g$ in the sense of the fine topology. Then there exists a sequence $\{ \langle \bar{R}_n, \bar{R}, f_n \rangle \}_{n=1}^\infty$ of deformations satisfying the following condition: For every neighbourhood K of $N(R_0)$ and positive ε , we can find an N such that

$$(i) \quad f_n^{-1}|_{(R_0 - K)} \text{ is } (1+\varepsilon)\text{-quasiconformal, and}$$

$$(ii) \quad \|\theta_{\bar{R}_n} \circ f_n^{-1} - \theta_{\bar{R}_0}\|_{(R_0 - K)} < \varepsilon$$

for every $n \geq N$. Here if $N(R_0) = \phi$, then we assume that $K = \phi$.

Proof. We can show this theorem by (3) in § 1 and the homeomorphism F in § 2. The details will appear elsewhere. q.e.d.

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