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The simplicity of the Green heart

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Lecture 1 : Groups of prime degree.

The subject of my lectures will be a new technique for studying finite permutation groups that is based on an idea of J. A. Green. I suppose it is best described as an application of modular representation theory to permutation groups: but such a grandiose description is unwise, in that it may frighten some mathematicians, and inaccurate, in that we hardly need anything from the modular representation theory proper. It is really no more than an intelligent use of linear algebra over finite fields.

The following notation is intended to remain fixed throughout these three lectures:

n	a natural number;
Ω	a set with n elements;
G	a permutation group on Ω (thus $G \leqslant \mathrm{Sym}(\Omega)$: the action will usually be assumed to be faithful);
p	a prime number, almost always a divisor of n ;
F	a field of characteristic p .

Generally G will be assumed to be transitive, indeed usually primitive on Ω , and so p will divide $|G|$. Our plan is to study the FG-module $F\Omega$, and to derive information about the action of G on Ω from knowledge of its submodules. To do

this, let

$$T := \{ \sum a_\omega \omega \mid a_\omega \in F \ , \ a_\rho = a_\sigma \quad \text{for all} \quad \rho, \sigma \in \Omega \} ,$$

$$S := \{ \sum a_\omega \omega \mid a_\omega \in F \ , \ \sum a_\omega = 0 \} \ .$$

Then T is obviously a 1-dimensional trivial submodule of $F\Omega$ (the "trace" submodule) and S is a submodule of codimension 1 (the "augmentation"). Moreover, given that p divides n, we have $T \leqslant S$. Following J. A. Green we define

$$H := S/T$$

and call this the heart of the FG-module $F\Omega$.

In order to give you a flavour of the subject and to show that it is not entirely superficial I would like to spend this first lecture proving the simplicity of the heart (remember, a module is said to be simple, or irreducible, if it has no non-zero proper submodules) for groups of prime degree. This was first proved by me in [TTMN (1972)], but is essentially due to Walter Feit [(1966)]. The elegant proof that I shall give here is due to Michael Klemm [(1977)].

Theorem 1. If G is insoluble and transitive of prime degree (that is, $n = p$), then H is a simple FG-module.

Proof. Let P be a Sylow p-subgroup of G. Certainly p divides $|G|$, but $|G|$ divides $p!$ and so p^2 does not divide $|G|$: thus P has order p and is cyclic, with generator a, say. We can identify Ω with the prime field F_p so that P is the group of translations and $a : \omega \longmapsto \omega + 1 \pmod p$.

Claim 1. There is a unique series

$$\{0\} = V_0 < V_1 < V_2 < \cdots < V_{p-1} < V_p = F\Omega$$

of P-invariant subspaces of $F\Omega$ such that $\dim V_r = r$.

This is elementary linear algebra: we have that $a^p = 1$, so $(a - 1)^p = 0$; on the other hand, because $F\Omega$ is a free FP-module, we have that $(a - 1)^{p-1} \neq 0$; it follows that in any such series $V_r = \ker(a - 1)^r$. It is simply the fact that the Jordan canonical form for a is the matrix

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & 0 \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix} .$$

Next we make an important notational observation: we can identify $F\Omega$ with the function space

$$F^{\Omega} := \{ f : \Omega \longrightarrow F \} .$$

The identification makes the function $f \in F^{\Omega}$ correspond to the linear combination $\sum_{\Omega} f(\omega).\omega \in F\Omega$. The action of G is easily seen to be given by the formula

$$f^g(\omega) = f(\omega g^{-1}) .$$

Notice that F^{Ω} is naturally an algebra under the usual laws of pointwise addition and multiplication of functions (we call this the Wielandt algebra) and that G acts as a group of automorphisms of it: if $f_1, f_2 \in F^{\Omega}$ then

$$f_1^g \, f_2^g = (f_1 f_2)^g$$

for all $g \in G$.

Claim 2. $F^{\Omega} \cong F[x] / (x^p - x)$.

This may be seen as follows. We have identified Ω with F_p

and F_p is of course a subfield of F, so polynomials do give functions on F_p, that is on Ω, to F. In fact

$$\{a_0 + a_1x + \cdots + a_{p-1}x^{p-1} \mid a_i \in F\}$$

is a p-dimensional vector space of functions and so it is all of F^{Ω}. The polynomial x is then the imbedding function $F_p \longrightarrow F$, and so $x^p = x$.

Claim 3. $V_r = \{a_0 + a_1x + \cdots + a_{r-1}x^{r-1} \mid a_i \in F\}$.

For, notice that $x^a = x - 1$ and so if $f(x) \in F^{\Omega}$ then $f(x)^a = f(x-1)$. Thus deg f = deg f^a, and so the space of polynomials of degree at most $r-1$ is P-invariant. Since there is only one P-invariant subspace of dimension r this space must be V_r.

We make one last general observation. For $f_1, f_2 \in F^{\Omega}$ define

$$\langle f_1, f_2 \rangle := \sum_{\omega \in \Omega} f_1(\omega)\, f_2(\omega)$$

Claim 4. The function $\langle\ ,\ \rangle$ is a non-singular, symmetric bilinear form and is G-invariant. Moreover,

$$V_r^{\perp} = V_{p-r} .$$

That $\langle\ ,\ \rangle$ is bilinear and symmetric should be clear. Also, if $f \neq 0$ then there exists $\alpha \in \Omega$ such that $f(\alpha) \neq 0$, and if $h \in F^{\Omega}$ is define by

$$h(\omega) := \begin{cases} 1 & \text{if } \omega = \alpha \\ 0 & \text{if } \omega \neq \alpha \end{cases},$$

then $\langle f, h \rangle = f(\alpha) \neq 0$: thus $\langle\ ,\ \rangle$ is non-singular. It is easy to see that $\langle\ ,\ \rangle$ is G-invariant. Finally, since V_r

is P-invariant so also is $V_r^{\perp}$, and since $\langle\ ,\ \rangle$ is non-singular $\dim V_r^{\perp} = p-r$, so it follows that $V_r^{\perp} = V_{p-r}$.

Up to this point I have simply been describing the notational ideas that were introduced by Wielandt in his Ohio Notes [1969]. Here comes Klemm's neat calculation which is the heart of the proof.

Let X be a G-invariant subspace of $F\Omega$ such that $T < X \leqslant S$ and such that X/T is a minimal G-submodule of the heart S/T. In our situation clearly $T = V_1$, $S = V_{p-1}$ and $X = V_r$ for some r such that $1 < r \leqslant p-1$. Let us assume that the heart is not simple. Then $X < S$ and $r < p-1$. It then follows from Claim 4 that $r < p/2$ because $X^{\perp}$ is also a G-invariant subspace lying properly between V_1 and V_{p-1} , so $X^{\perp} = V_{p-r}$ and, by minimality of X , we have $r \leqslant p-r$.

Now $r \geqslant 2$, so $V_2 \leqslant X$ and $x \in X$. Then $\langle x^g \mid g \in G \rangle$ (here $\langle\ \rangle$ denotes linear span) is a subspace of X and is G-invariant, so by minimality

$$X = \langle x^g \mid g \in G \rangle \ .$$

Consequently there exists $g \in G$ such that

$$x^g = a_0 + a_1 x + \cdots + a_{r-1}x^{r-1} \qquad \text{where } a_{r-1} \neq 0 \ .$$

Suppose, for the moment, that $r \geqslant 3$. Then $x^2 \in X$ and so $(x^2)^g \in X$. But

$$\begin{aligned} (x^2)^g &= (x^g)^2 \\ &= (a_0 + a_1 x + \cdots + a_{r-1}x^{r-1})^2 \\ &= a_0^{\,2} + 2a_0a_1x + \cdots + a_{r-1}^{\,2}\, x^{2r-2} \qquad \text{(since } 2r-2 < p\text{)} \\ &\notin V_r \qquad\qquad\qquad\qquad\qquad\quad \text{(since } 2r-2 > r\text{).} \end{aligned}$$

This is impossible. Therefore $r = 2$ and for all $g \in G$ there exist a_0, a_1 in F such that

$$x^g = a_0 + a_1 x \quad ,$$

and so G is soluble. Thus if G is insoluble then the heart of $F\Omega$ is simple.

Lecture 2 : Consequences of simplicity.

Let me begin this lecture with some generalities about modules and permutation groups. If X, Y are FG-modules we put

$$i(X, Y) := \dim_F \hom_{FG}(X, Y) \quad .$$

Lemma. If $U \leqslant X$ and $V \cong X/U$ then

$$i(X, Y) \leqslant i(U, Y) + i(V, Y) \quad .$$

This comes directly from the fact that the short exact sequence

$$0 \longrightarrow U \longrightarrow X \longrightarrow V \longrightarrow 0$$

gives the exact sequence

$$0 \longrightarrow \hom_{FG}(V,Y) \longrightarrow \hom_{FG}(X, Y) \longrightarrow \hom_{FG}(U, Y) \quad .$$

Corollary. If $0 = X_0 \leqslant X_1 \leqslant X_2 \leqslant \cdots \leqslant X_k = X$ then

$$i(X, Y) \leqslant \sum_{r:=1}^{k} i(X_{r/X_{r-1}}, Y) \quad .$$

The connection between this very general module theoretic concept and permutation groups is the following

Lemma. If Ω_1 and Ω_2 are G-spaces then

$$i(F\Omega_1, F\Omega_2) = \#\,\mathrm{orb}\,(G, \Omega_1 \times \Omega_2) \quad .$$

One can prove this as follows. For each orbit Δ of G in $\Omega_1 \times \Omega_2$ define

$$\varphi_\Delta : F\Omega_1 \longrightarrow F\Omega_2$$

by

$$\varphi_\Delta : \omega_1 \longmapsto \sum_{\omega_2 \in \Delta(\omega_1)} \omega_2 \quad ,$$

where $\Delta(\omega_1) := \{\omega_2 \in \Omega_2 \mid (\omega_1, \omega_2) \in \Delta\}$, for all $\omega_1 \in \Omega_1$.

One then needs to show — and I leave this to you as an exercise — that these maps φ_Δ are G-homomorphisms, that the different

φ_Δ as Δ ranges over the G-orbits in $\Omega_1 \times \Omega_2$ are linearly independent, and that they span $\hom_{FG}(F\Omega_1, F\Omega_2)$. The lemma has two well-known and important special cases (each of which can, of course, be proved very easily directly):

Corollaries. (i) $i(F, F\Omega) = \#\text{orb}\ (G, \Omega)$;

(ii) Our group G is 2-transitive on Ω if and only if $\dim_F \text{end}_{FG}\ (F\Omega) = 2$.

Notice that, if F has characteristic zero then these may be interpreted character theoretically and we retrieve the famous enumeration theorems of Frobenius (1887). For example, (i) becomes the formula

$$\frac{1}{|G|} \sum_{g \in G} |\text{fix}_\Omega(g)| = \#\text{orb}\ (G, \Omega) ,$$

which is nowadays so often attributed incorrectly to Burnside.

This is enough of preparation. We can come now to some consequences of the simplicity of the heart. I shall assume in what follows

(1) that G is transitive on Ω ;

(2) that $n \geq 4$ and p divides n ; and

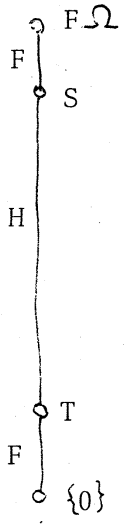
(3) that the heart H of $F\Omega$ is a simple module.

Then we have the

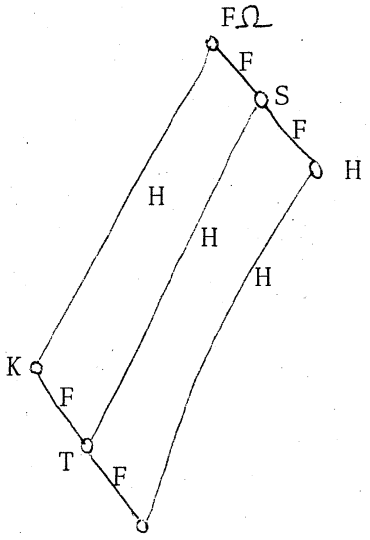
Proposition. Either G is 2-fold transitive or $p = 2$, $n = 2m$ where m is odd, and G is an imprimitive group having two blocks of imprimitivity of size m (and is of rank 3).

Proof. Since the module $F\Omega$ is self-dual (the fourth "Claim" in my argument in the first lecture was actually a proof

of this) so also is H, and moreover, H is a submodule of $F\Omega$ if and only if it is also a quotient module. Consequently there are only two possibilities for the structure of $F\Omega$: either T is the unique minimal submodule, in which case S is the unique maximal submodule; or $F\Omega$ has a submodule, hence also a quotient module, isomorphic to H, and there is then a 2-dimensional submodule K such that $F\Omega \cong K \oplus H$. The lattices of submodules of $F\Omega$ in the two cases look like this:



Case (1)



Case (2)

In case (1) it is very easy to see that dim end $(F\Omega) = 2$, and so G is doubly transitive on Ω.

In case (2) the linear transformations induced by G on K can be described by matrices of the form $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$, and so this group is an elementary abelian p-group. Therefore the derived group G' acts trivially on K and, as it is easy to see that $\mathrm{fix}_H(G') = \{0\}$, we have that

$$\dim_F \hom_{FG'}(F, F\Omega) = 2 ,$$

so G' has precisely 2 orbits Ω_1 and Ω_2 in Ω. These are blocks of imprimitivity for G and if N is the stabiliser of these two sets then N must have index 2 in G. Since N acts trivially on K it follows that $p = 2$. Also, H must split as a direct sum of two N-modules H_1 and H_2 (which, by Clifford's Theorem, must be irreducible), such that

$$\left.\begin{aligned} F \oplus H_1 &\cong F\Omega_1 \\ F \oplus H_2 &\cong F\Omega_2 \end{aligned}\right\} \quad \text{as FN-modules,}$$

and it follows that $n/2$ must be odd (also, that N is 2-transitive on Ω_1 and Ω_2, and that G has rank 3 on Ω (which is equivalent to the fact that N_α is still transitive on Ω_2 for $\alpha \in \Omega_1$)).

It is perhaps worth observing that the wreath product $S_m$wr S_2 when m is odd and $n = 2m$, is an example that illustrates case (2).

The significance of the proposition is that, in general, simplicity of the heart implies double transitivity: in particular, this is so if G is primitive, or if p is odd, or if $n \not\equiv 2 \pmod 4$. We should not expect the converse to be true, and indeed it is not: the affine groups $AGL(d, p^r)$ give the easiest counter-examples. Nevertheless, for the known doubly transitive groups it turns out surprisingly often that the heart is simple, although there are some interesting cases where it is not. Dr. Brian Mortimer (presently at Carleton University, Ottawa, Canada) has undertaken an extensive study of this phenomenon.

As a corollary of this elementary proposition and the theorem that I proved in the first lecture we have

Burnside's Theorem : An insoluble transitive group of prime degree is doubly transitive.

I feel that simplicity of the heart (when p divides n) is generally rather stronger than double transitivity. The following theorem, whose proof follows ideas of J. A. Green very closely (cf. also Klemm [1977]), is intended to illustrate what I mean.

Theorem 2. Suppose

(i) that the heart H of $F\Omega$ is a simple module and that p is odd ;

(ii) that G contains a transitive (hence regular) abelian subgroup A ; and

(iii) that G contains an element b such that $b^{-1}ab = a^{-1}$ for all $a \in A$.

The conclusion is : that G is 3-fold transitive.

Proof. I will say that the FG-module X is

negative if $\mathrm{fix}_X(A) \neq \{0\}$ and $xb = -x$ for all $x \in \mathrm{fix}_X(A)$;

positive if $\mathrm{fix}_X(A) \neq \{0\}$ and $xb = x$ for all $x \in \mathrm{fix}_X(A)$;

mixed otherwise.

Now we have

Observation 1 : The heart H is a negative module.

For this we need to see first that $\mathrm{fix}_H(A) \neq \{0\}$. Let P be the p-primary constituent of A and Q the maximal p'-subgroup, so that $A = P \times Q$. If

$$u := \sum_{x \in Q} x \in FA$$

and $U := (F\Omega)u$, then U is an A-invariant subspace of $F\Omega$ of dimension $|P|$, on which Q acts trivially and P acts regularly. Certainly $T \leqslant U$ and, if V/T is a minimal non-zero FA-submodule of U/T then both P and Q centralise V/T, so $V/T \leqslant \mathrm{fix}_H(A)$. Hence $\mathrm{fix}_H(A) \neq \{0\}$. Now let $x \in \mathrm{fix}_H(A)$. We write $x = y + T$ for some $y \in F\Omega$. If $a \in A$ then $xa = x$, and so there exists $z_a \in T$ such that

$$ya = y + z_a .$$

Then also $$ya^{-1} = y - z_a .$$

Therefore

$$\begin{aligned}(y + yb)a &= ya + ya^{-1}b \\ &= (y + z_a) + (y - z_a)b \\ &= y + yb .\end{aligned}$$

This is true for all a in A, that is, $y + yb \in \mathrm{fix}_{F\Omega}(A)$, and so, as A is transitive, $y + yb \in T$. Hence $x + xb = 0$ in $F\Omega/T$ and $xb = -x$, which is what I promised to show.

Observation 2 : The module $F\Omega^{\{2\}}$ is positive, where $\Omega^{\{2\}}$ denotes the G-space consisting of unordered pairs of elements of Ω.

To see this, let $D := \mathrm{gp}(A, b)$. Since A acts regularly on Ω we have that $C_G(A) = A$, and since b^2 centralises A therefore $b^2 \in A$. Hence $|D : A| = 2$. Let $\alpha \in \Omega$. The stabiliser D_α is a complement for A in D and without loss

of generality we may suppose that $b \in D_\alpha$: thus b is an involution and $\alpha \in \text{fix}\,(b)$. Now let Γ be an A-orbit in $\Omega^{\{2\}}$. Then Γ contains $\{\alpha, \beta\}$ for some β in Ω ; now $\beta = \alpha a$ for some $a \in A$; and

$$\begin{aligned}\{\alpha, \beta\}\, b &= \{\alpha b,\ \alpha ab\} \\ &= \{\alpha,\ \alpha b a^{-1}\} \\ &= \{\alpha,\ \alpha a^{-1}\} \\ &= \{\alpha a^{-1},\ \alpha\} \\ &= \{\alpha,\ \alpha a\}\, a^{-1} \\ &= \{\alpha, \beta\}\, a^{-1} \ ;\end{aligned}$$

thus $\{\alpha, \beta\}\, b \in \Gamma$. Since b normalises A it acts to permute the A-orbits in $\Omega^{\{2\}}$ and so the fact that $\Gamma \cap \Gamma b \neq \phi$ implies that $\Gamma = \Gamma b$. Thus every A-orbit in $\Omega^{\{2\}}$ is a D-orbit, and it follows that

$$\text{fix}_{F\Omega^{\{2\}}}(D) = \text{fix}_{F\Omega^{\{2\}}}(A) \ .$$

Hence $x = xb$ for all $x \in \text{fix}_{F\Omega^{\{2\}}}(A)$, and so $F\Omega^{\{2\}}$ is positive, as I claimed.

The proof of the theorem now goes as follows. Since H is simple a consequence of our two observations is that

$$i(H,\ F\Omega^{\{2\}}) = 0 \ .$$

By the corollary to our first lemma we have

$$i(F\Omega\ ,\ F\Omega^{\{2\}}) \leqslant 2\, i(F,\ F\Omega^{\{2\}}) + i(H,\ F\Omega^{\{2\}}) \ .$$

Now G is doubly transitive (by the proposition), so it is transitive on $\Omega^{\{2\}}$ and hence $i(F, F\Omega^{\{2\}}) = 1$. Therefore

$$i(F\Omega, F\Omega^{\{2\}}) \leqslant 2$$

and, by the second lemma, this means that

$$\#\,\mathrm{orb}\,(G, \Omega \times \Omega^{\{2\}}) \leqslant 2 .$$

Let

$$\textcircled{H} := \{ (\omega_1, \{\omega_1, \omega_2\}) \mid \omega_1, \omega_2 \in \Omega, \ \omega_1 \neq \omega_2 \}$$

and

$$\Phi := \{ (\omega_1, \{\omega_2, \omega_3\}) \mid \omega_1, \omega_2, \omega_3 \text{ all different} \} .$$

Then $\textcircled{H}$ and Φ are G-invariant subsets of $\Omega \times \Omega^{\{2\}}$. Since G has only two orbits on this set it follows that it is transitive on Φ. Let (β, γ) be a transposition of the involution b (which, recall, has α as a fixed point). Then, given ω_1, ω_2, ω_3, distinct members of Ω, there exists g in G such that

$$(\omega_1, \omega_2, \omega_3)g = \begin{cases} (\alpha, \beta, \gamma) & \underline{\text{or}} \\ (\alpha, \gamma, \beta) , \end{cases}$$

and so either g or gb takes $(\omega_1, \omega_2, \omega_3)$ to the triple (α, β, γ). Hence G is transitive on ordered triples and this is what we wanted to prove.

As a corollary, or perhaps a special case, we have the

Theorem (TMN [1972]). If G is an insoluble transitive group of prime degree p, and if the normaliser of a Sylow p-subgroup has even order, then G is 3-fold transitive.

Lecture 3 : Groups containing regular abelian subgroups.

In this last lecture we shall return to the theme of my first talk, and variations on it.

Theorem 3. (Klemm [1977]). If G is primitive and contains a transitive cyclic subgroup A, and if $n = p^m$ where $m \geqslant 2$, then the heart of $F\Omega$ is simple.

Proof. (L. L. Scott). Let a be a generator of A, so that a is a cyclic permutation $(\omega_0, \omega_1, \cdots, \omega_{p^m-1})$. As FA-module $F\Omega$ is uniserial : if $V_r := \ker (a - 1)^r$ then

$$\{0\} = V_0 < V_1 < V_2 < \cdots < V_{p^m} = F\Omega$$

is an FA-composition series, and it is unique: if U is any A-invariant subspace then $U = V_r$ where $r = \dim U$ (compare "Claim 1" in the first lecture).

Let X be a G-invariant subspace with $T \leqslant X \leqslant S$. Then of course X is A-invariant and so $X = V_r$ for some r in the range $1 \leqslant r \leqslant p^m - 1$.

Now let P^* be a Sylow p-subgroup of Sym (Ω) such that $A \leqslant P^*$. Certainly $F\Omega$ is an FP^*-module, and, as P^* is a p-group and char $F = p$, any composition series has 1-dimensional trivial factors. Such a composition series consists of A-invariant subspaces and so, since the FA-composition series is unique we see that

$$\{0\} = V_0 < V_1 < V_2 < \cdots < V_{p^m} = F\Omega$$

is the unique FP^*-composition series also. Consequently X is

P^*-invariant, and, if $G^* := \langle G, P^* \rangle \leq \mathrm{Sym}(\Omega)$, then X is G^*-invariant.

The group G^* is certainly primitive since $G^* \geq G$; and G^* contains a p-cycle having $p^m - p$ fixed points because P^* does. By a well-known theorem of Jordan (see, for example, Wielandt [1964], Theorems 13.2 or 13.9) it follows that $G^* \geq \mathrm{Alt}(\Omega)$. It is quite easy to show, by an inductive argument which I leave to you as an exercise (but be careful: since the induction is on n you will want to define the "heart" also in case $n \not\equiv 0 \pmod{p}$), that the heart of $F\Omega$ is simple as a module for $F.\mathrm{Alt}(\Omega)$. Hence $X = S$ or $X = T$ and so S/T is a simple FG-module as I claimed.

From the results of Lecture 2 we now have

Corollaries : (1) (Burnside, 1901). A primitive group of degree p^m (where $m \geq 2$) that contains a p^m-cycle is 2-transitive.

(2) (Klemm [1977]). With the hypotheses of the theorem, if there exists $b \in G$ such that $b^{-1}ab = a^{-1}$, and if $p > 2$, then G is 3-transtivie.

Burnside, on p.343 of the second edition of his book [1911], suggests that a theorem similar to the first corollary should be true for any primitive group G that contains a transitive abelian subgroup A. Although this is certainly false in general many interesting special cases have been proved, and work on it by Schur, Wielandt, Kochendörffer and Bercov has served to develope a very useful technique known as the theory of Schur-rings or S-rings. (See Wielandt [1964] for a survey). Two of the older

theorems in the area will serve very well as illustrations:

in 1933 Schur proved that if A is cyclic then G is 2-transitive (unless $n = p$ and G is soluble);

in 1935 Wielandt proved that if A has a cyclic Sylow p-group for some prime p (and $n \neq p$) then G is 2-transitive.

The module theory that I have been discussing throws new light on these theorems for we have the

Almost Theorem. If G is primitive and contains a transitive abelian subgroup A, if the Sylow p-subgroup of A is cyclic, and if $n > p$, then the heart of $F\Omega$ is simple (over any field F of characteristic p).

Using the method of proof of Theorem 3 I have proved this statement in case the Sylow p-subgroup of A has order p^m where $m \geqslant 2$, but I have not yet been able to complete the proof in case $m = 1$.

To finish these lectures I would like to mention to you two related matters where investigation of the Green heart is turning out to be instructive and profitable.

Mr. Prabir Bhattacharya, a student at Oxford, is, I believe, very close now to proving that if G is primitive of degree p^2 then the heart of $F\Omega$ is simple unless G is similar to a subgroup of the wreath product S_p wr C_2 or to a subgroup of the affine group $AGL(2, p)$. His proof is based on the methods of Wielandt [1969] and Klemm [1977]. It would be very interesting

to know what can be done for groups of higher prime-power degrees.

The second question is this: is the heart simple for a 2-transitive group of degree 3p ? In 1975 Jan Saxl and I showed that the heart is simple when G is 2-transitive of degree 2p . Our proof is quite short, but is based on consideration of the so-called "Brauer Tree" for the principal p-block of G : thus it uses some of the deeper theory of modular representations, and is therefore very different from the kind of argument that I have shown you in these lectures. Dr. Karin Erdmann, Professor Harvey Blau and I have tried to do a similar thing for groups of degree 3p , but the question appears to be related to problems about groups having several low-dimensional representations in characteristic p (where 'low' here means something like 'less than 2p'), and this is a subject of notorious difficulty. So, although my question about groups of degree 3p is a very special one, I see it as a test question which could provide a focus for further work on the modular representations of permutation groups.

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