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Representations of finite Chevalley groups
(after G. Lusztig)

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§1. Lefschetz number.

Let $X(\subset K^N)$ be an affine algebraic variety defined over $K = \bar{\mathbb{F}}_p$, and g an automorphism of X of finite order. Then, there exists a power q of p such that both X and g are defined over $\mathbb{F}_q$. If $F : X \longrightarrow X$ is the Frobenius map $(x_1, \ldots, x_N) \longmapsto (x_1^q, \ldots, x_N^q)$ of X, $|X^{F^n g}| < \infty$ for $n = 1, 2, \ldots$. (If σ is a transformation of a set S, S^σ denotes the set of σ-fixed points of S.) We consider the formal power series:

$$f_{X,g}(t) = -\sum_{n=1}^{\infty} |X^{F^n g}| t^n.$$

Theorem(Dwork). $f_{X,g}$ is a rational function in t with rational coefficients. Moreover, it has only simple poles and no pole at $t = \infty$.

Now, we define the Lefschetz number $L(g, X)$ by

(1) $$L(g, X) = f_{X,g}(\infty).$$

It can be shown that $L(g, X)$ is independent of the choice of rational structure of X.

There is another definition of $L(g, X)$ which is due to Grothendieck. Fix a prime number ℓ different from p. Let $H_c^i(X, \bar{\mathbb{Q}}_\ell)$ $(0 \leqq i \leqq 2 \dim X)$ be the ℓ-adic étale cohomology group of X with compact support. Then, since g is an automorphism of X, g acts on $H_c^i(X, \bar{\mathbb{Q}}_\ell)$ as a linear transformation, and L(g, X) can be expressed as

$$(2) \qquad L(g, X) = \sum_i (-1)^i \mathrm{Tr}(g, H_c^i(X, \bar{\mathbb{Q}}_\ell)).$$

(The equivalence of the formulae (1) and (2) follows from the Lefschetz fixed points formula:

$$|X^{F^n g}| = \sum_i (-1)^i \mathrm{Tr}(F^n g, H_c^i(X, \bar{\mathbb{Q}}_\ell)).)$$

By (1) and (2), we have:

Theorem. (a) L(g, X) is a rational integer. (b) If G is a finite group of automorphism of X, the $\mathbb{Z}$-valued function $L(\cdot, X): g \longmapsto L(g, X)$ on G is a generalized character of G.

§2. The Deligne-Lusztig construction (Ann. Math. 103, 1976).

Notations.

G = a connected reductive linear algebraic group defined over $\mathbb{F}_q$,

F = the corresponding Frobenius map of G,

T = a F-stable maximal torus of G,

θ = a $\mathbb{C}$-valued character of T^F,

B = a Borel subgroup containing T (not F-stable, in general),

U = the unipotent radical of B.

Let $X = \{g \in G \mid g^{-1}F(g) \in U\}$. This is an affine variety. The finite group $G^F \times T^F$ acts on X by

$$(g_0, t)g = g_0 g t^{-1} \qquad (g_0 \in G^F,\ t \in T^F,\ g \in X).$$

Hence, by the results in section 1,

$$(g_0, t) \longmapsto L((g_0, t), X)$$

is a generalized character of $G^F \times T^F$. Hence, if we define a $\mathbb{C}$-valued function $R_{T,\theta}^G$ on G^F by

$$R_{T,\theta}^G(g_0) = |T^F|^{-1} \sum_{t \in T^F} L((g_0, t)), X)\,\theta(t) \qquad (g_0 \in G^F),$$

this is a generalized character of G^F. (It can be shown that $R_{T,\theta}^G$ is independent of the choice of B.) In particular, if B is F-stable, $R_{T,\theta}^G = \mathrm{ind}_{B^F}^{G^F}(\tilde{\theta})$, where $\tilde{\theta} : B^F \longrightarrow B^F/U^F \cong T^F \xrightarrow{\theta} \mathbb{C}^*$.

In a sense, $R_{T,\theta}^G$'s are almost always irreducible. This can be seen from the following

Theorem. Let T and T' be F-stable maximal tori of G, and θ and θ' characters of T^F and T'^F respectively. Then,

$$\langle R_{T,\theta}^G, R_{T',\theta'}^G \rangle_{G^F} = \#\{w \in W(T, T')^F \mid \theta = \theta' \circ w\},$$

where $W(T, T') = T\ \{n \in G \mid n^{-1}T_n = T'\}$. In particular,

(a) If T and T' are not G^F-conjugate the above inner product is zero.

(b) If θ is in general position (i.e. $\{w \in W(T, T) \mid \theta \circ w = \theta\} = \{1\}$), then $R_{T,\theta}^G$ is irreducible up to signature.

Next, we are interested in the values of $R_{T,\theta}^G$.

Definition. Define a $\mathbb{C}$-valued function Q_T^G on the set of unipotent elements of G^F by:

$$Q_T^G(u) = R_{T,\theta}^G(u).$$

(It can be shown that the right hand side is independent of $\theta \in \hat{T}^F$.) Q_T^G's are called Green functions of G.

Using Green functions, we can state the following:

Theorem. Let $x = su = us$ be the Jordan decomposition of an element x of G^F (s = the semisimple part u = the unipotent part). Then,

$$R_{T,\theta}^G(x) = |Z_G^0(s)^F|^{-1} \sum_g Q_{g^{-1}Tg}^{Z^0(s)}(u)\theta(gsg^{-1}),$$

where the sum is over the set of $g \in G^F$ such that gsg^{-1} T and $Z_G^0(s)$ = the connected component of the centralizer of s in G. (It is known that $Z_G^0(s)$ is reductive. Hence $Q_{g^{-1}Tg}^{Z^0(s)}$ means Green function of $Z_G^0(s)$.)

The above theorem tells us that if one wants to know the values of $R_{T,\theta}$, one must calculate Q_T^G. Here are some known results about Q_T^G:

(a) $G = GL_n$. In this case, J. A. Green (Trans. A. M. S. 80, 1955) proved that $Q_T^G(u)$'s are polynomials in q and showed how to calculate them.

(b) $G = U_n$. R. Hotta and T. A. Springer (Inv. Math. 41, 1977) proved that $Q_T^{U_n}(u)(q) = Q_T^{GL_n}(u)(-q)$.

(c) G. Lusztig (Proc. London Math. Soc., 33, 1976) calculated

Q_T^G for the Coxeter torus T of the symplectic and orthogonal groups G.

§3. Classification of irreducible representations of finite classical groups.

Let G and T be as in section 2. We denote by X(T) the Z-module Hom(T, K*), and by Y(T) its dual Hom(K*, T). The Frobenius F acts on X(T) and Y(T) by

$$F(\alpha)(Ft) = \alpha(t)^q \qquad (\alpha \in X(T),\ t \in T)$$

and

$$F(h(x)) = (Fh)(x^q) \qquad (h \in Y(T),\ x \in K^*)$$

respectively. In X(T) (resp. Y(T)), we have the root system Σ (resp. coroot system Σ^*) of G with respect to T.

Definition. A connected reductive group G* defined over $\mathbb{F}_q$ is called the dual group of G if it has a F-stable maximal torus T* such that

(i) $X(T^*) \cong Y(T)$

and $Y(T^*) \cong X(T)$,

(ii) The action of Frobenius F on X(T*) and Y(T*) is given via the above isomorphism,

(iii) The root (resp. coroot) system of G* with respect to T* is Σ^* (resp. Σ).

Remarks. (a) G and its dual G* have the same Weyl group.

(b) $(G^*)^* \cong G$.

(c) There is a natural bijective correspondence between {F-stable max. tori of G}/G^F-conj. and {F-stable max. tori of G^*}/G^{*F}-conj. .

Examples.

G	G^*
GL_n	GL_n
U_n	U_n
Sp_{2n}	SO_{2n+1}
$SO^{\pm}_{2n}$	$SO^{\pm}_{2n}$
CSp_{2n}	G^0_{2n+1} } the connected compo. of Clifford groups
$CO^{\pm,0}_{2n}$	$G^{\pm,0}_{2n}$ } the connected compo. of Clifford groups

Here, $CSp_{2n} = \{g \in GL_{2n} \mid {}^t gJg = (\text{scalar})\cdot J\}$ for skew symmetric J.

Proposition. There exists a bijective correspondence between

$$\{(T, \theta) \mid T\text{: F-stable max. tori of } G,\ \theta \in \widehat{T^F}\}/G^F\text{-conj.}$$

and

$$\{(T', s) \mid T'\text{: F-stable max. tori of } G^*,\ s \in T'^F\}/G^{*F}\text{-conj.}$$

Hence, we can write $R_{T',s}$ for $R_{T,\theta}$ if $(T', s) \longleftrightarrow (T, \theta)$.

Theorem. Assume that the center of G is connected. For semisimple $s \in G^{*F}$, put

$$\rho_s = (-1)^{\sigma(G)-\delta_s} \sum_{\substack{(T',s') \bmod G^{*F} \\ s' \underset{G^{*F}}{\sim} s}} \frac{1}{\langle R_{T',s'}, R_{T',s'}\rangle_{G^F}} R_{T',s'},$$

where $\sigma(G)$ = the split rank of G and δ_s = the split rank of $Z_{G*}(s)$. Then,

(i) ρ_s are irreducible characters of G^F,

(ii) $\rho_s = \rho_{s'}$ if and only s and s' are $G*^F$-conjugate,

(iii) $\dim \rho_s = \left(\frac{|G*^F|}{|Z^0_{G*}(s)^F|} \right)_{p'}$

($(\cdot)_{p'}$ means the p'-part of $(\cdot)$.),

(iv) $\{\rho_s \mid s$ semisimple in $G*^F\}$ exhausts all the irreducible characters of G^F whose dimensions are prime to p.

Theorem. (i) For any irreducible character ρ of G^F, there exists some $R_{T,s}$ such that $\langle \rho, R_{T,s} \rangle_{G^F} \neq 0$.

(ii) If there exists an irreducible character ρ of G^F such that $\langle \rho, R_{T_1,s_1} \rangle \neq 0$ and $\langle \rho, R_{T_2,s_2} \rangle \neq 0$, then s_1 and s_2 are $G*^F$-conjugate.

Hence we have a surjection:

ϕ : {irreducible char. of G^F} $\longrightarrow$ {semisimple conj. class of $G*^F$}

defined by

$$\phi(\rho) = s \text{ such that } R_{T,s} \quad \rho \text{ for some } T.$$

(Note that $\phi^{-1}(s) \ni \rho_s$ for any semisimple s.)
Next we want to know what $\phi^{-1}(s)$ looks like. To answer this question we need the following notion:

Definition. An irreducible character ρ of G^F is called unipotent if $\langle \rho, R_{T,1} \rangle \neq 0$ for some T.

Theorem. (C. R. Acad. Sci. Paris 284, 1977, pp. 493-495; Irred. repr. of finite classical groups (to appear)). Let G be one of the following groups: GL_n, U_n, CSp_{2n}, $CO_{2n}^{\pm,0}$, SO_{2n+1} (These groups have connected center.). Let s be a semisimple element of G^{*F}. Then $\phi^{-1}(s)$ is in bijective correspondence with the set of unipotent irreducible characters of $(Z_{G^*}(s))^{*F}$ in such a way that

$$\dim \rho = (\dim \sigma) \times (\dim \rho_s)$$

if $\rho \in \phi^{-1}(s)$ corresponds to the unipotent irreducible character σ of $(Z_{G^*}(s))^{*F}$:

Remarks. (a) Lusztig also calculated the dimensions of the unipotent irreducible characters of classical groups. Hence the dimensions of the irreducible characters of the groups mentioned in the above theorem are now known.

(b) Lusztig conjectures that the above theorem will be true for any reductive G with connected center.

§4. A conjecture.

Lusztig has done much, but there remain many open problems. For example, the explicit calculation of Green functions, determination of values of unipotent characters and so on. Hotta and Springer (loc. cit) has proved the so called Ennola conjecture for finite unitary groups $U(n, q^2)$ for $p \geq n$, q large. Their result says that 'the character table of $U(n, q^2)$ can be obtained from that of $GL(n, q)$ by changing q into $-q$.'

By inspecting the character tables of $Sp_4(q)$ (Srinivasan, Enomoto) and $G_2(q)$ (Chang-Ree, Enomoto), one sees that the same principle also holds between $Sp_4(q)$ (resp. $G_2(q)$) and itself. Hence, although we have no rigorous formulation, we are tempted to believe that 'the generalized Ennola principle' holds for any finite reductive groups. The results of Lusztig in section 3 also give support to this 'conjecture'.