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Doubly transitive but not doubly primitive permutation groups

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My research interest in doubly transitive permutation groups which are not doubly primitive arose from some beautiful results of Michael O'Nan and a special problem of my own. Suppose that G is a 2-transitive permutation group on Ω, and that for $\alpha \in \Omega$, G_α has a nontrivial normal subgroup N. Michael O'Nan [2,3] showed that, if N satisfies any one of the following three properties, $PSL(n,q) \le G \le P\Gamma L(n,q)$ in its representation on the points or hyperplanes of the projective space, where $n \ge 3$ and q is a power of a prime.

(i) N is abelian and is not semiregular on $\Omega - \{\alpha\}$.

(ii) N is not faithful on its orbits in $\Omega - \{\alpha\}$.

(iii) N is 2-transitive on its orbits in $\Omega - \{\alpha\}$, $|N| > 2$, and N is intransitive on $\Omega - \{\alpha\}$.

Notice that the set Σ of orbits of N in $\Omega - \{\alpha\}$ is a complete set of blocks of imprimitivity for G_α in $\Omega - \{\alpha\}$ such that N is contained in the kernel of the action of G_α on Σ. Thus properties (ii) and (iii) are essentially properties of the kernel of G_α on Σ. Now in order to discuss my problem let us assume.

(*) G is a 2-transitive permutation group on Ω, and for $\alpha \in \Omega$, G_α has a set $\Sigma = \{B_1, \ldots, B_t\}$ of nontrivial blocks of imprimitivity in $\Omega - \{\alpha\}$, where $|\Sigma| = t > 1$, and $|B_i| = b > 1$ for $1 \le i \le t$.

I wanted to know if such a group could exist with $G_\alpha^\Sigma \supseteq A_t$ and t much larger than b. If $t > b + 1$ it is easy to show in this situation that G_α contains a non-identity element fixing B_1 pointwise. I was able to show in [4] :

Theorem 1. If (*) is true, $G_\alpha^\Sigma \supseteq A_t$ where $t \ge 3$, and G_α contains a non-identity element which fixes B_1 pointwise, then $t \le 5$ and G is a collineation group of a Desarguesian projective or affine plane of order $t - 1$.

Thus it seemed that perhaps the group G could be characterised if strong assumptions were made on either the kernel of G_α on Σ, (as O'Nan had done), or the way in which G_α acted on Σ. My best result in this direction is the following.

Theorem 2. ([5,6]) Suppose that (*) is true, G_α is 3-transitive on Σ of degree $t \geq 3$, and G_α contains a non-identity element which fixes B_1 pointwise. If either

(a) G_α is not faithful on Σ, or

(b) G_α is faithful and 3-primitve on Σ,

then G is a collineation group of a Desarguesian projective or affine plane of order $t - 1$.

I conjecture that this result is true with the restrictions (a), (b) removed. Now projective and affine planes are special examples of block designs with parameter $\lambda = 1$, (where by a block design I mean a set of v points and a set of blocks with a relation of incidence between points and blocks such that each block is incident with k points and each pair of distinct points is incident with λ blocks, where $v > k + 1 > 1$, $\lambda > 0$). If $\mathcal{D}$ is a block design with parameters $\lambda = 1$ and $k > 2$, and if G is an automorphism group of $\mathcal{D}$ which is 2-transitive on the points of $\mathcal{D}$, then G is not 2-primitive on points, (for if Δ is a block of $\mathcal{D}$ incident with a point α then the set of points incident with Δ and distinct from α is a nontrivial block of imprimitivity for G_α). Moreover M.D. Atkinson (see [1]) has conjectured that any 2-transitive but not 2-primitive group G either is a normal extension of a Suzuki simple group, or is an automorphism group of a block design with parameter $\lambda = 1$, or has a regular normal subgroup (with restrictions on the degree). With this conjecture in mind we could ask.

Given that (*) is true, under what conditions on G_α^Σ can we conclude that G is an automorphism group of a block design with parameter $\lambda = 1$ in which the blocks containing α are precisely $B_i \cup \{\alpha\}$ for $i = 1, \ldots, t$?

One result which partially answers this question is :

Theorem 3. ([6]) Suppose that (*) is true, that G_α is 2-transitive on Σ, and G_α contains a non-identity element which fixes B_1 pointwise. Then either G is an automorphism group of a block design with parameter $\lambda = 1$, the blocks of which are the translates under G of $B_1 \cup' \{\alpha\}$, or $PSL(n,q) \le G_\alpha^\Sigma \le P\Gamma L(n,q)$ on points or hyperplanes of the projective space, where $n \ge 3$ and q is a power of a prime, and G_α is faithful on Σ.

I conjecture, of course, that the second possibility can be removed. (I have shown in unpublished work that in the second case $t \le bq$.)

References:

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