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Transfer theorems for cohomological G-functors

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Maps and functors are usually on the right. G is a finite group, p a prime, k a commutative ring with identity element. M_R is the category of finite generated right R-modules. A k-algebra is a k-module P with a bilinear multiplication $(\alpha, \beta) \longrightarrow \alpha \cdot \beta$. A G-algebra over k is a k-algebra A, on which G acts as algebra automorphisms.

Definition (Green [2]) A G-functor over M_k is defined to be a quadraple $a = (a, \tau, \rho, \sigma)$, where a, τ, ρ, σ are families of the following kind :

$a = (a(H))$ assigns a k-module $a(H)$ for each $H \le G$;

$\tau = (\tau_H^K)$ assigns a k-homomorphism $\tau_H^K : a(H) \longrightarrow a(K) : \alpha \longrightarrow \alpha^K$ for each pair (H, K) such that $H \le K \le G$;

$\rho = (\rho_H^K)$ assigns a k-homomorphism $\rho_H^K : a(K) \longrightarrow a(H) : \beta \longrightarrow \beta_H$ for each pair (H, K) such that $H \le K \le G$;

$\sigma = (\sigma_H^g)$ assigns a k-homomorphism $\sigma_H^g : a(H) \longrightarrow a(H^g) : \alpha \longrightarrow \alpha^g$ for each pair (H, g) such that $H \le G$, $g \in G$.

τ^K_H, ρ^K_H, σ^g_H are sometimes simply denoted by τ^K, ρ_H, σ^g. These families must satisfy the following :

Axioms for G-functor. (In these axioms, D, H, K, L $\leq$ G; g, g' $\in$ G; $\alpha \in a(H)$, $\beta \in a(K)$).

(a) $\alpha^H = \alpha$, $(\alpha^K)^L = \alpha^L$ if $H \leq K \leq L$;

(b) $\beta_K = \beta$, $(\beta_H)_D = \beta_D$ if $D \leq H \leq K$;

(c) $\alpha^h = \alpha$ if $h \in H$, $(\alpha^g)^{g'} = \alpha^{gg'}$;

(d) $(\alpha^K)^g = (\alpha^g)^{K^g}$, $(\beta_H)^g = (\beta^g)_{H^g}$;

(e) (Mackey axiom) If $H \leq L$, $K \leq L$ and $\Gamma = \mathrm{Rep}(H\backslash L/K)$, then

$$(\alpha^L)_K = \sum_{g\in\Gamma} (\alpha^g{}_{K\cap H^g})^K.$$

Definition. A G-functor $a = (a, \tau, \rho, \sigma)$ is called cohomological if it satisfies the axiom C:

(C) If $H \leq K \leq G$ and $\beta \in a(K)$, then $\beta_H{}^K = |K : H|\, \beta$.

Definition. Let $a = (a, \tau, \rho, \sigma)$ and $a' = (a', \tau', \rho', \sigma')$ be G-functors over M_k. A morphism θ of G-functors $\theta : a \longrightarrow a'$ is a family $\theta = (\theta_H)_{H\leq G}$ of k-homomorphisms $\theta_H : a(H) \longrightarrow a'(H)$ such that for all H, K, g with $H \leq K \leq G$, $g \in G$,

(*) $\tau^K_H\, \theta_K = \theta_H\, \tau'^K_H$, $\rho^K_H\, \theta_H = \theta_K\, \rho'^K_H$, $\sigma^g_H\, \theta_{H^g} = \theta_H\, \sigma'^g_H$.

We denote by $M_k(G)$ the category whose objects are all G-functors over M_k and with morphisms as just defined. $M_k(G)$ is an abelian category. $M_k^c(G)$ denotes the full subcategory of $M_k(G)$ whose objects are all cohomological G-functors over M_k.

Remark and Definition. The original definition of G-functors by Green [2] contains the Frobenius axiom, that is, each $a(H)$ is a k-algebra and for all $H \leq K \leq G$, $\alpha \in a(H)$, $\beta \in a(K)$,

$$\text{(F)} \quad \alpha^K \cdot \beta = (\alpha \cdot \beta_H)^K, \; \beta \cdot \alpha^K = (\beta_H \cdot \alpha)^K.$$

The axiom of the multiplicative G-functor is as follows:

$$\text{(M)} \quad (\beta \cdot \beta')_H = \beta_H \cdot \beta'_H \quad \text{for} \quad H \leq K \leq G, \; \beta, \beta' \in a(K).$$

$A_k(G)$ denotes the category whose objects are all G-functors over A_k, the category of k-algebras and k-linear maps, which satisfy (F) and (M) and morphisms are morphisms $\theta = (\theta_H)$ between G-functors such that θ_H is multiplicative for each $H \leq G$. $A_k(G)$ is a subcategory of $M_k(G)$. $A_k^c(G)$ is the full subcategory of $A_k(G)$ whose objects are cohomological.

Examples of G-functors. In these examples, H and K are arbitrary subgroups of G such that $H \leq K$; g is an arbitrary element of G. Other examples are found in Green [2]. All G-functors in the examples are cohomological except for Example 1 and 10. V denotes always a kG-module.

Example 1. ch : the character ring functor.

ch(H) : the character ring of H;

$\tau_H^K : \alpha \longrightarrow \alpha^K$: the induced character;

$\rho_H^K : \beta \longrightarrow \beta_{|H}$: the restriction to H;

$\sigma_H^g : \alpha \longrightarrow \alpha^g$: the conjugate by g (i.e. $\alpha^g(y) = \alpha(gyg^{-1})$ for each $y \in H$).

This functor is in $A_Z(G)$. The Mackey axiom is the Mackey decomposition theorem. The Frobenius axiom is the Frobenius reciprocity.

Example 2. $H_V^* := \sum_{n=0}^{\infty} H_V^n$: the cohomology ring functor.

$H_V^*(H) := H^*(H, V) := \sum_{n=0}^{\infty} H^n(H, V)$: the cohomology group of H;

$\tau_H^K := cor_{H,K}$: the corestriction (transfer);

$\rho_H^K := res_{K,H}$: the restriction;

$\sigma_H^g := con_{H,g}$: the conjugation.

H_V^* is in $M_k^c(G)$. If V is a G-algebra over k, then H_V^* is in $A_k^c(G)$.

Example 3. $c_V := H_V^0$: the centralizer functor.

$c_V(H) := \{v \in V \mid vh = v \text{ for all } h \in H\}$;

$\tau_H^K : \alpha \longrightarrow \alpha^K := \sum \alpha g$, where g runs over Rep(H\K);

$\rho_H^K : \beta \longrightarrow \beta$ (inclusion);

$\sigma_H^g : \alpha \longrightarrow \alpha g$.

This functor is in $M_k^c(G)$ and $c_V = H_V^0$. If V is a G-algebra, then c_V is in $A_k^c(G)$.

Example 4. $H_V^* := \sum_{n \in Z} \hat{H}_V^n$: the Tate cohomology ring functor.

$\hat{H}_V^*(H) := \hat{H}^*(H, V) := \sum_{n \in Z} \hat{H}^n(H, V)$: the Tate cohomology group of H.

τ_H^K, ρ_H^K, σ_H^g are same as Example 2.

H_V^* is in $M_k^c(G)$. If V is a G-algebra over k, then it is in $A_k^c(G)$.

Example 5. $\hat{c}_V := \hat{H}_V^0$: the Tate centralizer functor.

This is a quotient functor of c_V in Example 3.

$\hat{c}_V(H) := c_V(H)/Vt_H$, where $t_H := \sum_{h \in H} h$.

This is in $M_k^c(G)$ and if V is a G-algebra over k, then this is in $A_k^c(G)$.

Example 6. ab : the abelian factor fanctor.

$ab(H) := H/H'$;

$\tau_H^K : xH' \longrightarrow xK'$: the natural map;

$\rho_H^K : yK' \longrightarrow T(y)H'$: group-theoretic transfer;

$\sigma_H^g : xH' \longrightarrow x^g(H^g)'$: the conjugation.

This is in $M_Z^c(G)$.

Example 7. $\hat{}$: the dual group functor.

$\hat{}(H) := \hat{H} := \mathrm{Hom}(H, Q/Z)$;

$\tau_H^K : \alpha \longrightarrow T_H^K(\alpha)$: the character-theoretic transfer;

$\rho_H^K : \beta \longrightarrow \beta_{|H}$: the restriction;

$\sigma_H^g : \alpha \longrightarrow \alpha^g$ $(\alpha^g(y) = \alpha(gyg^{-1})$ for $y \in H^g)$.

This functor is in $M_Z^c(G)$ and the dual functor of ab. See [3].

Example 8. ℓ_P : The Lie ring functor.

Assume that G acts a p-group P with a decending central series $P = P_0 \geq P_1 \geq \cdots$. Let $L(P) := \oplus\sum_i (P_i/P_{i+1})$ be the associated Lie ring on which G acts.

$\ell_P(H) := \sum_i C_{P_i}(H)P_{i+1}/P_{i+1} \subseteq L(P)$;

$\tau_H^K : \alpha \longrightarrow \sum \alpha^g$, where g runs over $\mathrm{Rep}(H\backslash K)$;

$\rho_H^K : \beta \longrightarrow \beta$: the inclusion;

$\sigma_H^g : \alpha \longrightarrow \alpha^g$.

This functor is in $A_k^c(G)$, where k is the ring of rational integers or p-adic integers, and this is a subfunctor of $c_{L(P)}$.

Example 9. $\hat{h}^0(a)$, $h^0(a)$: 0-dimensional cohomology "group" functors of a G-functor $a = (a, \tau, \rho, \sigma)$ over M_k. These are quotient fanctors of a such that

$\hat{h}^0(a)(H) := a(H)/\mathrm{Im}\,\tau_1^H + \mathrm{Ker}\,\rho_1^H$,

$h^0(a)(H) := a(H)/\mathrm{Ker}\,\rho_1^H$.

These functors are in $M_k^c(G)$. If a is in $A_k(G)$, then these are in $A_k^c(G)$.

Example 10. z : the center functor.

$z(H) := Z(kH)$: the center of the group ring kH;

$\tau_H^K : \alpha \longrightarrow \sum g^{-1}\alpha g$, where g runs over $\mathrm{Rep}(H\backslash K)$;

$\rho_H^K : \overline{C} \longrightarrow \overline{C \cap H}$, where C is a conjugate class of K and $\overline{C}$ is the class sum;

$\sigma_H^g : \alpha \longrightarrow g^{-1}\alpha g$.

This functor is in $M_k(G)$.

Transfer theorems. After this, $a := (a, \tau, \rho, \sigma)$ denotes always a cohomological G-functor over M_k.

Lemma 1. Let $H \leq G$. Assume that (*) $|G : H|\, 1_{a(G)}$ is an automorphism of $a(G)$. Then ρ_H^G is a monomorphism, τ_H^G is an epimorphism, and $a(H) = \mathrm{Im}\,\rho_H^G \oplus \mathrm{Ker}\,\tau_H^G$.

Proof. Set $\tau = \tau_H^G$, $\rho = \rho_H^G$. Then $\rho\cdot\tau = |G:H|1$, and so ρ is a mono and τ is an epi by (*). $0 \longrightarrow \mathrm{Ker}\,\tau \longrightarrow a(H) \xrightarrow{\tau} a(G) \longrightarrow 0$ is split, so $a(H) = \mathrm{Im}\,\rho \oplus \mathrm{Ker}\,\tau$.

Remark. If there is $|G:H|^{-1} \in k$, then (*) holds. If $(p, |G:H|) = 1$ and $\mathrm{ch}(k/J(k)) = p$, then (*) holds.

Lemma 2. (Maschke, complete reducibility). Let $a \in M_k(G)$ and let a' be a subfunctor of a such that $a(H)$ and $a'(H)$ are k-free for each $H \leq G$. Assume that $|G|k = k$. Then a' is a direct summand of a.

Lemma 3. Let $a \in M_k^c(G)$. Assume that $|G|k = k$. Then a is isomorphic to $c_{a(1)}$, where $a(1)$ is regarded as a G-module by $\alpha \cdot g = \alpha^g$, $\alpha \in a(1)$, $g \in G$.

The proofs of these lemmas are not short but easy. An analogue of Lemma 3 holds for non-cohomological G-functors, too.

Lemma 4. (Tate). Let $\bar{a} := a/J(k)a := (\bar{a}, \bar{\tau}, \bar{\rho}, \bar{\sigma})$ be a quotient functor of a such that $\bar{a}(H) = a(H)/J(k)a(H)$ for each $H \leq G$. Let $H \leq G$ and assume that $|G : H|\, 1_{a(G)}$ is an automorphism of $a(G)$.

(1) Let A be a k-submodule of $\operatorname{Ker} \tau_H^G$. If $A/J(k)A = \operatorname{Ker} \bar{\tau}_H^G$, then $A = \operatorname{Ker} \tau_H^G$.

(2) Let B be a k-submodule of $a(H)$ containing $\operatorname{Im} \rho_H^G$. Assume that $a(H)$ is Artinean. If $\operatorname{Soc}(B) = \operatorname{Soc}(\operatorname{Im} \rho_H^G)$, then $B = \operatorname{Im} \rho_H^G$.

Proof. Let $K := \operatorname{Ker} \tau_H^G$ and $J := J(k)$. By Lemma 1, we have that $K = A + JK$, so $K = A$ by Nakayama's lemma. $B = \operatorname{Im} \rho_H^G \oplus (B \cap K)$. Thus $\operatorname{Soc}(B \cap K) = 0$, and so $B \cap K = 0$.

<u>Hypothesis A</u>. $a = (a, \tau, \rho, \sigma) \in M_k^c(G)$, $H \leq G$, $ch(k/J(k)) = p$, $a(K)$ is k-free for each $K \leq G$, $(p, |G:H|) = 1$.

<u>Theorem 5 (Generalized co-focal subgroup theorem). Assume that Hypothesis A holds. Then</u>

$$\operatorname{Im} \rho_H^G = \{\alpha \in a(H) \mid (\alpha^g{}_{H \cap H^g} - \alpha_{H \cap H^g})^H = 0 \quad \text{for all} \quad g \in G\}$$

$$= \{\alpha \in a(H) \mid \alpha^g{}_{H \cap H^g} = \alpha_{H \cap H^g} \quad \text{for all} \quad g \in G\}$$

$$= \{\alpha \in a(H) \mid \alpha^G{}_H = |G:H|\alpha\}.$$

<u>Proof</u>. By the Mackey axiom, $\alpha^G{}_H - |G:H|\alpha = \sum(\alpha^g{}_{H \cap H^g} - \alpha_{H \cap H^g})^H$, where g runs over $\operatorname{Rep}(H\backslash G/H)$. If $\alpha^G{}_H = |G:H|\alpha$, then $\alpha = |G:H|^{-1}\alpha^G{}_H \in \operatorname{Im} \rho_H^G$. Let $\beta \in a(G)$ and $\alpha := \beta_H$, so that $\alpha \in \operatorname{Im} \rho_H^G$. By the axioms of G-functors, $\alpha^g{}_{H \cap H^g} = \alpha_{H \cap H^g}$ for each $g \in G$.

<u>Theorem 6 (Generalized focal subgroup theorem). Assume that Hypothesis A holds. Then</u>

$$\operatorname{Ker} \tau_H^G = \langle \alpha_{H \cap H^g}{}^{g^{-1}H} - \alpha_{H \cap H^g}{}^H \mid \alpha \in a(H),\ g \in G \rangle$$

$$= \langle \beta^{g^{-1}H} - \beta^H \mid \beta \in a(H \cap H^g),\ g \in G \rangle$$

$$= \{\alpha^G{}_H - |G:H|\alpha \mid \alpha \in a(H)\}.$$

This theorem is the dual of Theorem 5.

There are many other transfer theorems for cohomological G-functors which are generalizations of transfer theorems in finite group theory. Assume that $a \in M_k^c(G)$, $p \in J(k)$ and P is a Sylow p-subgroup of G. We can define the conjugation family for $a_{|P} \in M_k^c(P)$ by the same method as one in finite group theory. Using it, an analogue of Alperin's transfer theorem is proved. The principle proving Zappa-type theorems which are most primitive as transfer theorems in finite group theory seems to be Green's theorem([2, Theorem 2]). To generalize Grun-Wielandt type transfer theorems, we need to introduce the concept of singularities which is defined in [3] in the case of the dual group functor.

Definition. Let $a \in M_k^c(G)$, $S \leq G$, $\alpha \in a(S)$, $X \leq G$. Then (S, α, X) is called a singularity in G for a provided

(a) $\alpha^G{}_X \neq 0$,

(b) $\alpha_{S \cap Y^u} = 0$ for any $Y < X$ and $u \in G$.

The analogue of [3, Lemma 3.2] holds.

Lemma 7.(See [3, Lemma 4.1]). *Assume that Hypothesis A holds. Let B be a k-submodule of $a(H)$ which contains properly $\mathrm{Im}\ {}^G_H$. Then there is $\alpha \in B$ such that $\alpha \neq 0$ and $\alpha^G = 0$. Take a minimal subgroup X of H such that $\alpha_X \neq 0$. Then there is $g \in G - H$ such that $(S, \alpha^g{}_S - \alpha_S, X)$ is a* singularity in H for a_H, where $S := H \cap H^g$.

Remark. The above definition and lemma are not self-dual. Thus the co-singularities are similarly defined. This concept is used to study $\operatorname{Ker} \tau_H^G$, but it is not easy to treat. See Glauberman's lecture note (AMS).

Applications. Throughout the remainder of this note, we assume that the following holds :

Hypothesis B. P is a Sylow p-subgroup of G, k is a field of characteristic p, V is a kG-module, $E := \mathrm{End}_k(V)$. E is a kG-module by $v\phi^g := vg^{-1}\phi g$, $v \in V$, $\phi \in E$, $g \in G$.

Lemma 8. Assume that G is a p-group. Then V is kG-free if and only if $\hat{c}_V(G) = 0$.

Hall-Higman's theorem. The abelian case follows from Lemma 8 and the general focal subgroup theorem for $\hat{c}_V$. The extra-special case is reduced to the abelian case by the consideration of $\hat{c}_E$.

Coprime action. (1) If G is a p'-group, then $V = C_V(G) \oplus [V, G]$. (2) If a p'-group Q acts on a p-group P, then $P = C_P(Q)[P, Q]$.

These are proved by the application of Lemma 1 to c_V and ℓ_P.

<u>Cohomology groups</u>. If P is abelian, then $a(G) \simeq a(N_G(P))$, where $a = H_k^*$ or $\hat{H}_k^*$, k is a trivial kG-module.

This follows from the general focal subgroup theorem. This is a generalization of Johnson's theorem for elementary abelian P.

<u>Maschke-Higman-Gaschütz theorem</u>. V is P-projective. In particular, if G is a p'-group, then V is complete reducible. If $\chi \in H^*(G, V)$ and $res_{G,P}(X) = 0$, then $X = 0$ (Gaschütz).

Apply Lemma 1 to c_E and H_V^*.

<u>Groups with cyclic</u> P. Assume that P is of order p, $N := N_G(P)$. If $\dim V \leq (p - 1)/2$ and V is indecomposable, then V_N is also indecomposable. (This result can be more generalized).

Since $\tau_1^P = 0$ for c_E, the co-focal subgroup theorem yields that $c_E(G) \simeq c_E(N)$, and so $c_E(N)$ is also a local ring.

<u>Finite groups</u>. Let apply transfer theorems for G-functor to the functors ab and ^. Lemma 1 yields that $(P \cap G')/P'$ is a direct summand of P/P'. This is proved Thompson. Lemma 4

yields Tate's theorem. Applying general focal (resp. cofocal) subgroup theorem to ab (resp. ^), we have the focal subgroup theorem. By Lemma 7, we have that if P has no quotient groups isomorphic to $Z_p \wr Z_p$, then $P \cap G' = P \cap N_G(P)'$. Green's theorem ([3, Theorem 2]) yields Zappa's theorem : If W is weakly closed in P, then $\Omega_1(C_P(W)) \cap G' = \Omega_1(C_P(W)) \cap N_G(W)'$.

Concluding remark.

Definition. Let $\mathcal{F}$ be a weak conjugation family for a Sylow p-subgroup P. Then $\mathcal{F}$ is called a conjugation family for $a_{|P}$ provided whenever $g \in G$, $\alpha \in a(P)$, $Q = P \cap P^g$, and $R = gQg^{-1}$, then R is $\mathcal{F}$-conjugate to R^g via g' and $\alpha^g{}_Q = \alpha^{g'}{}_Q$. Since a conjugation family for P is a conjugation family for $a_{|P}$, there is a conjugation family for $a_{|P}$ by Alperin's theorem. By the general focal subgroup theorem, we have the following :

Theorem 9. Let P be a Sylow p-subgroup, $a \in M_k^c(G)$, $P \leq H \leq G$, k a field of characteristic p, $\mathcal{F}$ a conjugation family for $a_{|P}$. Then $\mathrm{Im}\, \rho_P^G$ consists of all $\alpha \in \mathrm{Im}\, \rho_P^H$ such that $(\alpha^n{}_F - \alpha_F)^g{}_Q{}^P = 0$ for each $(F, N) \in \mathcal{F}$, $n \in N$, $g \in G$, $Q = P \cap F^g$.

We observed that many transfer theorems in finite group are generalized to some for cohomological G-functors. There are also some transfer theorems for general G-functors. They are usually called induction theorems (e.g. Artin's theorem, Brauer's theorem, Green's theorem [2, Theorem 2], etc.). Some of them are equivalent to the vanishing theorems for (relative) cohomology "groups" for G-functors which are defined by the similar method as sheaf cohomology. The state of affairs seems like a part of sheaf cohomology in analytic function theory of several variables. Not only (relative)cohomology group functors of G-functors but also simple G-functors are almost cohomological. For the reason, I believe that the transfer theory for cohomological G-functors is the guide principeple of induction theorems for general G-functors.

I give only three fields in which G-functors seems to be applicable.

1. <u>Representation theory</u>. I am interested to reconstruct the modular representation theory of finite groups by application of (cohomological) G-functors. In particular, how can the Brauer's theorems be rewritten? There are some theorems which can be regarded as transfer theorems. I remark that the Brauer's first main theorem is also proved by the use of the functor z. (Green proved it by the functor c_A, where $A := kG$ regarded as G-algebra by the conjugation). Since $M_k(G)$ is like M_{kG}, where $ch(k) \nmid |G|$, I am also interested to study $M_k(G)$

along the ordinary representation theory.

2. Class field theory. The use of G-functors can rewrite the axioms of abstract class field theory, so we reach the concept of Galois G-functors.

3. Topology. Reseach objects in topologies of some kind are often naturally acted by groups. For example, remember the spirit of Erlangen problem and the covering spaces on which their monodoromy groups acts. Thus we are interested in spaces on which groups acts. In order to study such spaces, we can define a general equveriant cohomology theory with G-functors (or "sheaves" on G) as coefficients. This can be regarded as a functor of CW-pairs on which G acts to $M_k(G)$. In general, given a space X on which G acts and a functor of the category of spaces to M_k, we obtain some sheaves or co-sheaves on G, e.g. $H \longrightarrow h(X^H)$, $h(X/H)$, etc. Furthermore, there are some special functors (e.g. H^*: cohomology groups functor) on G-spaces which give G-functors by the similar method as Atiyah-Hirzebruch ([2, Example 5.4]), e.g. $H \longrightarrow H^*(X \times E_G/H)$.

Can we extend the definition of G-functors to (locally) compact groups ? The answer was given by Dress [1]. He introduced the concept of the Mackey functors and has been applying it to various fields. I expect that his theory will be perhaps equal to the category theory in the future mathematics. For example, his theory is applied to topology by Oliver, tom Dieck, etc. I have no good knowledge of topology, but I believe that finite group theory is useful for this field.

I don't know how cohomological Mackey functors are defined and whether transfer theorems for cohomological G-functors can be generalized to Mackey functors. Both Green and Dress do not attach much important to cohomological G-functor in my view.

References

[1] Dress, A.W.M. : Contributions to the theory of induced representations, in Algebraic K-theory II, Lecture Notes in Math., 342 (Springer, 1973).

[2] Green, J.A. : Axiomatic representation theory for finite groups. J. pure appl. Algebra 1, 41 - 77 (1971).

[3] Yoshida, T.: Character-theoretic transfer, J. Algebra 52 (1978)

Note. Prof. Neumann told me after the symposium that Holt proved the following surprising result :

Theorem. (Holt). *If a Sylow p-subgroup* P *of* G *is of class at most* $p/2$, *then* $H^2(G, k) \cong H^2(N_G(P), k)$, *where* $k = Z_p$, *trivial G-module*.

I think that he proved the following lemma, probably :

Lemma ? Let P *be a p-group of class at most* $p/2$. *Then* P

has no proper singularity for the functor H^2_k.

If so, Holt's theorem is probably generalized as follows :

Theorem ? Let P *be a Sylow p-subgroup of* G *and let* Q *be a strongly closed subgroup of* P. *If* Q *is of class at most* p/2, *then* $H^2(G, k) \simeq H^2(N_G(Q), k)$.

This is an analogue of Glaberman's theorem ([3, Corollary 4.6.2]). Comparing with [3, Lemma 3.7], it seems that there is much room for improvement of Lemma ?.