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On approximate Sufficiency

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H. Kudo defined the notion of approximate sufficiency as follows [1]. Let $(X, \mathfrak{A}, \{P, Q\})$ be a statistical structure and $\{\mathfrak{A}_n\}$ be a sequence of sub-σ-algebra such that $\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \cdots$ and $\bigvee_{n=1}^{\infty} \mathfrak{A}_n = \mathfrak{A}$. A sequence $\{\mathcal{B}_n\}$ $(\mathcal{B}_n \subset \mathfrak{A}_n)$ is called approximately sufficient for $\{P, Q\}$ if there exist probability measures P_n, Q_n $(n=1,2,\cdots)$ on $\mathfrak{A}_n$ satisfying

$$(1) \quad \|P - P_n\|_{\mathfrak{A}_n} \to 0, \quad \|Q - Q_n\|_{\mathfrak{A}_n} \to 0 \quad (n \to \infty)$$

$$(2) \quad \mathcal{B}_n \text{ is sufficient for } \{P_n, Q_n\}.$$

Here $\|P - P_n\|_{\mathfrak{A}_n}$ means $\sup_{B \in \mathfrak{A}_n} |P(B) - P_n(B)|$.

In [3] the author extends the notion of approximate sufficiency to general statistical structures. Let $(X, \mathfrak{A}, \mathcal{P} = \{P_\theta \mid \theta \in \Omega\})$ be a statistical structure. Let $\{\mathfrak{A}_n\}$ be the same as defined above. A sequence $\{\mathcal{B}_n\}$ $(\mathcal{B}_n \subset \mathfrak{A}_n)$ is called approximately sufficient for $\mathcal{P}$ if there exist families of probability measures $\mathcal{P}_n = \{P_{\theta,n} \mid \theta \in \Omega\}$ on $\mathfrak{A}_n$ $(n=1,2,\cdots)$ satisfying

(1) $\|P_\theta - P_{\theta,n}\|_{\mathfrak{A}_n} \to 0 \ (n \to \infty) \ (\forall \theta \in \Omega)$

(2) $\mathcal{B}_n$ is sufficient for $\mathcal{P}_n$ $(n=1,2,\cdots)$.

Then we have results corresponding to the well-known results about sufficiency obtained by Halmos-Savage and Bahadur [4] [5].

Theorem 1 Let $\mathfrak{A}$ be σ-generated and $\mathcal{P}$ be dominated. Then, by choosing a dominating probability measure λ_0 suitably, the following four assertions are equivalent [3].

(a) $\{\mathcal{B}_n\}$ is approximately sufficient for $\mathcal{P}$

(b) $\tilde{\rho}_{\lambda_0}(f_\theta, L_{\lambda_0}(\mathcal{B}_n)) \to 0 \quad (n \to \infty) \ (\forall \theta \in \Omega)$

(c) $\rho_{\lambda_0}(f_\theta, E_{\lambda_0}(f_\theta | \mathcal{B}_n)) \to 0 \quad (n \to \infty) \ (\forall \theta \in \Omega)$

(d) $\mathcal{B}_0 = \lambda_0\text{-}\liminf \mathcal{B}_n$ is sufficient for $\mathcal{P}$.

Here ρ_{λ_0} denotes the distance in $L^1(X, \mathfrak{A}, \lambda_0)$ and $L_{\lambda_0}(\mathcal{B}_n)$ denotes the set of all $\mathcal{B}_n$-measurable elements in $L^1(X, \mathfrak{A}, \lambda_0)$. $\tilde{\rho}_{\lambda_0}(f_\theta, L_{\lambda_0}(\mathcal{B}_n))$ denotes the distance between $f_\theta = \frac{dP_\theta}{d\lambda_0}$ and the set $L_{\lambda_0}(\mathcal{B}_n)$ in $L^1(X, \mathfrak{A}, \lambda_0)$ and $\lambda_0\text{-}\liminf \mathcal{B}_n$ denotes $\{A \in \mathfrak{A} \mid \exists B_n \in \mathcal{B}_n : \lambda_0(A \triangle B_n) \to 0\}$. $\lambda_0\text{-}\liminf \mathcal{B}_n$ is called the lower limit of $\{\mathcal{B}_n\}$. It is proved in [2] that $\lambda_0\text{-}\liminf \mathcal{B}_n$ is a σ-algebra.

In [2] λ_0-liminf $\mathcal{B}_n$ is characterized as the σ-algebra $\mathcal{B}_0$ having the following properties.

(i) $\mathcal{B}_0$ satisfies

$$\liminf_{n\to\infty} \int |E_{\lambda_0}(f|\mathcal{B}_n)|\, d\lambda_0 \geqq \int |E_{\lambda_0}(f|\mathcal{B}_0)|\, d\lambda_0$$

for every bounded $\mathcal{A}$-measurable f .... [A]

(ii) any σ-algebra $\mathcal{B}$ satisfying [A] is contained in $\mathcal{B}_0$.

If, for every $\theta_1, \theta_2 \in \Omega$, $\{\mathcal{B}_n\}$ is approximately sufficient for $\{P_{\theta_1}, P_{\theta_2}\}$, $\{\mathcal{B}_n\}$ is said to be pairwisely approximately sufficient for $\mathcal{P}$.

Theorem 2 Under the same assumptions as those in the above theorem, approximate sufficiency and pairwise approximate sufficiency are equivalent.

Theorem 3 If $\{\mathcal{B}_n\}$ is approximately sufficient for $\mathcal{P}$, for every $\mathcal{A}$-measurable, bounded f, there exist a sequence $\{h_n\}$ of $\mathcal{A}$-measurable, bounded functions and versions $\tilde{E}_\theta(f|\mathcal{B}_n)$ of $E_\theta(f|\mathcal{B}_n)$ such that $\rho_{\lambda_0}(\tilde{E}_\theta(f|\mathcal{B}_n), h_n) \to 0 \quad (n\to\infty)$ for every $\theta\in\Omega$.

The converse of this theorem is an open

problem. But, when $\{\mathcal{B}_n\}$ is monotone increasing, the converse holds. The question naturally arises whether the diameter of $\{\tilde{E}_\theta(f \mid \mathcal{B}_n) \mid \theta \in \Omega\}$ tends to 0 as $n \to \infty$ by choosing suitable versions $\tilde{E}_\theta(f \mid \mathcal{B}_n)$. The answer to this question is generally negative. But, if Ω is compact with respect to the metric $d(\theta_1, \theta_2) = \sup_{B \in \mathcal{O}\!\mathcal{L}} |P_{\theta_1}(B) - P_{\theta_2}(B)|$ and $\mathcal{P}$ is homogeneous, the answer is positive.

Let $\mathcal{C}$ be a σ-algebra satisfying $\mathcal{B} \subset \mathcal{C} \subset \mathcal{O}\!\mathcal{L}$. If, for every $\mathcal{C}$-measurable f, there exists a conditional expectation $E(f \mid \mathcal{B})$ common to every P_θ, $\mathcal{B}$ is said to be $\mathcal{C}$-sufficient for $\mathcal{P}$.

Theorem 4 If there exists a sequence of σ-algebras $\{\mathcal{C}_n\}$ such that $\mathcal{B}_n \subset \mathcal{C}_n \subset \mathcal{O}\!\mathcal{L}$, λ_0-$\liminf \mathcal{C}_n = \mathcal{O}\!\mathcal{L}$ and $\mathcal{B}_n$ is $\mathcal{C}_n$-sufficient for $\mathcal{P}$, then $\{\mathcal{B}_n\}$ is approximately sufficient for $\mathcal{P}$.

The converse of this theorem is an open problem.

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