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SMALLESTNESS AND MINIMALITY OF PAIRWISE SUFFICIENT SUBFIELDS

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This is a continuation of the preceding article[1] by J.K. Ghosh. The same definitions and notations as in that article will be used here, except that the basic space Ω is now replaces by $X = \{x\}$.

1. The smallest subfield with pairwise sufficiency and containment of carriers.

We begin with a simple example which, however, retains all the essential features of the discrete case in general.

[Example 1] Let X be an uncountable space, $\underline{A}$ the sigma-field of all the subsets of X and $\underline{P}$ be the family of all one-point probability measures on X. Define $\underline{C}$ to be the family of all the countable and cocountable sets and for each x in X define $\underline{C}(x)$ to be the family of all the countable sets which do not contain x and all the cocountable sets which contain x. Clearly both of these families are sub-sigma-fields(simply,"subfields") of $\underline{A}$. It follows from Theorem 5 of [1] that $\underline{C}(x)$ is minimal pairwise sufficient(MPS, in short) and that $\underline{C}$ is pairwise minimum sufficient(PSS, as I would rather call it pairwise smallest sufficient).

I would like to point out that $\underline{C}(x)$ is also PSS. In fact, all the subfields which shares the same partition as a PSS subfield are also PSS, because in the discrete case, if two subfields have a same partition they are equivalent in terms of the partial order (II) defined in [1]. Hence there are great many PSS in this case: All the separating subfields are PSS. Here, of course, a subfield $\underline{B}$ is called separating if for any two points in X there is a set in $\underline{B}$ which contains one and only one of them. (Here the example ends).

To single out one "smallest" subfield, one other concept seems to me more convenient: The smallest subfield with pairwise sufficiency and containment of supports (SPSC). This is defined as the smallest (minimum, in terms of [1]) one wrt. the partial order (I), among all the pairwise sufficient subfields which contain supports of all P in $\underline{P}$ relative to $\underline{P}$ itself, according to Definition 1 of [1]. In the foregoing example, $\underline{C}$ is SPSC, as well as the particular PSS written $\underline{A}_{pms}$ constructed in Theorem 1 of [1]. This coincidence is not an accident, as the following theorem shows.

[Theorem 1] Suppose for each P in $\underline{P}$ the supports relative to $\underline{P}$ exists. Then $\underline{A}_{pms}$ is SPSC.

[Outline of the proof] Assume that $\underline{B}$ is pairwise sufficient. then the functions $\psi_{P_1P_2}$, defined in Theorem 1 of [1] must be B-measurable. If we assume that $\underline{B}$ contains the supports of all P in $\underline{P}$, then the functions I_P which appear in the same Theorem are $\underline{B}$-measurable. Hence $\underline{B}$ includes $\underline{A}_{pms}$. (End)

Thus the existence of SPSC and its being PSS is proved under the same generality as the existence of the latter has thus far been proved. A possible criticism of this concept might be that while pairwise sufficiency is a "pairwise consept", that is, one preserved by the equivalence in termes of (II), the concept of support is not,and the definition of SPSC has a sort of inconsistency in its combining these two concepts belonging to two different categories. On the other hand, SPSC emerges quite naturally from the following theorem which holds under a slightly more general conditions than what is called weak domination.

[Generalized Neyman Factorization Theorem](Yamada and Morimoto) A subfield $\underline{B}$ is pairwise sufficient and contains the supports of all P in $\underline{P}$ if and only if every P in $\underline{P}$ has a $\underline{B}$-measurable density wrt. a pivotal measure.

Under the same generality, the existence of SPSC immediately follows: The subfield generated by all the versions of the densities of all P in $\underline{P}$ wrt. a pivotal measure is SPSC.

I would not state explicitly the conditions for the theorem or the definition of a pivotal measure here, because Neyman factorization is not the main subject here, and the existence of SPSC has been proved in [1] under a more general condition, that is, the existence of supports.

2. Characterization of minimal pairwise sufficient subfields in the discrete case.

I state in this section recent results by Namba[2]. I again take up Example 1, although the results are easily rephrased for the discrete case in general. Theorem 4 of [1] is now specialized to: A subfield is piarwise sufficient if and only if it is separating. Thus our problem is to decide whether a given separating subfield is a minimal one of that kind or not. Suppose that $\underline{B}$ is separating and let $\underline{F}=\{F_i;\ i\in I\}$ be a family of sets which generates $\underline{B}$. Define 2^I to be the space of all functions on I to $\{0,1\}$. Here I is the set of indices attached to the sets in $\underline{F}$. Points in this space are written $y=(y(i);\ i\in I)$, $z=(z(i);\ i\in I)$ etc. We define a mapping f on X onto a subset Y of 2^I as follows: A point x in X is mapped to a point y=f(x) which satisfies y(i)= 1 if x belongs to F_i and y(i)=0 otherwise. By f, $\underline{F}$ is carried to the family of all such sets that are written as $\{y;\ y(i)=1\}$ for some i in I. And $\underline{B}$ is carried by f to the sigma-field generated by it. We conveniently denote them by $\underline{F}$ and $\underline{B}$ again. A neighbourhood of y in Y is defined to be a set N(y;K), where K is any countable subset of I, which is the totality of those points z in Y which satisfies z(i)= y(i) for all i in K. The neighbourhoods, when y ranges over Y and K assumes to be all countable subsets of I, give rise to a topology on Y. Y is called ω_1-compact wrt. this topology if the following condition is satisfied: Assume that to each y in Y there corresponds a neighbourhood N(y;K(y)). Then one can choose a countable number of points $y_0, y_1, \dots y_k, \dots$ such that $\bigcup_{k=0}^{\infty} N(y;K(y_k)) = Y$.

We now state a theorem of Namba[2] which gives a complete characterization of minimality of a separating subfield.

[Theorem 2] $\underline{B}$ is minimal separating if and only if $\underline{Y}$ is ω_1-compact.

Let us see how this theorem works with the subfields given in Example 1.

[Example 2] Under the framework of Example 1, take the following generators $\underline{F}$ and $\underline{F}(x)$ of $\underline{C}$ and $\underline{C}(x)$, respectively:

$\underline{F}$ = all the singletons in X.

$\underline{F}(x)$= all the singletons except x .

Corresponding sets of indices I for these generators are X and $X - \{x\}$, respectively. By the correspondence f, the space X is mapped to one of the following two spaces, depending on cases:

Y = all y such that y(i)= 1 for one single i in I=X.

Y(x)= all y such that Y(i) = 1 for on single i in $I=X -\{x\}$,

and 0.

Here, 0 denotes the point of 2^I such that 0(i)= 0 for all i in I. Notice that f(x) = 0.

The ω_1-compactness of Y(x) is proved as follows: Suppose that N(y;K(y)) corresponds to y, for each y in Y. If a point y does not belong to N(0,K(0)), the neighbourhood corresponding to 0, then there exists i in K(0) such that y(i)=1. As K(0) is countable and as each y can assume the value 1 for at most one i, there are a countable number of points which do not belong to N(0,K(0)). Take them as $y_1, y_2, \ldots, y_k, \ldots$ and 0 as y_0. Then it is clear that the neighbourhoods corresponding these points collectively cover Y.

This proof does not work for Y, because it does not contain 0. On the other hand it is easy to disprove ω_1-compactness.

There exists an example of a minimal separating subfield which does not contain any singletons:

[Example 3] Let I be an uncountable set of indices i and for each non-negative integer n let Y_n be the set of all functions I to $\{0,1\}$ which assume the value 1 for at most n indices in I. Put $Y = \bigcup_{n=1}^{\infty} Y_n$, the set of all functions assuming the value 1 for a finite number of indices i in I. Let $\underline{F}$ be the family of all sets of the form: $\{y ; y(i) = 1\}$ for some i in $\underline{I}$. Define $\underline{B}$ as the subfield generated by $\underline{F}$, which is equal to the totality of the sets B for which there exists a countable subset K(B) of I such that y B and Y(i)=z(i) for all i in K(B) imply $y \in B$. Then it is clear that $\underline{B}$ is separating and $\underline{B}$ does not contain any singletons. To prove that it is minimal, we are sufficed to prove the ω_1-compactness of Y.

The proof is similar in nature to that given in the previous example, except only that we need induction over n.

References

[1] Ghosh, J.K., Minimality of pairwise sufficient σ-fields, in this volume.

[2] Namba, K., Representation theorem for minimal σ-algebras, to appear in the Proceedings of the Conference on Logic and Set Theorey in Belgrade, August-September, 1977.