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Stratification of Thom-Boardman Singularities

Takuo FUKUDA

Dept. Math. Chiba Univ.

Contrary to the case of functions, the analytic or C^{∞} classification of deformations of holomorphic or C^{∞} map-germs is almost impossible : [J.Mather]it is not generic that a map-germ has a finite dimensional versal deformation. R.Thom and F.Pham ($[T_1]$,$[Ph_1]$,$[Ph_2]$) propose a classification called topological unversal classification.Briefly speaking, two map-germs are topologically universally equivalent if they have topologically versal deformations which are topologically equivalent. This classification is finer than that of equisingularities in the sense of O.Zariski ($[Ph_2]$). Now this classification is possible in the following sense:[R. Thom] ([T]), let $\mathcal{O}_{n,p}$ be the set of all holomorphic (or C^{∞})map-germs of C^n into C^p (or R^n into R^p), then there exist an infinite codimensional subset Σ of $\mathcal{O}_{n,p}$ and a stratification of $\mathcal{O}_{n,p} - \Sigma$ such that any two map-germs belonging to the same stratum are topologically universally equivalent.

So we want to construct explicitely such a stratification so that given a map-germ one can determine by calculation to which stratum it belongs.

CONJECTURE (optimistic): One can obtain such a stratification by substratifying " canonically " the Thom-Boardman singularities. Here the term "canonically" means " only according to the adherence relations between them".

It seems that in general the adherence relations between the Thom-Boardman singularities are very complicated. However it seems also that their adherence relations are simple and neat between those of low codimensions. Here we report some observation on the simplest case.

Let $J^{\infty}(n,n)$ be the infinite jet space of C^n into C^n (or R^n into R^n). Let $\Sigma^{2,0}$ and $\Sigma^{1^k,0}$ denote the Thom-Boardman singularities with symbol (2,0) and with $(\underbrace{1,1,\ldots,1}_{k \text{ times}},0)$ respectively. Set

$$\Sigma_k^{2,0} = \Sigma^{2,0} \cap \text{ the closure of } \Sigma^{1^k,0}$$

$$\Sigma_{\infty}^{2,0} = \bigcap_{k=1}^{\infty} \Sigma_k^{2,0} .$$

Then our result is

THEOREM 1. $\Sigma_k^{2,0}$'s are submanifolds of $J^{\infty}(n,n)$ and the collection

$$\left\{ \Sigma^{1^k,0} , (\Sigma_i^{2,0} - \Sigma_{i+1}^{2,0}) \right\}$$

is a Whitney stratification of $J^{\infty}(n,n)$ - (the closure of $\Sigma^{2,1} \cup \Sigma_{\infty}^{2,0}$) satisfying the following conditions:

1) if a map-germ with $j^{\infty}f(0) \in \Sigma^{2,0}$ is multi-transversal to this stratification, then f is topologically stable.

2). Any two map-germs multi-transversal to the stratification and whose jets belong to the same stratum are topologically equivalent.

Moreover we have the explicite defining equations of $\Sigma_k^{2,0}$. (see theorem 2).

Since there is a natural submersion of $J^\infty(n,n) - \overline{\Sigma^{2,1} \cup \Sigma_\infty^{2,0}}$ into $J^\infty(2,2) - \overline{\Sigma^{2,1} \cup \Sigma_\infty^{2,0}}$ which preserves the Thom-Boardman singularities with the same symbols, in order to prove that $\{\Sigma^{1^k,0}, \Sigma_i^{2,0} - \Sigma_{i+1}^{2,0}\}$ is a Whitney stratification of $J^\infty(n,n) - \overline{\Sigma^{2,1} \cup \Sigma_\infty^{2,0}}$ and in order to determine the defining equations of $\Sigma_i^{2,0}$, it is enough to prove

THEOREM 2. $\{\Sigma^{1^k,0}, \Sigma_i^{2,0} - \Sigma_{i+1}^{2,0}\}$ is a Whitney stratification of $J^\infty(2,2)$ - the closure of $\Sigma^{2,1} \cup \Sigma_\infty^{2,0}$. Moreover for a map-germ $F = (f,g)$, $j^\infty F(0) \in \Sigma_k^{2,0}$ if and only if under suitable coordinates

$$\begin{cases} \partial f/\partial x(0) = \partial g/\partial x(0) = 0 \\ \partial^i f/\partial y^i(0) = 0 \quad \text{for } i = 1,2,\ldots,k-1 \\ \partial^j g/\partial y^j(0) = 0 \quad \text{for } j = 1,2,\ldots,k. \end{cases}$$

The properties (1) and (2) in theorem 1 follow from the following more general theorem: For a sequence of integers $I = (i_1,\ldots i_k)$, let $\Sigma^{I,0}$ denote the Thom-Boardman singularity of symbol $(i_1,i_2,\ldots,i_k,0)$. Then we have a decomposition of the jet space $J^\infty(n,p)$ into Thom-Boardman singularities

$$(*) \qquad J^\infty(n,p) = \bigcup_I \Sigma^{I,0} \quad :$$

THEOREM 3. Let $\mathcal{S}$ be a Whitney stratification of $J^\infty(n,p)$ which is a refinement of the decomposition (*). Then the following properties hold:

(1) If the jet extension $j^\infty f$ of a map-germ f is multitransversal to every stratum of $\mathcal{S}$, then f is topologically stable.

(2) Any two map-germs whose jets are multi-transversal and belong to the same stratum are topologically equivalent.

REMARK. Our result is far from to be called a new result . Nevertherless, our result gives some interesting examples: J.Mather classified in $[M_1]$ all analytically stable singularities of types $\Sigma^{2,0}$ and $\Sigma^{2,1,0}$ and we know that the set of analytically non-stable map-germs of types $\Sigma^{2,0}$ and $\Sigma^{2,1,0}$ is of codimension greater than the dimension of the target space. So his classification is complete for the mapgerms of type $\Sigma^{2,0}$ and $\Sigma^{2,1,0}$. So in order to obtain new classification results we have to continue our observation furthermore on $\Sigma^{2,1,0}$,$\Sigma^{2,1,1,0}$ and so on.

Neverthe̶less ,comparing with Mather's classification, we know that our classification gives examples of map-germs which are not analytically stable but are topologically stable and also examples of analytically stable map-germs which are not analytically equivalent to each other but are topologically equivalent. From $[M_1]$, analytically stable map-germs $F = (f_1,f_2)$ and $G = (g_1,g_2)$ of C^2 into C^2 are analytically equivalent if and only if the associated algebras $Q(F) = C[[x,y]]/(f_1,f_2)$ and $Q(G) = C[[x,y]]/(g_1,g_2)$ are isomorphic, and if F is analytically stable and of type $\Sigma^{2,0}$ then Q(F) is isomorphic to one of the following algebras which are not isomorphic to each other:

$I_{a,b}$: $C[[x,y]]/(xy,\ x^a + y^b)$, $b \geqslant a \geqslant 2$

$II_{a,b}$: $C[[x,y]]/(x^a,\ y^b,\ xy)$, $b \geqslant a \geqslant 2$

III_a : $C[[x,y]]/(x^2 + y^2,\ x^a)$, $a \geqslant 3$

IV_a : $C[[x,y]]/(x^2 + y^2,\ x^a,\ yx^{a-1})$, $a \geqslant 3$

For example, observe $I_{a,b}$. Let F and G be analytically stable map-germs of types $I_{a,b}$ and $I_{a',b}$, $a \neq a'$, respectively. Then F and G are not analytically equivalent, for $(a,b) \neq (a',b)$. On the other hand, our theorems 1 and 2 assert that they are topologically equivalent, for both of them are of type $\sum^{2,0}_{b-2} - \sum^{2,0}_{b-1}$.(See the defining equations in theorem 2).

We do not give here the proofs of the theorems. Someday when the auther obtains more results, they will appear somewhere.We have not given defintions of some terminologies, either. We are referred to $[Ph_1]$,$[Ph_2]$ or $[F_1]$ for a definition oftopological universal classification , to $[M_3]$ or $[F_2]$ for stratifications and [B] and $[M_2]$ for the Thom-Boardman singularities.

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