数理解析研究所講究録
第 328 巻 1978 年 55-70



# Some global topological properties of complex hypersurfaces.

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## §1. Introduction.

Local topological properties of a complex hypersurface $V$ in a compact complex $(n+1)$-manifold $M$ at a singular point of $V$ have been studied by many authors during the last decade. In this note we shall study some global topological properties of $V$ and a pair $(M, V)$ focusing on the normal euler class $\chi(V, M)$ of $V$ in $M$.

In §2, we shall recall a formula for the total Chern-Macpherson homology class of $V$ with isolated singularity. Existence of a nice almost complex resolution of $V$ in $M$ was basic for us to obtain the formula. We shall formulate a generalization of this notion for $V$ with possibly non-isolated singularity and give an example which admit no nice almost complex resolution in $M$.

In §3, micro-equivalence classification of PL embeddings of a compact oriented ordered $m$-subpolyhedron $V$ into closed oriented manifolds of codimension two with isolated singularity will be given. The complete set of invariants consists of the normal euler class and the singularity. This is a generalization of Noguchi [7]. Our purpose here is to prove

Theorem. Let $V$ and $V'$ be complex irreducible curves in a compact complex surface $M$. Then there are open neighborhoods $U$, $U'$ of $V$, $V'$ in $M$ so that $(U, V)$ and $(U', V')$ are "PL" homeomorphic if $V$ and $V'$ represent the same integral homology class in $M$ and have the same sets of link types at singular points.

Here we understand that a "PL" homeomorphism means a PL homeomorphism of compatible triangulations of $(U, V)$ and $(U', V')$ in the sense of [1].

§ 2. Normal euler classes and Chern-Macpherson classes.

Let $V$ be a complex hypersurface in a compact complex $(n+1)$-manifold $M$ with singular set $\Sigma V$. The complex structures of $V - \Sigma V$ and $M$ determines the fundamental classes $[V] \in H_{2n}(V;\mathbb{Z})$ and $[M] \in H_{2n+2}(M;\mathbb{Z})$. We have the Poincaré duality isomorphism $P = (\cap[M]) : H^*(M;\mathbb{Z}) \to H_{2n+2-*}(M;\mathbb{Z})$ and a homomorphism $\cap[V] : H^*(V;\mathbb{Z}) \to H_{2n-*}(V,\mathbb{Z})$, ( for some properties of $\cap[V]$, see [A] ).

Definition. Normal euler class $X(V,M)$ of $V$ in $M$ is defined by $X(V,M) = i^* \circ P^{-1} \circ i_* [V]$, where $i : V \to M$ is an inclusion map.

We shall say that $V$ admits a nice almost complex resolution in $M$, if for any open neighborhood $U$ of the singular set $\Sigma V$ of $V$ in $M$, there exists a complex analytic Whitney stratification $\mathcal{S}$ with a tubular neighborhood system $\{\pi_A : T_A \to A \mid A \in \mathcal{S}\}$ of $\Sigma V$ in $M$ and a smooth $2n$-submanifold $\hat{V}$, called a nice almost complex resolution of $V$ in $M$, satisfying the following two conditions;

(1) the tangent bundle $\tau(\hat{V})$ of $\hat{V}$ and the normal bundle $\nu(\hat{V}, M)$ of $\hat{V}$ in $M$ admit complex reductions $\tau_{\mathbb{C}}(\hat{V})$ and $\nu_{\mathbb{C}}(\hat{V}, M)$ as continuous vector bundles so that the Whitney sum

$$\tau_{\mathbb{C}}(\hat{V}) \oplus \nu_{\mathbb{C}}(\hat{V}, M) \cong \tau_{\mathbb{C}}(M)\,|_{\hat{V}}$$

as continuous complex vector bundles[, where $\tau_{\mathbb{C}}(M)$ is the complex tangent bdle of $M$], and

(2) if we put $T = \bigcup_{A \in \mathcal{S}} T_A$, then

$$T \subset U, \quad V - T = \hat{V} - T \quad \text{and}$$

for each $A \in \mathcal{S}$, if we put $T_{\partial A} = \bigcup_{B < A} T_B$, then $\pi_A$ restricted to $(T - T_{\partial A}) \cap \hat{V}$ is a locally trivial fibration over $A - T_{\partial A}$, and for each point $x \in A - T_{\partial A}$, the fiber $\pi_A^{-1}(x) \cap \hat{V}$ has the same euler number as a Milnor fiber of $L \cap V$ in $L$ at $x$ for a local transversal plane $L$ to $A$ in $M$.

The condition (1) says that by definition $\hat{V}$ is an almost complex submanifold of $M$ and the condition (2) says that $\hat{V}$ approximates $V$ nicely. The following theorem is proved in [2].

<u>Theorem 1.</u> Suppose that $V$ has isolated singularity. Then $V$ admits a nice almost complex resolution $\hat{V}$ in $M$.

The total Chern-Macpherson class $c_*(V)$ of $V$ is defined as an element of $H_{2*}(V;\mathbb{Z})$ by Macpherson [6].

<u>Corollary to Theorem 1.</u> If $V$ has isolated singularity, we have a formula

$$c_*(V) = \left(\frac{i^*c^*(M)}{c^*(V,M)}\right) \cap [V] + (-1)^{n-1}\mu(V,M) ,$$

where $c^*(M) \in H^{2*}(M;\mathbb{Z})$ is the total chern class of the complex $(n+1)$-manifold $M$, $c^*(V,M)$ is the total normal chern class $1+X(V,M) \in H^0(V;\mathbb{Z}) + H^2(V;\mathbb{Z})$ and $\mu(V,M)$ is the sum of all the Milnor numbers $\mu_x$ of $V$ in $M$ at $x \in \Sigma V$.

<u>Example 1.</u> If $M = \mathbb{P}^{n+1}$ ( complex projective $(n+1)$-space ) and $V$ is of degree $d$, then we have an explicit formula for $c_*(V)$ by means of homology classes $[V_m]$ of multiple hyperplane sections $V_m = V \cap L^{(m+1)}$,

$(m = 0, 1, \cdots, n)$ in $\mathbb{P}^{n+1}$;

$$c_*(V) = \sum_{m=0}^{n} \left( \sum_{k=0}^{n-m} \binom{n+2}{n-m-k} \cdot (-d)^k \right) [V_m] + (-1)^{n-1} \mu(V, \mathbb{P}^{n+1})$$

Example 2. Let $V$ be a complex irreducible curve in a compact complex surface $M$, and let $\tilde{V}$ be a normalization of $V$. Then the topological type of $\tilde{V}$ is completely determined by the homology class $i_*[V] \in H_2(M; \mathbb{Z})$ and the link types of $V$ in $M$ at singular points. In fact, we put

$r_x$ = the number of connected components of a link $K_x$ at $x$,

$\mu_x$ = the Milnor number of $V$ in $M$ at $x$,

we have that the euler number $\chi(\tilde{V})$ of $\tilde{V}$ is given by

$$\begin{aligned} \chi(\tilde{V}) &= \chi(V) + \sum_{x \in \Sigma V} (r_x - 1) \\ &= c^1(M) \cap i_*[V] - \langle i_*[V], i_*[V] \rangle \\ &\quad + \sum_{x \in \Sigma V} (\mu_x + r_x - 1) \end{aligned}$$

, since by Corollary to Theorem 1

$$\chi(V) = c_0(V) = (i^* c^1(M) - X(V, M)) \cap [V] + \sum_{x \in \Sigma V} \mu_x ,$$

where $\langle \ , \ \rangle$ stands for the intersection number.

By the Milnor's formula $2\delta_x = \mu_x + r_x - 1$, the formula for $\chi(\tilde{V})$ above is a topological expression of the so-called adjunction formula;

$$2 - 2\cdot g(\tilde{V}) = V\cdot(K+V) - 2\cdot\sum_{x\in\Sigma V}\delta_x ,$$

where $g(\tilde{V})$ is the genus of $\tilde{V}$, $V$ is a complex line bundle over $M$ determined by the divisor $V$ in $M$ and $K$ is the canonical line bundle of $M$, see Kodaira ( [5], p. 30, (3.10)) or Serre ( [8], p. 75, Proposition 5. ).

At this point, we remark that not every hypersurface admits a nice almost complex resolution in $M$;

Example 3. Let $V$ be a singular Enriques surface in $\mathbb{P}^3$ defined by

$$(z_0 z_1 z_2)^2 + (z_1 z_2 z_3)^2 + (z_2 z_3 z_0)^2 + (z_1 z_2 z_3)^2$$
$$+ (z_0 z_1 z_2 z_3)\cdot(\text{generic quadratic form}) = 0 .$$

The singular surface $V$ admits no nice almost complex resolution in $\mathbb{P}^3$.

In fact, $\Sigma V = 6$ lines $z_i = z_j = 0$ $(i \neq j)$, the "minimal" complex analytic Whitney stratification of $\Sigma V = \{ \Sigma V - (4$ triple points $\cup$ $24$ cusp (or Whitney umbrella) points$)$, these $28$ points $\}$. From this, if we suppose that there exists a nice almost complex resolution $\hat{V}$ of $V$ in $\mathbb{P}^3$, then we have that (from the condition (2) for $\hat{V}$)

$$\begin{aligned}\chi(\hat{V}) &= \chi(V) - \chi(\Sigma V) + 24\cdot 2 \\ &= \chi(V) - (6\cdot 2 - 2\cdot 4) + 24\cdot 2 \\ &= \chi(V) + 44.\end{aligned}$$

By the arguments of Kodaira ([5], pp. 41–51, (4.13)), we have that $\chi(V) = 28$. Hence we have that

$$\chi(\hat{V}) = 72.$$

On the other hand, by the condition (1) for $\hat{V}$,

$\chi(\hat{V})$ = the euler number of a non-singular surface of degree 6 in $\mathbb{P}^3$. Thus we should have

$\chi(\hat{V}) = 108 \ (\neq 72)$. This is a contradiction.

<u>Remark</u>. There is a simpler example of $V$ in $\mathbb{P}^3$ admitting no nice almost complex resolution in $\mathbb{P}^3$; two planes $(z_0 z_1 = 0)$ in $\mathbb{P}^3$. But this is not irreducible.

## §3. Normal euler classes, singular types and micro-equivalence.

<u>Definition</u>. A PL embedding $f: V \to M$ of

a polyhedron $V$ of dimension $m$ into a PL $(m+c)$-manifold $M$ has isolated singularity, if there is a 0-dim. subpolyhedron $R$ of $V$ such that for each point $x$ of $V - R$, there is an open neighborhood $U_x$ of $f(x)$ in $M$ such that $(U_x, U_x \cap V)$ is PL homeomorphic to $(\mathbb{R}^{m+c}, \mathbb{R}^m \times 0)$.

The minimal $R$ is called the singular set of $f$ and denoted by $\Sigma V$ $(= \Sigma_f V)$.

<u>Remark</u>. A restriction of $f$ $f | V - \Sigma V : V - \Sigma V \to M - f(\Sigma V)$ is locally flat. If $V$ is a PL $m$-manifold, and if $f$ has isolated singularity, then $f$ is called 1-flat by Noguchi [6].

In the following we shall assume that ($f : V \to M$ has isolated singularity,) $V$ is a compact polyhedron of dim. $m$ such that $V - \Sigma V$ is oriented and $M$ is a closed oriented PL $(m+c)$-manifold, when otherwise stated.

Taking divisions $L$, $K$ of $M$, $V$, we make $f : K \to L$ simplicial. For the second barycentric sub-divisions $L''$, $K''$ of $L$, $K$, we put

$D_x = st(f(x), L'')$, $st(x, K'') = C_x$, $lk(f(x), L'') = S_x$, $lk(x, K'') = K_x$ and

$f_x = f \mid C_x : C_x \to D_x$, $\dot{f}_x = f \mid K_x : K_x \to S_x$.

Then $D_x$, $C_x$ are cones $f(x) * S_x$, $x * K_x$ and $f_x$ is a cone extension of a locally flat embedding $\dot{f}_x : K_x \to S_x$ of the orientable closed $(m-1)$-manifold $K_x$ into the $(m+c-1)$-sphere. We put

$D_\Sigma = \bigcup_{x \in \Sigma V} D_x$, $C_\Sigma = \bigcup_{x \in \Sigma V} C_x$, $S_\Sigma = \bigcup_{x \in \Sigma V} S_x$,

$K_\Sigma = \bigcup_{x \in \Sigma V} K_x$, $V_0 = (V - C_\Sigma) \cup K_\Sigma$, $M_0 = (M - D_\Sigma) \cup S_\Sigma$,

$f_\Sigma = \bigcup_{x \in \Sigma V} f_x : C_\Sigma \to D_\Sigma$, $\dot{f}_\Sigma = \bigcup_{x \in \Sigma V} \dot{f}_x : K_\Sigma \to S_\Sigma$

and give orientations $S_\Sigma$, $K_\Sigma$ induced from $M_0$, $V_0$ as their "boundaries". Let $V_{0,i}$, $i = 1, \cdots, r$, be all connected components of $V_0$ and let

$K_{\Sigma,i} = K_\Sigma \cap V_{0,i}$ and $V_i = V_{0,i} \cup x * K_{\Sigma,i}$, $i = 1, \cdots, r$. Thus we may regard of $V$ as an ordered union $V_1 \cup \cdots \cup V_r$ of irreducible components $V_1, \cdots, V_r$ of $V$. In the same way, $K_\Sigma$ is regarded as an ordered union $K_{\Sigma,1} \cup \cdots \cup K_{\Sigma,r}$.

Thus we regard of $f$ and $f_\Sigma$ as ordered oriented PL embeddings.

Definition. Two ordered oriented PL embeddings $\varphi : W \to N$ and $\varphi' : W \to N'$ are equivalent, if there is an orientation preserving PL homeomorphism $h : N \to N'$ (called an equivalence,) such that $h \circ \varphi | W_i = \varphi' | W_i$ for each $i$. Two ordered oriented PL embeddings $\varphi$ and $\varphi'$ are micro-equivalent, if for some open neighborhoods $U$, $U'$ of $\varphi(W)$, $\varphi'(W)$ in $N$, $N'$, $\varphi : W \to U$ and $\varphi' : W \to U'$ are equivalent. ~~An~~ The equivalence $g : U \to U'$ is called a micro-equivalence.

In case $c \leqq 2$, we shall call the equivalence class ~~$f_\Sigma$~~ of an ordered ~~oriented~~ locally flat PL embedding $f_\Sigma : K_\Sigma \to S_\Sigma$ as to be the (ordered oriented) singularity of $f$ and denote it by $\sigma(f)$.

(Ordered) (normal) euler class $X(f)$ of $f : V \to M$ is defined by $X(f) = f^* \circ P^{-1} \circ f_* [V] = f^* \circ P^{-1} \circ f_* \left(\sum_{i=1}^{r} [V_i]\right) \in H^c(V; \mathbb{Z}) \cong \sum_{i=1}^{r} H^c(V_i; \mathbb{Z})$ $(m \geqq 2)$, where $P$ is the Poincaré duality isomorphism $\cap [M] : H^*(M; \mathbb{Z}) \to H_{m+c-*}(M; \mathbb{Z})$ and $[V]$, $[M]$ are the fundamental classes of $V$, $M$ determined by the orientations of $V - \Sigma V$, $M$.

*Theorem 2.* Let $\boxed{\text{two PL embeddings}}$ $f : V \to M$, $f' : V \to M'$ be with isolated singularity from a compact oriented ordered $m$-polyhedron $V$ into closed oriented $(m+c)$-manifolds $M, M'$.

(1) If $c = 1$, then $f$ and $f'$ are micro-equivalent if and only if $\sigma(f) = \sigma(f')$.

(2) If $c = 2$, then $f$ and $f'$ are micro-equivalent if and only if $\sigma(f) = \sigma(f')$ and $X(f) = X(f')$.

( Compare with Noguchi [7] ).

The proof of Theorem 2 will be given in the forthcoming paper [3].

Let $V$ be a compact oriented ordered $m$-subpolyhedron of a closed oriented PL $(m+c)$-manifold $M$ such that an inclusion map $i : V \to M$ has isolated singularity. We define $\sigma(V, M) = \sigma(i)$ and $X(V, M) = X(i)$. Two such pairs $(M, V)$ and $(M', V')$ are micro-equivalent, if there is a PL homeomorphism $h : V \to V'$ such that $i$ and $i' \circ h$ are micro-equivalent. In this case, putting $h^* \sigma(V', M') = \sigma(i' \circ h)$, we have that

$$\sigma(V, M) = h^* \sigma(V', M') \quad \text{and} \quad X(V,M) = h^* X(V', M').$$

Hence we have by Theorem 2

Corollary to Theorem 2. An orientation (and order) preserving PL homeomorphism $h : V \to V'$ extends to an orientation preserving PL homeomorphism of (some) neighborhoods $U$ and $U'$ of $V$ and $V'$ in $M$ and $M'$ (, called a micro-equivalence) if and only if $h^* \sigma(V', M') = \sigma(V, M)$ and $h^* X(V', M') = X(V, M)$.

Now we are ready to prove Theorem stated in the introduction.

Proof of Theorem. We regard of $(M, V)$ and $(M, V')$ as to be compatibly triangulated.

The only if part is immediate from Theorem 2.

We are going to prove that if there is a bijection $\alpha : \Sigma V \to \Sigma V'$ such that two oriented pairs $(S_x, K_x)$ and $(S'_{\alpha(x)}, K'_{\alpha(x)})$ are PL homeomorphic (preserving orientations) and if $V$ and $V'$ represent the same homology class in $M$, i.e., $i_*[V] = i'_*[V']$, then $(M, V)$ and $(M, V')$ are micro-equivalent.

Let $H_\Sigma : (S_\Sigma, K_\Sigma) \to (S'_\Sigma, K'_\Sigma)$ be an orientation preserving ) PL homeomorphism such that $H_{\Sigma}(S_x, K_x) = (S'_{\alpha(x)}, K_{\alpha(x)})$ for each $x \in \Sigma V$. By §2, Example 2, normalizations $\tilde{V}$ and $\tilde{V}'$ are PL homeomorphic. Since $V_0$ and $V_0'$ are obtained from $\tilde{V}$ and $\tilde{V}'$ by deleting the interiors of disks on $\tilde{V}$ and $\tilde{V}'$, it follows from the homoginity of disks on a surface that $H_{\Sigma} / K_{\Sigma}$ canbe extended to an orientation preserving PL homeomorphism $h_0 : V_0 \to V_0'$. By cone extensions, $h_0 : V_0 \to V_0'$ can be extended to a PL homeomorphism $h : V \to V'$. From the construction, we have that $h^* \sigma(V', M) = \sigma(V, M)$. Moreover, $V$ and $V'$ represent the same homology class in $M$ (and $h_*[V] = [V']$). Hence $h^* X(V', M) = X(V, M)$. It follows from Corollary to Theorem 2 that $(M, V)$ and $(M, V)$ are micro-equivalent, completing the proof.

<u>Remark</u>. Theorem still holds without hypothesis of irreducibility of $V$, if we consider "orders" of $V$ and link types.

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