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Local models of degenerate varieties

M.-N. Ishida and T. Oda
(Mathematical Institute Tohoku University)

Introduction

In connection with the compactification problem of moduli spaces, we encounter many examples of "degenerate varieties". For instance, (i) stable curves of Deligne-Mumford [DM], (ii) degenerate Jacobian varieties of Oda-Seshadri [OS] or, more generally degenerate abelian varieties of Namikawa and Nakamura [N1] [N3], (iii) degenerate hyper-elliptic surfaces of Tsuchihashi [T1] and (iv) degenerate forms of Hopf surfaces and other surfaces of class VII_0, for instance, by Kodaira [K1], Miyake-Oda [MO] and Nakamura [N5].

These degenerate varieties are reduced and connected. But in general they are reducible and their irreducible components need not cross normally. The singularities are formally isomorphic to those of affine varieties defined by ideals generated by monomials, which, however, are usually too many in number to give rise to (local) complete intersections.

Thus it is rather hard to deal with them through their defining equations. Fortunately, however, we have a way of dealing systematically with monomials by means of the theory of torus embeddings by Demazure [D1], Mumford et al. [TE] and Miyake-Oda [MO], which we briefly recall in §2.

As the first reasonable local models of degenerate varieties, we thus study in §3 closed invariant reduced subschemes of smooth (or more generally normal) torus embeddings. Basing ourselves on our results for local models, we attempt to formulate in §6 a global theory of degenerate varieties, which, in a sense, is a generalization of the theory of toroidal embeddings by Mumford et al. [TE].

We are able to get the following:

(1) (Ishida) The description in §3 of the dualizing complex $K^{\bullet}_{Y}(\underline{O})$ of a closed invariant reduced subscheme Y of a normal torus embedding Z and the criterion for Y to be Cohen-Macaulay or Gorenstein. The results include, as special cases, those of Hochster [H1], Mumford et al. [TE], Reisner [R1] and Goto-Watanabe [GW].

(2) The part of the "tangent complex" necessary for the formal deformation theory, i.e. $\underline{\mathrm{Ext}}^{i}_{\underline{O}_Y}(L^{Y}_{\cdot}, \underline{O}_Y)$ for $i = 0, 1$ (the case $i = 2$ being yet to be computed), where Y is a Gorenstein closed invariant reduced subscheme of a smooth torus embedding Z, $\underline{\mathrm{Ext}}^{i}$ are the local hyperextension groups and $L^{Y}_{\cdot}$ is the cotangent complex studied by Lichtenbaum-Schlessinger, Grothendieck, Rim and Illusie. (§4, §5)

(3) The Picard group of a global degenerate variety X when it has some good properties. (§6)

Our results grew out of our effort to understand and generalize those of Nakamura [N3], who obtained (1) and (2) in the case of degenerate abelian varieties.

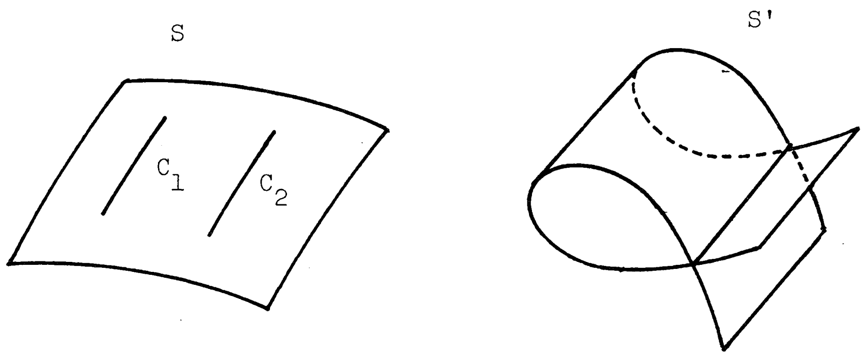
All the varieties we consider here are either reduced algebraic varieties over the field $\mathbb{C}$ of complex numbers or reduced complex analytic spaces. They may be reducible.

§1 Singularities of degenerate varieties

We give here some examples of singularities of degenerate varieties.

1. The case of curves. Deligne and Mumford introduced the notion of stable curves in order to compactify the moduli space of smooth curves. Stable curves have at worst ordinary double points. Note that an ordinary double point is the singularity formally isomorphic to that of the curve $Y = \{xy = 0\} \subset \mathbb{C}^2 = Z$ at the origin.

2. If there are disjoint and isomorphic non-singular curves C_1, C_2 on a non-singular surface S, we get a new surface S' by glueing C_1 and C_2 through an isomorphism $C_1 \xrightarrow{\sim} C_2$.



This surface S' has singularities formally isomorphic to $Y = \{xy = 0\} \subset \mathbb{C}^3 = Z$. This type of surface actually appears as a degenerate abelian surface and a degenerate Hopf surface.

3. Let Y be the union of n planes in $\mathbb{C}^n = Z$ $(n \geq 3)$ with coordinates (x_1, x_2), (x_2, x_3), ..., (x_{n-1}, x_n) and (x_n, x_1). Y is called the elliptic polygonal n-cone in Mumford [M1], and is semi-stable for $n \leq 6$ in his sense. Y is defined by equations as follows :

$$n = 3, \quad Y = \{x_1x_2x_3 = 0\} \subset \mathbb{C}^3$$

$$n = 4, \quad Y = \{x_1x_3 = x_2x_4 = 0\} \subset \mathbb{C}^4$$

$$n \geq 5, \quad Y = \left\{ \begin{array}{ll} x_ix_j = 0 & i = 1, \ldots, n \\ & j \neq i-1, i, i+1 \pmod{n} \end{array} \right\} \subset \mathbb{C}^n$$

These singularities for $n = 4$ and 6 actually appear in degenerate abelian surfaces.

Note that the first and the second examples and the third example for $n = 3$ and 4 are complete intersections, but the third example for $n \geq 5$ is not. However all these examples have non-singular normalizations. The following example is a little bit more complicated.

4. Define an involution ι on $\mathbb{C}^4$ by $\iota(x_1, x_2, x_3, x_4) = (-x_1, -x_2, -x_3, -x_4)$, and let Y be the quotient of $\{x_1x_3 = x_2x_4 = 0\} \subset \mathbb{C}^4$ by ι. Y is a 2-dimensional subvariety of $Z = \mathbb{C}^4/\iota$, and is the union of 4 copies of the irreducible variety $\{uv - w^2 = 0\} \subset \mathbb{C}^3$. To describe Y , we need 8 variables and 4 equations of the type (monomial) = (monomial) as well as 16 equations of the type (monomial) = 0. Thus it is far from being a complete intersection. This singularity actually appears in degenerate hyperelliptic surfaces.

§2 A brief review of torus embeddings

Let $T = \underbrace{\mathbb{C}^* \times \mathbb{C}^* \times \cdots \times \mathbb{C}^*}_{r}$ be an algebraic torus of dimension r.

Definition 2.1. A normal variety Z of dimension r is a T-embedding if (1) Z contains T as a Zariski open subset, and (2) the action of T on itself by multiplication can be extended to an algebraic action of T on Z.

$$\begin{array}{ccc} T \times Z & \xrightarrow{\text{regular}} & Z \\ \cup & \text{multi.} & \cup \\ T \times T & \longrightarrow & T \end{array}$$

By the results of [TE], to every T-embedding Z is associated a rational partial polyhedral decomposition Δ (= a collection of cones in the real vector space $\mathbb{R}^r$ together with the lattice $\mathbb{Z}^r$). Many geometric properties (completeness, projectivity, non-singularity etc.) are described easily in terms of Δ. The most important for us is the following one-to-one correspondence.

$$\left\{\begin{array}{l} \text{T-invariant reduced} \\ \text{closed irreducible} \\ \text{subvarieties of } Z \end{array}\right\} \xleftrightarrow{\ 1\,:\,1\ } \Delta$$
$$V(\sigma) \longleftarrow\!\!\!\shortmid \ \sigma$$

Furthermore, $\dim V(\sigma) + \dim \sigma = r$ for every $\sigma \in \Delta$ and $V(\tau)$ is contained in $V(\sigma)$ if and only if σ is a face of τ. Each $V(\sigma)$ has a unique open T-orbit $V^{\circ}(\sigma)$.

Definition 2.2. For a subset Σ of Δ, we say

$$\Sigma \text{ is } \begin{cases} \underline{\text{star closed}} \text{ if } \Sigma \ni \sigma\ , \Delta \ni \tau \text{ and } \tau \succ \sigma \text{ imply } \Sigma \ni \tau \\ \underline{\text{locally star closed}} \text{ if } \Sigma \ni \rho, \sigma\ , \Delta \ni \tau \text{ and } \rho \succ \tau \succ \sigma \\ \qquad\qquad \text{imply } \Sigma \ni \tau \end{cases} .$$

For a locally closed subset Φ of Δ , we set $\Phi_i = \{\sigma \in \Phi\ ;\ \dim \sigma = i\}$ and $C^i(\Phi, \mathbb{Z}) = \mathrm{Map}(\Phi_i, \mathbb{Z})$. We define a coboundary map $\delta : C^i(\Phi, \mathbb{Z}) \longrightarrow C^{i+1}(\Phi, \mathbb{Z})$ by $(\delta f)(\tau) = \sum_{\sigma \in \Phi_i} [\sigma, \tau] f(\sigma)$ for every $f \in C^i(\Phi, \mathbb{Z})$ and every $\tau \in \Phi_i$, where $[\sigma,\tau]$ ($= 0, +1$ or -1) is the incidence defined by fixing an orientation for each cone in Δ . Thus we get the complex $C^{\cdot}(\Phi, \mathbb{Z})$ of $\mathbb{Z}$-modules. We denote by $H^i(\Phi, \mathbb{Z})$ its i-th cohomology group.

§3 Local models of degenerate varieties

We take a T-embedding Z as an ambient space, and take possibly reducible T-invariant reduced closed subvarieties Y of Z as local models of degenerate varieties. Let Δ be the polyhedral decomposition associated to Z . Then to such $Y \subset Z$, is associated a subset Σ of Δ by $\Sigma = \Sigma_Y = \{\sigma \in \Delta : V(\sigma) \subset Y\}$. Clearly Σ is a star closed subset of Δ , i.e. σ in Δ belongs to Σ if a face of σ is in Σ . Let h be the codimension of Y in Z . Then for each integer i with $h \le h+i \le r = \dim Z$, we set $Y^i = \bigcup_{\sigma \in \Sigma_{h+i}} V(\sigma)$, the union of $(r-h-i)$-dimensional T-invariant closed subvarieties contained in Y , where $\Sigma_j = \{\sigma \in \Sigma\ ;\ \dim \sigma = j\}$. If Y is equidimensional, these varieties induce a filtration $Y^0 \supset Y^1 \supset \ldots \supset Y^n$ ($n = r-h = \dim Y$) on Y. Clearly, the normalization $\tilde{Y}^i$ of Y^i is the disjoint union

$\coprod_{\sigma \in \Sigma_{h+i}} V(\sigma)$. Consider the usual sheaves $\mathbb{Z}_W$, $\mathbb{C}_W$, $\mathbb{C}^*_W$, $\underline{O}_W$ and $\underline{O}^*_W$ on a variety W . Let $\underline{F}$ represent any one of the symbols $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{C}^*$, $\underline{O}$ and $\underline{O}^*$. We set $K^i_Y(\underline{F}) = \lambda_{i*}\underline{F}_{\tilde{Y}^i}$ where λ_i is the composite map $\tilde{Y}^i \longrightarrow Y^i \hookrightarrow Y$. Since $\underline{F}_{\tilde{Y}^i} = \bigoplus_{\sigma \in \Sigma_{h+i}} \underline{F}_{V(\sigma)}$, we can define the coboundary map $\delta : K^i_Y(\underline{F}) \longrightarrow K^{i+1}_Y(\underline{F})$ by

$$\delta((f_\sigma)_{\sigma \in \Sigma_{h+i}}) = (\sum_{\sigma \in \Sigma_{h+i}} [\sigma, \tau] Q^\tau_\sigma(f_\sigma))_{\tau \in \Sigma_{h+i+1}} ,$$

where $Q^\tau_\sigma : \underline{F}_{V(\sigma)} \longrightarrow \underline{F}_{V(\tau)}$ is the restriction map for $\sigma \prec \tau$ and 0 otherwise and $[\sigma, \tau]$ is the incidence. Thus we have a sequence $K^\cdot_Y(\underline{F}) = (\ldots \to 0 \longrightarrow K^0_Y(\underline{F}) \longrightarrow K^1_Y(\underline{F}) \longrightarrow K^2_Y(\underline{F}) \longrightarrow \ldots)$, which can easily be shown to be a complex of sheaves on Y .

<u>Theorem</u> 3.1. $K^\cdot_Y(\underline{O})$ is the dualizing complex of Y .

Since we always have $\underline{H}^0(K^\cdot_Y(\underline{F})) \neq 0$, we have the following corollary.

<u>Corollary</u> 3.2. The following are equivalent.

(1) The cohomology sheaf $\underline{H}^i(K^\cdot_Y(\underline{O}))$ vanishes for every $i > 0$.

(2) The cohomology sheaf $\underline{H}^i(K^\cdot_Y(\mathbb{C}))$ vanishes for every $i > 0$.

(3) $H^i(\Sigma(\rho), \mathbb{C}) = 0$ for every $i > h$ and every ρ in Σ , where $\Sigma(\rho) = \{\sigma \in \Sigma ; \sigma \prec \rho\}$.

(4) Y is Cohen-Macaulay.

Furthermore, Y is Gorenstein if and only if these conditions are satisfied and $\underline{H}^0(K^\cdot_Y(\underline{O}))$ is an invertible $\underline{O}_Y$-module.

(3) is a combinatorial property of the abstract complex $\Sigma = \Sigma_Y$, and is a generalization of the result of Reisner [R1] for Y em-

bedded in the affine space $\mathbb{C}^r$ with the natural $(\mathbb{C}^*)^r$-action. Hochster [H1] showed every torus embedding is Cohen-Macaulay, and Mumford et al. [TE] gave an explicit description of its dualizing sheaf. These results also follow from our theorem.

For a map $\alpha : \Sigma_h = \{\sigma \in \Sigma\ ;\ \dim \sigma = h\} \longrightarrow \{-1, 1\}$, we define a morphism $\tilde{\alpha} : \underline{F}_Y \longrightarrow K_Y^0(\underline{F}) = \bigoplus_{\sigma \in \Sigma_h} \underline{F}_{V(\sigma)}$ by $\tilde{\alpha}(f) = \sum_{\sigma \in \Sigma_h} \alpha(\sigma) f|_{V(\sigma)}$.

<u>Defintion</u> 3.4. We say Y is $\underline{F}$-spherical if there exists α such that $\tilde{\alpha} : \underline{F}_Y \longrightarrow K_Y^0(\underline{F})$ induces an exact sequence

$$0 \longrightarrow \underline{F}_Y \longrightarrow K_Y^0(\underline{F}) \longrightarrow K_Y^1(\underline{F}) \longrightarrow \cdots .$$

The α above depends on the orientation of cones σ in Σ_h. By changing the orientation of σ if necessary, however, we may assume $\alpha(\sigma) = 1$ for every σ.

Since $\underline{F}$ can be any one of the five variants, we get five diffrent versions of sphericity.

<u>Theorem</u> 3.5. The following implications hold.

$$\begin{array}{ccccc} & \nearrow & \underline{O}^*\text{-spherical} & & \\ \mathbb{Z}\text{-spherical} & \Rightarrow & \mathbb{C}\text{-spherical} & \Leftrightarrow & \underline{O}\text{-spherical} \\ & \searrow & \mathbb{C}^*\text{-spherical} & & \end{array}.$$

We can interpret $\mathbb{C}$-sphericity ($= \underline{O}$-sphericity) by a combinatorial condition as follows.

<u>Proposition</u> 3.6. The following conditions on Y are equivalent.

(i) Y is $\mathbb{C}$-spherical.

(ii) Y is Cohen-Macaulay, $H^h(\Sigma(\rho), \mathbb{C}) \simeq \mathbb{C}$ for every $\rho \in \Sigma$ and $H^h(\Sigma, \mathbb{C}) \simeq \mathbb{C}$.

(iii) Y is Cohen-Macaulay, $H^h(\Sigma(\rho), \mathbb{C}) \simeq \mathbb{C}$ for every $\rho \in \Sigma_{h+1}$ and $H^h(\Sigma, \mathbb{C}) \neq 0$.

If Y is $\underline{O}$-spherical, then $\underline{H}^0(K_Y^{\cdot}(\underline{O})) \simeq \underline{O}_Y$. Thus we have;

Corollary 3.7. Y is Gorenstein if Y is $\underline{O}$-spherical.

Each Z in Examples 1 through 4 in §1 has the natural structure of a torus embedding and Y is a closed subvariety invariant under the action of the torus. We see easily that Y in Examples 1, 3 and 4 are $\mathbb{Z}$-spherical. Y in Example 2 is also $\mathbb{Z}$-spherical if we replace $Z = \mathbb{C}^3$ by $Z = \mathbb{C}^2 \times \mathbb{C}^*$. Hence every Y in §1 are Gorenstein.

Example 3.8. When $Z = \mathbb{C}^r$ with the natural $(\mathbb{C}^*)^r$-action, to $Y \subset Z$ is associated a simplicial complex (see Reisner [R1] or Hochster [H2]). Hochster showed Y is Gorenstein if the simplicial complex is a triangulation of a sphere. We can show such a Y is $\mathbb{Z}$-spherical.

Example 3.9. For an arbitrary T-embedding $\bar{Z}$, $Y = \bar{Z} \setminus T$ is $\mathbb{Z}$-spherical. Since a local ring R is Gorenstein if and only if its completion $\hat{R}$ is, it follows that the boundary of any toroidal embedding is Gorenstein.

§4 $\underline{Ext}^0_{\underline{O}_Y}(\Omega^1_{\underline{O}_Y}, \underline{O}_Y)$ of local models

According to Rim[R2] and Illusie [I3], the global hyperextension groups $\mathrm{Ext}^i_{\underline{O}_X}(L^X_\cdot, \underline{O}_X)$ for $i = 0, 1$ and 2 play important roles for the deformation theory of the variety X, where $L^X_\cdot$ is the cotangent complex introduced and studied by Lichtenbaum-Schlessinger [LS] and Illusie [I3]. In order to compute these groups, we want to know the sheaves $\underline{\mathrm{Ext}}^i_{\underline{O}_X}(L^X_\cdot, \underline{O}_X)$ for $i = 0, 1$ and 2, In this section we define a complex $K^\cdot_Y(\Theta)$ for a T-invariant Y contained in a non-singular torus embedding Z, and relate it to $\underline{\mathrm{Ext}}^0_{\underline{O}_Y}(\Omega^1_Y, \underline{O}_Y)$, which is equal to $\underline{\mathrm{Ext}}^0_{\underline{O}_Y}(L^Y_\cdot, \underline{O}_Y)$.

Let Y be a T-invariant closed subvariety of a non-singular T-embedding Z. Then for σ in the associated Σ, $V(\sigma)$ is non-singular and $D(\sigma) = V(\sigma) \setminus V^O(\sigma)$ is a divisor with normal crossing on $V(\sigma)$, where $V^O(\sigma)$ is the open T-orbit in $V(\sigma)$. Let $\Theta_{V(\sigma)}(\log D(\sigma))$ be the subsheaf of the tangent sheaf $\Theta_{V(\sigma)}$ consisting of sections which send the ideal of $D(\sigma)$ into itself. Let $V(\tau)$ be an irreducible component of $D(\sigma)$. Then for an open subset U of $V(\sigma)$, and for a section s of $\Theta_{V(\sigma)}(\log D(\sigma))$ on U, the derivation $s : \underline{O}_U \longrightarrow \underline{O}_U$ induces a derivation $\bar{s} : \underline{O}_{U \cap V(\tau)} \longrightarrow \underline{O}_{U \cap V(\tau)}$. By the fact that $D(\tau) = \left(\bigcup_{\substack{\tau' \in \Sigma_{h+i+1} \\ \tau' \neq \tau}} V(\tau') \right) \cap V(\tau)$ $(h+i = \dim \sigma)$, it is easy to see that $\bar{s}$ is a section of $\Theta_{V(\tau)}(\log D(\tau))$ on $U \cap V(\tau)$. Hence the restriction map Q^τ_σ : $\Theta_{V(\sigma)}(\log D(\sigma)) \longrightarrow \Theta_{V(\tau)}(\log D(\tau))$ can be defined. Set $K^i_Y(\Theta) = \lambda_{i*}\Theta_{\tilde{Y}_i}(\log D^i)$ where $D^i = \coprod_{\sigma \in \Sigma_{h+i}} D(\sigma)$ is the divisor with normal crossing on $\tilde{Y}^i = \coprod_{\sigma \in \Sigma_{h+i}} V(\sigma)$. Using the above restriction map, we can define the coboundary map $\delta : K_Y^i(\Theta) \longrightarrow K_Y^{i+1}(\Theta)$ and we get a complex $K_Y^{\bullet}(\Theta)$ similary as in §3.

Theorem 4.1. If Y is Cohen-Macaulay, then $\underline{H}^i(K_Y^{\bullet}(\Theta)) = 0$ for every $i > 0$ and $\underline{H}^0(K_Y^{\bullet}(\Theta)) = \underline{\mathrm{Ext}}^0_{\underline{O}_Y}(\Omega^1_{\underline{O}_Y}, \underline{H}^0(K_Y^{\bullet}(\underline{O})))$.

Since $\underline{H}^0(K_Y^{\bullet}(\underline{O}))$ is the dualizing sheaf by Theorem 3.1, we have the following corollary.

Corollary 4.2. If Y is Gorenstein, then $\underline{\mathrm{Ext}}^0_{\underline{O}_Y}(\Omega^1_{\underline{O}_Y}, \underline{O}_Y) = \underline{H}^0(K_Y^{\bullet}(\Theta)) \otimes \underline{H}^0(K_Y^{\bullet}(\underline{O}))^{-1}$. In particular, if Y is $\mathbb{C}$-spherical, then there exists an exact sequence

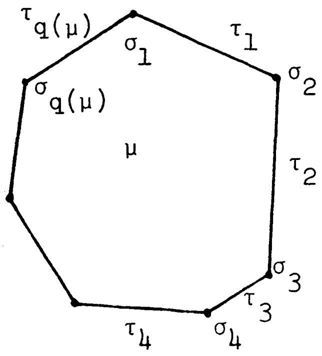
$$0 \longrightarrow \underline{\mathrm{Ext}}^0_{\underline{O}_Y}(\Omega^1_{\underline{O}_Y}, \underline{O}_Y) \longrightarrow K_Y^0(\Theta) \longrightarrow K_Y^1(\Theta) \longrightarrow \cdots$$

This is a generalization of the exact sequence obtained by Nakamura [N3] in the case of stable quasi-abelian varieties.

§5 $\underline{\mathrm{Ext}}^1_{\underline{O}_Y}(\Omega^1_Y, \underline{O}_Y)$

Let Y be a $\mathbb{C}$-spherical T-invariant closed subvariety of an affine non-singular T-embedding Z . Then we can calculate the extension group $\underline{\mathrm{Ext}}^1_{\underline{O}_Y}(\Omega^1_Y, \underline{O}_Y)$ which is equal to $\underline{\mathrm{Ext}}^1_{\underline{O}_Y}(L^Y_{\bullet}, \underline{O}_Y)$ since Y is reduced.

Let μ be a cone in Σ_{h+2} ($h = \mathrm{codim}_Z Y$). Then the $\mathbb{C}$-sphericity implies that each τ in Σ_{h+1} has exactly two faces in Σ_h , and for each face $\sigma \prec \mu$ in Σ_h there are exactly two cones τ' , τ'' with $\sigma \prec \tau' \prec \mu$ and $\sigma \prec \tau'' \prec \mu$. We denote by $q(\mu)$ the

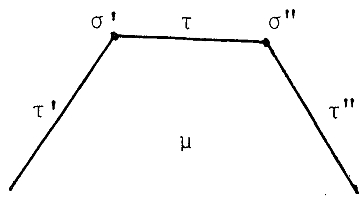


number of faces in Σ_{h+1} of μ.

For τ in Σ_{h+1}, we define an invertible sheaf $\underline{L}_\tau$ on the codimension one subvariety $V(\tau)$ of Y by

$$\underline{L}_\tau = \underline{O}_{V(\tau)}\Big(\sum_{\substack{q(\mu)=3\\ \tau\prec\mu\in\Sigma_{h+2}}} V(\mu) - \sum_{\substack{q(\mu)\geq 5\\ \tau\prec\mu\in\Sigma_{h+2}}} V(\mu)\Big)\otimes N_{V(\sigma')/V(\tau)}\otimes N_{V(\sigma'')/V(\tau)},$$

where σ' and σ'' are the faces of τ in Σ_h and $N_{V(\sigma)/V(\tau)}$ is the normal bundle of $V(\tau)$ in $V(\sigma)$, hence is an invertible $\underline{O}_{V(\tau)}$-module. Let μ be a cone in Σ_{h+2} with $\mu\succ\tau$, and let τ' (resp. τ'') be a cone in Σ_{h+1} with $\mu\succ\tau'\succ\sigma'$ (resp. $\mu\succ\tau''\succ\sigma''$). Then the restriction of $\underline{L}_\tau$ to the open T-orbit $V^o(\mu)$ of $V(\mu)\subset V(\tau)$ is



$$\{N_{V(\tau)/V(\mu)}\otimes N_{V(\tau')/V(\mu)}\otimes N_{V(\tau'')/V(\mu)}\}|V^o(\mu) \qquad \text{if } q(\mu)=3$$

$$\{N_{V(\tau')/V(\mu)}\otimes N_{V(\tau'')/V(\mu)}\}|V^o(\mu) \qquad \text{if } q(\mu)=4 \text{ and}$$

$$\{N^{-1}_{V(\tau)/V(\mu)}\otimes N_{V(\tau')/V(\mu)}\otimes N_{V(\tau'')/V(\mu)}\}|V^o(\mu) \qquad \text{if } q(\mu)\geq 5\ .$$

Now we set

$$A = \{\{\mu,\tau',\tau''\} : \mu\in\Sigma_{h+2},\ q(\mu)=4,\ \mu\succ\tau',\ \tau''\in\Sigma_{h+1},\ \tau' \text{ and } \tau'' \text{ have no common face in } \Sigma_h\}$$

and

$$B = \{\{\mu,\tau\} : \mu\in\Sigma_{h+2},\ q(\mu)=3,\ \mu\succ\tau\in\Sigma_{h+1}\}\ .$$

For $\alpha = \{\mu, \tau', \tau''\}$ in A, we set $\underline{M}_\alpha = \left\{ N_{V(\tau')/V(\mu)} \otimes N_{V(\tau'')/V(\mu)} \right\} \Big| V^\circ(\mu)$, and for $\beta = \{\mu, \tau\}$ in B, we set $\underline{N}_\beta = \left\{ N_{V(\tau)/V(\mu)} \otimes N_{V(\tau')/V(\mu)} \otimes N_{V(\tau'')/V(\mu)} \right\} \Big| V^\circ(\mu)$ where τ' and τ'' are the other cones in Σ_{h+1} with $\mu \succ \tau', \tau''$. We define a homomorphism

$$\phi : \bigoplus_{\tau \in \Sigma_{h+1}} \underline{L}_\tau \longrightarrow \left(\bigoplus_{\alpha \in A} \underline{M}_\alpha \right) \oplus \left(\bigoplus_{\beta \in B} \underline{N}_\beta \right)$$

of quasi-coherent sheaves on Y as follows: For $\tau \in \Sigma_{h+1}$ and $\alpha = \{\mu, \tau_1', \tau_1''\} \in A$ (resp. $\beta = \{\mu, \tau_1\} \in B$) the map $\underline{L}_\tau \longrightarrow \underline{M}_\alpha$ (resp. $\underline{L}_\tau \longrightarrow \underline{N}_\beta$) is the zero map if $\mu \not\succ \tau$ or $\tau = \tau_1'$ or τ_1'' (resp. if $\mu \not\succ \tau$ or $\tau = \tau_1$) and $[\tau, \mu]$-times the restriction map if $\mu \succ \tau$ and $\tau \neq \tau_1', \tau_1''$ (resp. if $\mu \succ \tau$ and $\tau \neq \tau_1$).

Theorem 5.1. $\mathrm{Ext}^1_{\underline{O}_Y}(\Omega^1_Y, \underline{O}_Y) = \mathrm{Ker}\ \phi$.

§6 Global degenerate varieties

Let Z be a torus embedding and let Y be an equidimensional closed subvariety of Z which is invariant under the torus action. Thus we have the filtration $Y = Y^0 \supset Y^1 \supset \ldots \supset Y^n$ ($n = \dim Y$).

Definition 6.1. An abstract degenerate variety is a connected analytic space X with a filtration $X = X^0 \supset X^1 \supset \ldots \supset X^n$ ($n = \dim X$) by closed subvarieties such that for every point $x \in X$, there exist an open neighborhood U_x of x, a torus embedding Z_x, an equidimensional closed subvariety Y_x of Z_x invariant under the torus action and an open immersion $\phi_x : U_x \longrightarrow Y_x$ with

$\phi_x^{-1}(Y_x^i) = \psi_x^{-1}(X^i)$ for every $i \geq 0$, where $\psi_x\colon U_x \longrightarrow X$ is the inclusion map. In the category of algebraic spaces, the notion of an abstract degenerate variety can be defined in a similar manner with ϕ_x and ψ_x taken to be étale morphisms.

We occasionally impose some of the following additional conditions.

1) X is Cohen-Macaulay or, equivalently, we can take Y_x Cohen-Macaulay for every point $x \in X$.

2) X is Gorenstein or, equivalently, we can take Y_x Gorenstein for every point $x \in X$.

3) X is locally $\mathbb{Z}$-spherical, i.e. we can take $\mathbb{Z}$-spherical Y_x for every point x of X.

4) The normalization of X is non-singular or, equivalently, we can take Y_x embedded in a non-singular torus embedding Z_x for every $x \in X$.

5) X is globally oriented, i.e. there exists a complex $K_X^{\cdot}(\mathbb{Z})$ of sheaves on X such that the restriction $K_X^{\cdot}(\mathbb{Z})|_{U_x}$ is equal to the restriction $K_{Y_x}^{\cdot}(\mathbb{Z})|_{U_x}$.

6) X is $\mathbb{Z}$-spherical, i.e. X is globally oriented and there is a morphism from the constant sheaf $\mathbb{Z}_X$ to $K_X^0(\mathbb{Z})$ which induces an exact sequence $0 \longrightarrow \mathbb{Z}_X \longrightarrow K_X^{\cdot}(\mathbb{Z})$.

We obviously have the implications $6) \Rightarrow 3) \Rightarrow 2) \Rightarrow 1)$.

Degenerate abelian varieties (= SQAV) of Namikawa and Nakamura are $\mathbb{Z}$-spherical and their normalizations are non-singular at least when dimension ≤ 4.

If an abstract degenerate variety X is globally oriented, we can also consider the complexes of sheaves $K_X^{\bullet}(\mathbb{C})$, $K_X^{\bullet}(\mathbb{C}^*)$, $K_X^{\bullet}(\underline{O})$ and $K_X^{\bullet}(\underline{O}^*)$ as in the case of local models. Furthermore if X is $\mathbb{Z}$-spherical, they give rise to the resolutions of sheaves $\mathbb{C}_X$, $\mathbb{C}^*{}_X$, $\underline{O}_X$ and $\underline{O}^*{}_X$, respectively. Using the resolution $0 \longrightarrow \underline{O}^*{}_X \longrightarrow K_X^0(\underline{O}^*) \longrightarrow K_X^1(\underline{O}^*) \longrightarrow \dots$, we obtained the following theorem on the Picard group of X .

<u>Theorem</u> 6.2. If X is a compact $\mathbb{Z}$-spherical abstract degenerate variety, then $\mathrm{Pic}^{\circ}(X)$ is written as an extension

$$0 \longrightarrow H^1(K_X^{\bullet}(\mathbb{C}^*)) \longrightarrow \mathrm{Pic}^{\circ}(X) \longrightarrow \mathrm{Ker}[\mathrm{Pic}^{\circ}\tilde{X}^0 \longrightarrow \mathrm{Pic}^{\circ}\tilde{X}^1] \longrightarrow 0 \ ,$$

where $\mathrm{Pic}^{\circ}\tilde{X}^0 \longrightarrow \mathrm{Pic}^{\circ}\tilde{X}^1$ is the map induced by the coboundary map $K_X^0(\underline{O}^*) \longrightarrow K_X^1(\underline{O}^*)$.

Note that $\mathrm{Ker}[\mathrm{Pic}^{\circ}\tilde{X}^0 \longrightarrow \mathrm{Pic}^{\circ}\tilde{X}^1]$ is an abelian variety and $H^1(K_X^{\bullet}(\mathbb{C}^*))$ is an algebraic torus. This is a generalization of the description by Oda and Seshadri of the generalized Jacobian varieties for curves with at worst ordinary double points. This theorem is the first step to the generalization of the theory of Oda and Seshadri [OS] and Ishida [I2].

REFERENCES

[D1] M. Demazure, Sous-groupes algébriques de rang maximum du groupe de Cremona, Ann. scient. Éc. Norm. Sup., 4^e série, t. 3 (1970) 507-588.

[DM] P. Deligne and D. Mumford, The irreducibility of the space of given genus, Publ. Math. I.H.E.S., 36 (1970).

[G1] A. Grothendieck, Catégorie cofibrées additives et Complex cotangent relatif, Lecture Notes in Math. 79, Springer-Verlag Berlin Heidelberg New York (1968).

[GW] S. Goto and K. Watanabe, On graded rings I-II, to appear.

[H1] M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, Ann. of Math. 96 (1972), 318-337.

[H2] M. Hochster, Cohen-Macaulay rings, combinatorics, and simplicial complexes, Ring theory II, Lecture Notes in Pure and App. Math 26 (1977), Dekker, 171-223.

[I1] M.-N. Ishida, Torus embeddings and Dualizing complexes, to appear.

[I2] M.-N. Ishida, Compactifications of a family of generalized Jacobian varieties, to apear in Proc. of International simposium on Algebraic Geometry, Kyoto, 1977.

[I3] L. Illusie, Complex Cotangent et Déformations, I - II., Lecture Notes in Math. 239, 283, Springer-Verlag,Berlin Heideberg New York (1971).

[K1] K. Kodaira, On the structure of compact complex analytic surfaces, I~IV, Amer. J. Math. 86 (1964), 751-798; 88 (1966), 682-721; 90 (1968), 55-83 and 1048-1066.

[LS] S. Lichtenbaum and M. Schlessinger, The cotangent complex of a morphism, Trans. Amer. Math. Soc. 128 (1967), 41-70.

[M1] D. Mumford, Stability of projective varieties, Monographie N°24 de L'Enseignment Math. Imprimerie Kundig Genève, (1977).

[MO] K. Miyake and T. Oda, Torus embeddings and applications. Tata Inst. Lecture Notes, to appear.

[N1] Y. Namikawa, A new compactification of the Siegel space and the degeneration of abelian varieties, I - II, Math. Ann. 221 (1976), 97-141 and 201-241.

[N2] Y. Namikawa, Toroidal Degeneration of Abelian Varieties, Complex Analysis and Algebraic Geometry, Iwanami Shoten Publishers and Cambrige Univ. Press (1977) 227-237.

[N3] I. Nakamura, On moduli of stable quasi-abelian varieties, Nagoya Math. J., 58 (1975) 149-214.

[N4] I. Nakamura, Relative Compactification of the Néron Model and its application, Complex Analysis and Algebraic Geometry, Iwanami Shoten Publishers and Cambrige Univ. Press (1977) 207-225.

[N5] I. Nakamura, these proceedings.

[OS] T. Oda and C. S. Seshadri, Compactifications of the generalized Jacobian variety, to appear in Trans. AMS.

[R1] G. A. Reisner, Cohen-Macaulay quotients of polynomial rings, Advances in Math. 21 (1976), 30-49.

[R2] D. S. Rim, Formal Deformation Theory, Groupes de Monodromie en Géométrie Algébrique. (SGA7), Lecture Notes in Math. 288, Springer-Verlag,Berlin Heidelberg New York, (1972).

[S1] H. W. Schuster, Deformationen analytichen Algebren, Inv. Math. 6 (1968), 262-274.

[T1] H. Tsuchihashi, Compactifications of the moduli spaces of hyperelliptic surfaces, Master's thesis, Tohoku Univ. (1978).

[TE] G. Kempf, F, Knudsen, D. Mumford and B. Saint-Donat, Toroidal Embeddings I, Lecture Notes in Math. 339, Springer-Verlag,Berlin Heidelberg New York (1973).