数理解析研究所講究録
第 331 巻 1978 年 21-29

ASYMPTOTIC CYCLES ON TWO-DIMENSIONAL MANIFOLDS

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INTRODUCTION

In 1957, S.Schwartzman introduced the concept of asymptotic cycles. This concept represents how the trajectory of a flow rounds around the phase space in the homological meaning. Let us recall the definition.

Let M be a closed C^∞ Riemannian manifold and φ_t a C^1-flow on it. Choose p a point of M and consider a one cycle $\hat{C}_{T,p} = C_{T,p} + C'_{T,p}$, where $C_{T,p}$ denotes the trajectory from p to $\varphi_T(p)$ and $C'_{T,p}$ a minimal geodesic from $\varphi_T(p)$ to p .

<u>DEFINITION</u> The <u>asymptotic cycle</u> of p , denoted by A_p , is defined by

$$A_p = \lim_{T\to\infty} \frac{1}{T} [\hat{C}_{T,p}]$$

when the limit exists. (Here [ ] denotes the homology class.)

It is easy to check that A_p is invariant under the flow φ_t and is independent of the choice of Riemannian metrics.

Here we study the relations between asymptotic cycles and the behaviour of trajectories on closed orientable two-manifolds. In Section 1 we give fundamental notations and statements of results, in Section 2 we describe the outline of the proof of Theorem 2 and Section 3 is a note for Theorem 3 and the remainning problem.

1. NOTATIONS AND STATEMENTS OF RESULTS

Throughout this paper, we suppose M is a closed orientable two-manifold and φ_t is a C^1 flow on M . And for simplicity, we assume that φ_t has <u>only a finite number of equilibrium points</u>.

If p is a point of M , $L_+(p)$ denotes the positive semi-trajectory departing at p and $\omega(p)$ the ω-limit set of p .

We call $L_+(p)$ <u>exceptional</u> and $\overline{L_+(p)}$ an <u>exceptional domain</u> if $L_+(p)$ is contained in $\omega(p)$, nowhere-dense, and neither an equilibrium point nor a periodic trajectory.

A subset C of M is called a circuit if C is a unicursal diagram consisting of equilibrium points and trajectories connecting them.

<u>THEOREM 1</u> One and only one of the following eight cases occurs.

(1) $L_+(p)$ is an equilibrium point.

(2) $L_+(p)$ approaches one equilibrium point.

(3) $L_+(p)$ winds around a circuit from one side.

(4) $L_+(p)$ is a periodic trajectory.

(5) $L_+(p)$ winds around a periodic trajectory from one side.

(6) $L_+(p)$ is locally dense (namely, $\overline{L_+(p)}$ contains a non-empty open set).

(7) $L_+(p)$ is exceptional.

(8) $L_+(p)$ approaches one exceptional domain.

To state Theorem 2 and Theorem 3 , we need the following definition.

DEFINITION

(i) $\alpha \in H_1(M;\mathbb{R})$ is <u>rational</u> if $\alpha \neq 0$ and there exist $k \in \mathbb{R}$ and $\alpha' \in H_1(M;\mathbb{Z})$ such that $\alpha = k\alpha'$.

(ii) $\alpha \in H_1(M;\mathbb{R})$ is <u>irrational</u> if α is neither 0 nor rational.

THEOREM 2 Suppose A_p exists and is rational, then $L_+(p)$ is either of type (4) or of type (5).

THEOREM 3 If M is a two-dimensional torus, then A_p exists for all $p \in M$, and

if $L_+(p)$ is of type (1),(2) or (3), then A_p is 0.

if $L_+(p)$ is of type (4) or (5), then A_p is rational or 0.

if $L_+(p)$ is of type (6),(7) or (8), then A_p is irrational or 0.

Moreover $\omega(p_1) = \omega(p_2)$ implies $A_{p_1} = A_{p_2}$.

2. ASYMPTOTIC CYCLES OF SEMI-TRAJECTORIES OF TYPE (6)

It is immediate that the asymptotic cycle A_p is zero for a semi-trajectory $L_+(p)$ of type (1),(2) or (3). For $L_+(p)$ of type (4) or (5), A_p is given by $A_p = \frac{1}{\tau}[C]$, hence is rational or zero. (Here $[C]$ denotes the homology class of the periodic trajectory and τ is its minimal period.)

The rest of this section is devoted to show that asymptotic cycles of semi-trajectories of type (6) are not rational. If $L_+(p)$ is of type (7) or (8), we can perform a similar computation and obtain that A_p is also never rational. These results and the previous observation imply Theorem 2.

Let p be a point of M with $L_+(p)$ locally dense, then by the orientability of M, we can construct a simple closed curve C which is transverse to the flow φ_t and is contained in $\overline{L_+(p)}$. Consider the Poincaré map $\mathcal{G}$ of this flow with respect to C. $\mathcal{G}$ is defined on $L_+(p) \cap C$, a dense subset of C, hence $\mathcal{G}$ may not be defined at a point x only if x is of type (2), and the finiteness assumption for equilibrium points implies that the cardinal number of such points is at most finite.

Now we define P-transformations.

<u>DEFINITION</u> $\mathcal{G}$ is called a <u>P-transformation</u>, if there exist distinct k points $p_1, \ldots, p_k$ and distinct k points $q_1, \ldots, q_k$ in S^1, such that $\mathcal{G}$ is an orientation-preserving homeomorphism from $S^1 \setminus \{p_1, \ldots, p_k\}$ to $S^1 \setminus \{q_1, \ldots, q_k\}$.

For a P-transformation $\mathcal{G}$, $\mathcal{G}_R$ (resp. $\mathcal{G}_L$) denotes a right (resp. left) continuous extention of $\mathcal{G}$, and

$\bigcup_{n\in\mathbb{Z}} \varphi_R^n(\{p_1,\ldots,p_k\})$ is denoted by $S(\varphi)$. We call a point of $S(\varphi)$ <u>singular</u> and a point of $S^1 \setminus S(\varphi)$ <u>regular</u>.

<u>LEMMA</u> Let $\varphi : S^1 \longrightarrow S^1$ be a P-transformation with a regular point x_0 satiesfying $\overline{\{\varphi^n(x_0)\}_{n\geq 0}} = S^1$. Then the only closed invariant subsets under φ_R or φ_L are whole S^1 and the empty set.

<u>PROPOSITION</u> Let $\varphi : S^1 \longrightarrow S^1$ be a P-transformation, then φ_R or φ_L has a non-trivial invariant measure on S^1.

<u>COROLLARY</u> Let $\varphi : S^1 \longrightarrow S^1$ be a P-transformation with a regular point x_0 satiesfying $\overline{\{\varphi^n(x_0)\}_{n\geq 0}} = S^1$. Then every φ_R (or φ_L) invariant measure μ satiesfies the condition that $\mathrm{supp}.\mu = S^1$ and $\mu(S(\varphi)) = 0$. And for every regular point x , any cluster point of the sequence $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{\varphi^k(x)}$ gives a φ_R (hence also φ_L and φ) invariant measure. (Where δ denotes the Dirac measure.)

What we must prove is that if the asymptotic cycle exists for a semi-trajectory of type (6), then it is not rational.

For a semi-trajectory $L_+(p)$ of type (6), take a transeverse curve C_0 as before and the P-transformation φ induced by the Poincaré map with respect to C_0 . Without loss of generality, we can assume p is contained in C_0 .

Let τ denote the first return time with respect to C_0 .

$$\tau(x) = \inf.\{ t>0 \; : \; \varphi_t(x) \in C_0 \}$$

Then the n-th return time of p is given by

$$T(n) = \sum_{k=0}^{n-1} \tau(\varphi^k(p)) \; .$$

It is enough to show that every cluster point of the following sequence is irrational or zero :

$$\frac{1}{T(n)} [\hat{C}_{T(n),p}] \; .$$

Assume the contrary, then there exists a sequence n_i such that

$$\alpha = \lim_{i \to \infty} \frac{1}{T(n_i)} [\hat{C}_{T(n_i),p}]$$

is rational.

By the previous corollary, we can suppose that the following sequence converges to a φ-invariant measure, taking a subsequences if necessary.

$$\mu = \lim_{i \to \infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} \delta_{\varphi^k(p)}$$

Using this invariant measure, we will be led to a contradiction.

Let γ_0 be the homology class represented by C_0 , then the intersection number of α and γ_0 is given as follows:

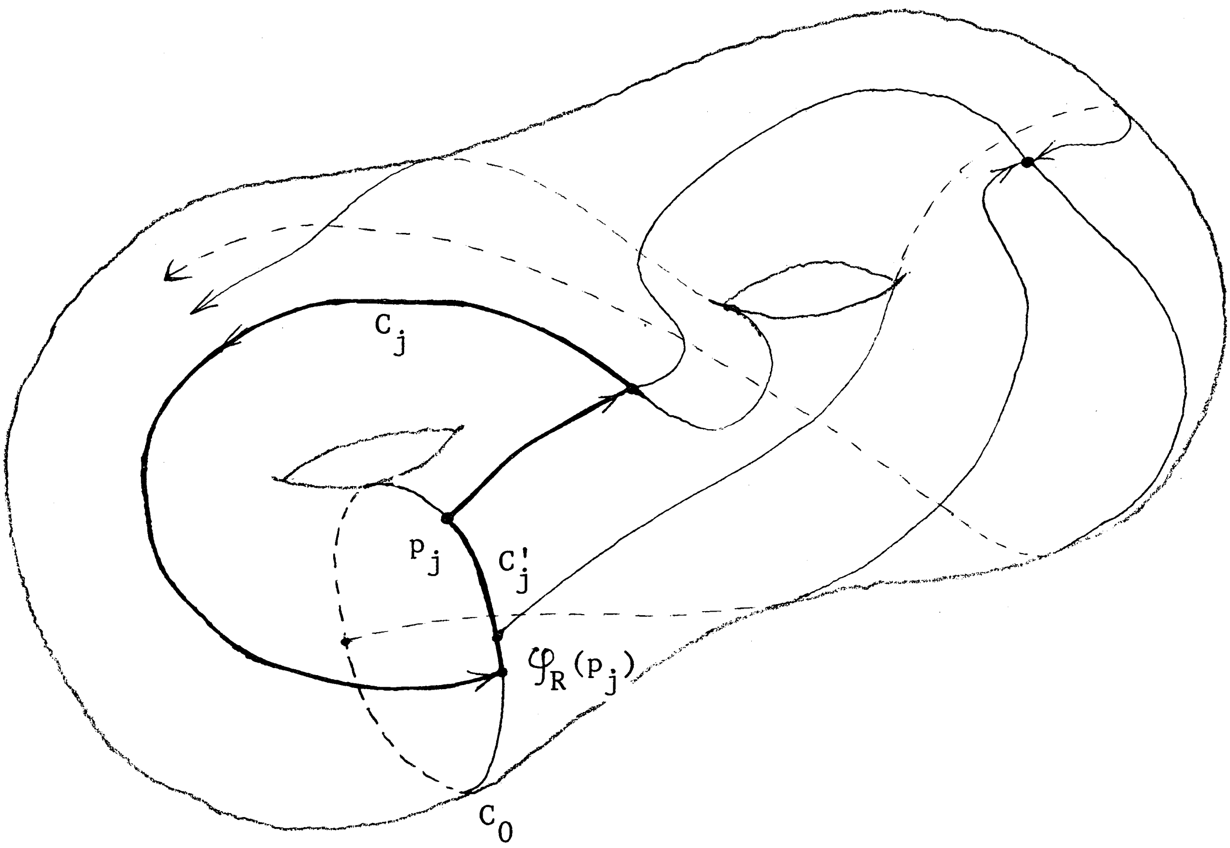
$$\alpha \circ \gamma_0 = \lim_{i \to \infty} \frac{1}{T(n_i)} [\hat{C}_{T(n_i),p}] \circ [C_0]$$

$$= \lim_{i \to \infty} \frac{1}{T(n_i)} n_i$$

$$= \left(\int_{C_0} \tau(x)\, d\mu(x) \right)^{-1}$$

As in the definition of P-transformations, let $p_1, \ldots, p_k$ be points in C_0 where φ is not defined and the ordering is compatible to the orientation of C_0 .

Consider the integral homology class $\gamma_j = [\hat{C}_j] = [C_j + C_j']$, where C_j denotes the 'trajectory' from p_j to $\varphi_R(p_j)$ (more precisely, C_j is the limit of the segment of the trajectory from x to $\varphi(x)$ as x approaches p_j from the right side) and C_j' is the segment from $\varphi_R(p_j)$ to p_j in C_0 .



The intersection number of α and γ_j is given by

$$\alpha \circ \gamma_j = \lim_{i\to\infty} \frac{1}{T(n_i)} \sum_{k=0}^{n_i - 1} \chi_{(\varphi_R(p_j),\ p_j)}(\varphi^k(p))$$

where $\chi_{(\varphi_R(p_j),\ p_j)}$ denotes the characteristic function of the open interval $(\varphi_R(p_j),\ p_j)$.

So, we obtain the following equation.

$$\alpha \circ \gamma_j = \left(\int_{C_0} \tau d\mu\right)^{-1} \cdot \mu((\varphi_R(p_j),\ p_j))$$

If we put $a_j = \mu((\varphi_R(p_j),\ p_j))$, then this equation becomes

$$a_j = \frac{\alpha \circ \gamma_j}{\alpha \circ \gamma_0} .$$

Hence, by the assumption that α is rational, a_j is a rational number for all j .

Let us introduce the coordinate in C_0 by measure μ . Since $\varphi\big|_{(p_j,\ p_{j+1})}$ is continuous and preserves μ , it follows that $\varphi\big|_{(p_j,\ p_{j+1})}(x) = x - a_j$. But all a_j's are rational numbers, this contradicts the fact that φ has a regular point with a dense orbit. Thus we obtain the desired result.

3. CONVERGENCES OF ASYMPTOTIC CYCLES

In the case of a two-dimensional torus, every P-transformation induced by a semi-trajectory of type (6) has a continuous extention on S^1. Then Theorem 3 is obtained from the fact that every homeomorphism of S^1 with a dense orbit is uniquely ergodic.

We expect that the result of Theorem 3 holds also for a surface of higher genus. This is essentially reduced to the next problem.

PROBLEM Is every P-transformation with a dense positive-orbit uniquely ergodic ?

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