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Ergodic theory of diffeomorphisms

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1. A "almost everywhere" stable manifold theorem

Theorem. Let M be a compact differentiable manifold and $f : M \mapsto M$ a diffeomorphism of class $C^{r,\theta}$ (r integer ≥ 1, $\theta \in (0,1]$). Let d be a Riemann metric on M.

There is a Borel set $\Gamma \subset M$ with the following properties

(a) $f\Gamma \subset \Gamma$ and $\sigma(\Gamma) = 1$ for every f-invariant probability measure σ on M

(b) Let $x \in \Gamma$ and $\lambda_x^{(1)} < \cdots < \lambda_x^{(r)}$ be the strictly negative characteristic exponents of Tf. Define $\mathcal{V}_x^{(1)} \subset \cdots \subset \mathcal{V}_x^{(r)}$ by

$$\mathcal{V}_x^{(p)} = \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^n x, f^n y) \leq \lambda_x^{(p)}\}$$

for $p = 1,\cdots,r$. Then $\mathcal{V}_x^{(p)}$ is the image of $V_x^{(p)}$ by an injective $C^{r,\theta}$ immersion tangent to the identity at x.

- Characterictic exponents will be explained later.

- This is an "almost everywhere" stable manifold theorem, where several stable manifolds with different rates of convergence may be present at a point.

Corollary. If ρ is ergodic and all the characteristic exponents of Tf are < 0 a.e., then ρ is carried by an attracting periodic orbit.

- The proof of the theorem can be reduced to proving the existence of local stable manifolds.

- Using exponential maps, the local theorem can be formulated as a theorem about invariant manifolds for a non linear vector bundle map T over $\tau : M \mapsto M$. The differentiability of $T_x : E_x \mapsto E_{\tau x}$ is used, but $\tau : M \mapsto M$ is just assumed to be a measure preserving transformation.

- In particular, one can take the vector bundle to be trivial, i.e. one studies the ergodic properties of nonlinear maps F_x, $x \in M$, such that F_x maps the unit ball of $\mathbb{R}^m$, into $\mathbb{R}^m$ and $F_x 0 = 0$.

- The linear version of this problem is the multiplicative ergodic theorem which we have to study first.

2. The multiplicative ergodic theorem.

Let (M,Σ,ρ) be a fixed probability space, and $\tau : M \mapsto M$ a measurable map preserving ρ. We denote by f^+ the positive part of a real function f.

Theorem. *Let $T : M \mapsto \underline{M}_m$ be a measurable function to the real $m \times m$ matrices such that*

$$\log^+ \| T(\cdot) \| \in L^1(M,\rho)$$

and write $T_x^n = T(\tau^{n-1}x)\ldots T(\tau x)T(x)$.

There is $\Gamma \subset M$ *such that* $\tau\Gamma \subset \Gamma$ *and* $\rho(\Gamma) = 1$. *Furthermore, if* $x \in \Gamma$, $u \in \mathbb{R}^m$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_x^n u\| = \chi(x,u)$$

exists, finite or $-\infty$.

The values of $\chi(x,u)$ for $u \neq 0$ are called characteristic exponents. Notice that

$$V_x^\lambda = \{u \in \mathbb{R}^m : \chi(x,u) \leq \lambda\}$$

is a linear subspace of $\mathbb{R}^m$.

Complement.

Let $*$ denote matrix transposition. One may take Γ such that, if $x \in \Gamma$,

(a) $\lim_{n\to\infty} (T_x^{n*} T_x^n)^{1/2n} = \Lambda_x$ exists

(b) the characteristic exponents $\lambda_x^{(r)}$ are the log's of the eigenvalues of Λ_x. The space V_x^λ is the sum of the eigenspaces $U_x^{(r)}$ of Λ_x corresponding to the eigenvalues $\leq \lambda$. The functions $x \to \lambda_x^{(r)}$, $x \mapsto m_x^{(r)} = \dim U_x^{(r)}$ are τ-invariant.

The proof can be obtained in two steps.

I. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \log \|(T_x^n)^{\wedge q}\| \qquad (*)$$

exists almost everywhere (this follows from a "subadditive ergodic theorem" and insures the existence of a limit for the sum of the largest q eigenvalues of $(T_x^{n*}T_x^n)^{1/2n}$.

II. From

$$\limsup_{n\to\infty} \frac{1}{n} \log \|T(\tau^n x)\| \leq 0 \qquad (**)$$

and (*) for $q = 1,\dots,m$ one obtains without further assumption the existence of the limits asserted by the multiplicative ergodic theorem.

3. Proof of a local stable manifold theorem

To prove the desired nonlinear version of the multiplicative ergodic theorem, we put

$$F_x^n = F_{\tau^{n-1}x} \circ \cdots \circ F_{\tau x} \circ F_x$$

and assume that

$$\int \rho(dx) \log^+ \|F_x\|_{r,\theta} < +\infty .$$

We want to prove the existence of a measurable set $\Gamma \subset M$ with $\tau\Gamma \subset \Gamma$, $\rho(\Gamma) = 1$, and measurable functions $\beta > \alpha > 0$ on Γ such that if $x \in \Gamma$, and $\lambda < 0$ is not a characteristic exponent of $T = Tf$ at x,

$$D_x = \{u \in \mathbb{R}^m : \|u\| \leq \alpha(n),\ \|F_x^n u\| \leq \beta(x)e^{n\lambda} \text{ for all } n \geq 0\}$$

is a $C^{r,\theta}$ submanifold of the ball $\|u\| \leq \alpha(x)$, tangent at 0 to V_x^λ.

If F_x is replaced by its linear part $T(x) = T_x f$, this follows from the multiplicative ergodic theorem. The idea of the proof of the nonlinear theorem is to consider F_x as a perturbation of $T(x)$. If $u \in D_x$, $F_x^n u$ tends exponentially fast (with n) to 0, therefore the deviation of $F_{\tau^n x}$ from $T(\tau^n x)$, at the relevant point $F_x^n u$, goes exponentially to zero. The heart of the proof reduces thus to the following fact.

If (T_x') is a sequence of $n \times n$ matrices and

$$\sup_n \|T_n' - T(\tau^{n-1}x)\| \, e^{n\eta}$$

is sufficiently small (for some $\eta > 0$, and $T(\tau^{n-1}x)$ such that the limits (*) exist and (**) holds), then, if we write

$$T'^n = T_n' \cdots T_1'$$

the limit

$$\lim_{n\to\infty} (T'^{n*} T^n)^{1/2n} = \Lambda_x'$$

exists and has the same eigenvalues (including multiplicity) as Λ_x. The eigenspaces depend continuously on the perturbation.

4. Abstract results about matrix products

1. Theorem. Let $T = (T_n)_{n>0}$ be a sequence of real $m \times m$

matrices such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|T_n\| \leq 0$$

We write $T^n = T_n \cdots T_2 \cdot T_1$ and assume that the limits

$$\lim_{n\to\infty} \frac{1}{n} \log \|(T^n)^{\wedge q}\|$$

exist for $q = 1,\cdots,m$.

(a) $$\lim_{n\to\infty} (T^{n*}T^n)^{1/2n} = \Lambda$$

exists, where $*$ denotes matrix transposition.

(b) Let $\exp \lambda^{(1)} < \cdots < \exp\lambda^{(s)}$ be the eigenvalues of Λ [real $\lambda^{(r)}$, possibly $\lambda^{(1)} = -\infty$], and $U^{(1)},\cdots,U^{(s)}$ the corresponding eigenspaces. Writing $V^{(0)} = \{0\}$ and $V^{(r)} = U^{(1)}+\cdots+U^{(r)}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|T^n u\| = \lambda^{(r)} \quad \text{when} \quad u \in V^{(r)} \setminus V^{(r-1)} \quad \text{for}$$

$r = 1,\ldots,s$.

① The eigenvalues of $(T^{n*}T^n)^{1/2n}$ send to limits

$$\lambda^{(1)} < \cdots < \lambda^{(s)}$$

- Let $U_n^{(r)}$ be the space spanned by the eigenvectors of $(T^{n*}T^n)^{1/2n}$ corresponding to eigenvalues sending to $\lambda^{(r)}$.

② Lemma. Given $\delta > 0$ $\exists K > 0$ s.t., for all $k > 0$,

$$\max\{|(u,u')|: u \in U_n^{(r)},\ u' \in U_{n+k}^{(r')},\ \|u\| = \|u'\| = 1\}$$

$$\leq K \exp[-n(|\lambda^{(r')} - \lambda^{(r)}| - \delta)]$$

Equivalently: if $\lambda^{(r)} = \lambda$, $\lambda^{(r')} = \lambda'$, $U_n^{(r)} = U_n$, $U_n^{(r')} = U_n'$, if v is the orthogonal projection in U_{n+k}' of $u \in U_n$, then

$$\|v\| \leq K\| u \|\exp[-n(|\lambda' - \lambda| - \delta)]$$

- If $\lambda < \lambda'$, $k = 1$, then for large n

$$\|v\|\exp[(n+1)(\lambda' - \tfrac{\delta}{4})] \leq \|T^{n+1}u\|$$

$$\leq \|T_{n+1}\|\|T^n u\| \leq \exp[C + (n+1)\tfrac{\delta}{2}]\cdot\|u\|\exp[n(\lambda + \tfrac{\delta}{4})]$$

$$\Rightarrow \quad \|v\| \leq \exp[C - \lambda' + \tfrac{3}{4}\delta]\cdot\exp[-n(\lambda' - \lambda - \delta)]\cdot\|u\|$$

- Induction on k $(\lambda < \lambda')$
- Orthogonality

③ $(U_n^{(r)})_{n>0}$ is Cauchy $\Rightarrow$ (a) $U_n^{(r)} \to U^{(r)}$

$$\Rightarrow \quad \max\{|(u,u')|: u \in U^{(r)},\ u' \in U_n^{(r')},\ \|u\| = \|u'\| = 1\}$$

$$\leq K \exp[-n(|\lambda^{(r')} - \lambda^{(r)}| - \delta)]$$

$\Rightarrow$ (b)

2. Theorem. Let the notation and assumptions be as in theorem 1. Furthermore, assume that $\det \Lambda \neq 0$.

Let $\eta > 0$ be given and, for $T' = (T'_n)_{n>0}$, write

$$\|T'-T\| = \sup_n \|T'_n - T_n\| \, e^{3n\eta}$$

and $T'^n = T'_n \cdots T'_2 \cdot T'_1$. Then there are $\delta, A > 0$ and, given $\varepsilon > 0$, there are $B_\varepsilon > 0$, $B'_\varepsilon > 1$ with the following properties. If $\|T'-T\| < \delta$

$$\lim_{n\to\infty} (T'^{n*} T'^n)^{1/2n} = \Lambda' \tag{1}$$

exists and has the same eigenvalues as Λ (including multiplicity). Furthermore, if $P^{(r)}(T')$ denotes the orthogonal projection of Λ' corresponding to $\exp \lambda^{(r)}$, and $\|T''-T\| < \delta$, we have

$$\|P^{(r)}(T') - P^{(r)}(T'')\| \leq A \, \|T'-T''\| \tag{2}$$

$$B_\varepsilon \exp n(\lambda^{(r)}-\varepsilon) \leq \|T'^n P^{(r)}(T')\| \leq B'_\varepsilon \exp n(\lambda^{(r)}+\varepsilon) \tag{3}$$

① To prove the existence of (1) and spectrum Λ = spectrum Λ', it suffices to show

$$\lim_{n\to\infty} \frac{1}{n} \log \|(T'^n)^{\wedge q}\| = \lim_{n\to\infty} \frac{1}{n} \log \|(T^n)^{\wedge q}\|$$

In fact it suffices to do this for $q = 1$. Equivalently, it suffices to find an open set $U \subset \mathbb{R}^m$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|T'^n u\| = \lambda^{(s)} \qquad \text{for} \quad u \in U$$

This results from the following

② Lemma. Let $(\xi_1,\ldots,\xi_m)$ be an orthonormal basis of $\mathbb{R}^m$ diagonalizing Λ, with ξ_m corresponding to the largest eigenvalue $\exp \lambda^{(s)}$. There is then $\delta > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|T'^n u\| = \lambda^{(s)}$$

whenever $0 < \alpha \leq 1$, $\|T'-T\| \leq \delta\alpha$, and $u \in U$, where

$$U = \{ \sum_{k=1}^{m-1} u_k \frac{\xi_k}{\alpha} + u_m \xi_m : \max_{k<m} |u_k| < |u_m| \}$$

③ The lemma implies $\| P^{(r)}(T')-P^{(r)}(T) \| \leq A \|T'-T\|$

④ Proof of the lemma ($\alpha = 1$ for simplicity).

Let $\xi_k^{(n)}$: unit vector $\sim T^n \xi_k$, and $\xi^{(n)}$ the matrix with columns $\xi_k^{(n)}$. Then $\| \xi_k^{(n)} \| < \sqrt{m}$ and

$$D_\varepsilon = \sup_n e^{-n\varepsilon} \|\xi^{(n)-1}\| < +\infty \qquad \text{if} \quad \varepsilon > 0$$

$$T_n \xi_k^{(n-1)} = t_k^{(n)} \xi_k^{(n)} \qquad \lim_{n\to\infty} \frac{1}{n} \log \prod_{j=1}^{n} t_k^{(j)} = \lambda^{(r(k))}$$

where we may assume $r(k)$ increasing with k.

- For any $u \in \mathbb{R}^m$ ($u \neq 0$), let $T'^n u = \sum_k u_k^{(n)} \xi_k^{(n)}$

Let μ be the smallest integer such that

$$(\forall n) \qquad \max_{j\leq\mu} |u_j^{(n)}| \geq \max_{k>\mu} |u_k^{(n)}|$$

Assuming $\|T'-T\| \leq \delta$, we estimate the $u_k^{(n)}$ recursively

- $$|u_k^{(n)}| \leq t_k^{(n)}|u_k^{(n-1)}| + D\delta e^{-2n\eta}\sum_{\ell}|u_{\ell}^{(n-1)}|$$

Replace the $t_k^{(n)}$ by $t_k^{(n)*}$ so that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{r=1}^{N}\log t_k^{(n)*} = \lambda^{(r(\mu))} \quad \text{for} \quad k \leq \mu$$

$$t_\mu^{(n)*} = t_\mu^{(n)}$$

Choose C such that (for all $\nu \geq 0$, $N > \nu$, $k,\ell \leq \mu$)

$$\prod_{n=\nu+1}^{N-1} t_\ell^{(n)*} / \prod_{n=\nu+1}^{N} t_k^{(n)*} \leq Ce^{N\eta}$$

Then, if $U^{(\nu)} = \max_{\ell} |u_\ell^{(\nu)}|$,

$$|u_k^{(n)}| \leq \prod_{n=\nu+1}^{N} t_k^{(n)*} \cdot \prod_{n=\nu+1}^{N} (1+mCD\delta e^{-n\eta})U^{(\nu)}$$

Choosing $\delta = \frac{1}{mCD}\prod_{n=1}^{\infty}(1-e^{-n\eta})^2$ yields, for $N > \nu$

$$\boxed{|u_k^{(N)}| \leq C' \prod_{n=\nu+1}^{N} t_k^{(n)*} \cdot \prod_{n=\nu+1}^{N}(1-e^{-n\eta})\cdot U^{(\nu)}} \qquad (4)$$

with $C' \leq \frac{1}{mCD\delta}$

- Choose ν so that $|u_\mu^{(\nu)}| = \max_k |u_k^{(\nu)}| = U^{(\nu)}$, then

$$\boxed{|u_\mu^{(N)}| \geq \prod_{n=\nu+1}^{N} t_\mu^{(N)} \cdot \prod_{n=\nu+1}^{N}(1-e^{-n\eta})\cdot U^{(\nu)}} \qquad (5)$$

- $\lim_{n\to\infty} \frac{1}{n} \log \|T'^n u\| = \lambda^{(r(\mu))} \Rightarrow$ lemma $(r(\mu)=s)$.

⑤ (2) and (3) be obtained from (4) and (5).

For instance the second half of (3) follows from (4).