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Exact Solutions to Nonlinear Difference Equations.

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A method for constructing discrete analogues of certain nonlinear evolution equations that exhibit exact solutions, is presented. Using the dependent variable transformation, we transform the nonlinear evolution equation into the bilinear differential equation of the following form

$$F(D_t, D_x)\, f\cdot g = 0,$$

where the operators D_t, D_x operating on $f\cdot g$ are defined by

$$D_x^{\ n} D_t^{\ m} f(x,t)\cdot g(x,t)$$
$$= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m f(x,t)g(x',t')\Big|_{\substack{x=x'\\ t=t'}} .$$

We then replace the bilinear differential equation by the associated bilinear difference equation, which is transformed back into the nonlinear difference equation by the associated dependent variable transformation. The nonlinear difference equation is a difference analogue of the original nonlinear differential equation.

In the following, we list some of the results: the nonlinear difference equations that exhibit exact solutions.

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(i) Anharmonic Oscillation

$$\frac{d^2}{dt^2} x(t) = -x(t) - \beta x(t)^3 ,$$

$$\Delta_t^2 x(t) = -x(t) - (\beta/2)[x(t+\delta) + x(t-\delta)] x^2(t),$$

$$x(t) = x_0 \mathrm{cn}(\Omega t,k),$$

where

$$(2 \delta^{-2})[1 - \mathrm{cn}\delta\Omega / \mathrm{dn}^2\delta\Omega] = 1$$

$$2 \delta^{-2} k^2 \mathrm{sn}^2\delta\Omega = \beta x_0^2 \mathrm{dn}^2\delta\Omega ,$$

and

$$\Delta_t^2 f(t) = \delta^{-2}[f(t+\delta) + f(t-\delta) - 2f(t)].$$

(ii) Pendulum

$$\frac{d^2}{dt^2} \theta = -\sin 2\theta ,$$

$$\delta^{-2}[\sin(\theta(t+\delta) - \theta(t)) - \sin(\theta(t) - \theta(t-\delta))]$$

$$= -(1/2)[\sin(\theta(t+\delta) + \theta(t)) + \sin(\theta(t) + \theta(t-\delta))],$$

$$\theta(t) = \mathrm{am}(\Omega t,k)$$

where

$$\mathrm{dn}^2(\Omega\delta) = (1 - \delta^2/2)^2 / (1 + \delta^2/2)^2.$$

(iii) Two-Waves Interaction

$$\frac{\partial u}{\partial \xi} = -uv, \qquad \frac{\partial v}{\partial \eta} = uv,$$

$$u(\xi + \delta,\eta) - u(\xi - \delta,\eta) = -2\delta u(\xi + \delta,\eta)v(\xi - \delta,\eta)$$

$$v(\xi, \eta+\delta) - v(\xi, \eta-\delta) = 2\delta v(\xi, \eta+\delta)u(\xi, \eta-\delta),$$

$$u = g_1(\eta) \;/\; [F_1(\eta) + F_2(\xi)]$$

$$v = g_2(\xi) \;/\; [F_1(\eta) + F_2(\xi)],$$

where

$$g_1(\eta - \delta) = -(2\delta)^{-1}[F_1(\eta + \delta) - F_1(\eta - \delta)]$$
$$g_2(\xi - \delta) = (2\delta)^{-1}[F_2(\xi + \delta) - F_2(\xi - \delta)].$$

(iv) Korteweg-de Vries Equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

$$\Delta_t \frac{W_n(t)}{1 + W_n(t)} = W_{n-1/2}(t) - W_{n+1/2}(t),$$

N-soliton solutions.

(v) Toda Equation

$$\frac{\partial^2}{\partial t^2} \log(1 + V_n(t)) = V_{n+1}(t) + V_{n-1}(t) - 2V_n(t),$$

$$\Delta_t^2 \log(1 + V_n(t)) = \hat{V}_{n+1}(t) + \hat{V}_{n-1} - 2\hat{V}_n(t)$$

where

$$\hat{V}_n(t) = \delta^{-2}\log(1 + \delta^2 V_n(t)),$$

N-soliton solutions.

(vi) Sine-Gordon Equation

$$\phi_{xx} - \phi_{tt} = \sin\phi,$$

$$\sin\{[\ \phi(x+2\delta,t) + \phi(x-2\delta,t) - \phi(x,t+2\delta) - \phi(x,t-2\delta)\ ]\ /4\}$$

$$= \delta^2 \sin\{[\ \phi(x+2\delta,t) + \phi(x-2\delta,t) + \phi(x,t+2\delta) + \phi(x,t+2\delta)\ ]\ /4\},$$

N-soliton solutions.