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Quasi-periodic solutions of the Sine-Gordon equation and the massive Thirring model

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In this note we consider quasi-periodic solutions of the sine-Gordon equation

$$u_{tt} - u_{tt} + \sin u = 0 \tag{0.1}$$

and the field equation of the massive Thirring model

$$\begin{aligned} - iu_t - iu_x + 2v + 2|v|^2u &= 0 \\ - iv_t + iv_x + 2u + 2|u|^2v &= 0. \end{aligned} \tag{0.2}$$

Quasi-periodic solutions of the sine-Gordon equation were given by Kozel-Kotlyarov[1)] in terms of the Riemann theta functions. We obtained a similar result for the massive Thirring model[2)].

In sections 1 and 2, we descrive the result of Kozel-Kotlyarov in a modified form. First we derive a system of solvable differential equations which generates a family of solutions of the sine-Gordon equation. This system is derived from the linear operators which were given by Ablowitz-Kaup-Newell-Segur[3)] and Zakharov-Takhatzhian-Faddeev[4)] for the sine-Gordon equation. Next that system is integrated by employing the theory of abelian integrals on a hyperelliptic curve, which gives an explicit formula for the solution in terms of the Riemann theta functions.

In section 3, we briefly summarize our result for the massive Thirring model. In this case the corresponding system of the differential equations are derived from the linear operators given by Mikhairov[5)].

§1. Sine-Gordon equation (1)

Let ϕ_1 and ϕ_2 be solutions of the following equations

$$2i\phi_1' + u'\phi_1 + \zeta\phi_2 = 0$$
$$2i\phi_2' + \zeta\phi_2 - u'\phi_1 = 0$$
$$2i\dot{\phi}_1 + \zeta^{-1}\exp(iu)\phi_2 = 0$$
$$2i\dot{\phi}_2 + \zeta^{-1}\exp(-iu)\phi_1 = 0$$

where $\phi' = \partial\phi/\partial\xi$, $\dot{\phi} = \partial\phi/\partial\eta$, $\xi = t + x$, $\eta = t - x$ and ζ is a parameter. These linear equations are essentially the same as those which were given by Ablowitz-Kaup-Newell-Segur and Zakharov-Takhtadzhian-Faddeev for the sine-Gordon equation. Then functions $f = \phi_1\phi_2$, $g = \phi_1^2$, $h = \phi_2^2$ satisfy the following equations

$$\begin{aligned} f' &= 2^{-1}i\zeta(g + h) \\ g' &= iwg + i\zeta f \qquad w = u' \\ h' &= i\zeta f - iwh \\ \dot{f} &= (2\zeta)^{-1}i\exp(-iu)g + (2\zeta)^{-1}i\exp(iu)h \\ \dot{g} &= \zeta^{-1}i\exp(iu)f \\ \dot{h} &= \zeta^{-1}i\exp(-iu)f \end{aligned} \tag{1.1}$$

We consider the system (1.1) in which u and w are regarded as arbitrary functions. We require that the above system have solutions of the forms

$$f = \sum_{j=1}^{N} f_j\zeta^{2j-1}, \quad g = \sum_{j=0}^{N} g_j\zeta^{2j}, \quad h = \sum_{j=0}^{N} h_j\zeta^{2j}. \tag{1.2}$$

where N is any natural number. This requirement determines coefficients u and w, as will be shown below. Putting (1.2) in (1.1), we have the following relations for coefficients:

$$
\begin{aligned}
& f'_j = 2^{-1}i(\, g_{j-1} + h_{j-1}\,) && j = 1,\cdots, N \\
& g'_j = iwg_j + if_j && j = 0,\cdots, N \quad f_0 = 0 \\
& h'_j = if_j - iwh_j && j = 0,\cdots, N \\
& \dot{f}_j = 2^{-1}i\exp(-iu)g_j + 2^{-1}i\exp(iu)h_j && j = 1,\cdots, N \\
& \dot{g}_j = i\exp(iu)f_{j+1} && j = 0,\cdots, N \quad f_{N+1} = 0 \\
& \dot{h}_j = i\exp(-iu)f_{j+1} && j = 0,\cdots, N \\
& g_N + h_N = 0 \\
& \exp(-iu)g_0 + \exp(iu)h_0 = 0
\end{aligned}
\tag{1.3}
$$

Further we see that the following relations hold:

$$g_N = -h_N = \text{constant},$$

$$g_0h_0 = \text{constant}.$$

Therefore coefficients u and w are expressed as

$$
\begin{aligned}
& u = -i\log(-h_0^{-1}g_0)^{1/2}, \\
& w = -g_N^{-1}f_N.
\end{aligned}
\tag{1.4}
$$

Substituting these relations (1.4) into (1.3), we get the following system of differential equations

$$
\begin{aligned}
& f'_j = 2^{-1}i(\, g_{j-1} + h_{j-1}\,) && j = 1,\cdots, N \\
& g'_j = -if_Ng_j + if_j && j = 0,\cdots, N-1 \\
& h'_j = if_j + if_Nh_j && j = 0,\cdots, N-1 \\
& \dot{f}_j = 2^{-1}i(-g_0h_0)^{-1/2}(\, g_0h_j - g_jh_0\,) && j = 1,\cdots, N \\
& \dot{g}_j = i(-g_0h_0)^{-1/2}g_0f_{j+1} && j = 0,\cdots, N-1 \\
& \dot{h}_j = -i(-g_0h_0)^{-1/2}h_0f_{j+1} && j = 0,\cdots, N-1
\end{aligned}
\tag{1.5}
$$

where we put $g_N = -h_N = -1$ without loss of generality.

Accordingly the existence of polynomial solutions of (1.2) is equivalent to the solvability of the system (1.5). Kozel-Kotlyarov derived the system of this type as the deformation equations of the monodromy matrix of the linear operators.

Reversing arguments, we begin our discussion with the system (1.5). First we can show that the system (1.5) is completely integrable. Therefore for any initial condition $\{ f_j(0,0),\ (j = 1,\cdots, N)\ \ g_j(0,0),\ h_j(0,0)\ (j = 0,\cdots, N-1)\}$, there exists a unique solution $\{ f_j(\xi,\eta),\ g_j(\xi,\eta),\ h_j(\xi,\eta)\}$.

Define polynomials f, g, h by (1.2) and coefficients u, w by (1.4) with solutions f_j, g_j, h_j. Then polynomials f, g, h are solutions of (1.1). Using the system (1.5), we know that g_0h_0 is a constant and the function u defined by (1.4) is a solution of the sine-Gordon equation. Further by (1.1), we see that

$$P(\zeta) = f^2 - gh = \sum_{j=0}^{2N} p_j \zeta^{2j} \tag{1.6}$$

is a polynomial with constant coefficients.

Let $\pm\zeta_j(\xi,\eta)$ ($j = 1,\cdots, N$) be the roots of the equation $g = 0$. Then the polynomial g is expressed as

$$g = \prod_{j=1}^{N} (\zeta^2 - \zeta_j^2). \tag{1.7}$$

The solution u is expressed in terms of ζ_j^2 as

$$u = -i\log((-1)^N p_0^{-1/2} \prod_{j=1}^{N} \zeta_j^2). \tag{1.8}$$

Next we derive differential equations for ζ_j^2. By (1.7), we have

$$g' = -\sum_{j=1}^{N} (\zeta_j^2)' \prod_{k\neq j} (\zeta^2 - \zeta_k^2) \tag{1.9}$$

where $\prod_{k\neq j} = \prod_{k=1;\neq j}^{N}$.

Putting $\zeta = \zeta_j$ in the equation

$$g' = iwg + i\zeta f,$$

we have by (1.9),

$$-(\zeta_j^2)'\prod_{k\neq j}(\zeta_j^2 - \zeta_k^2) = i\zeta_j f(\zeta_j).$$

Since $f(\zeta_j)^2 = P(\zeta_j)$ by (1.6), we have

$$(\zeta_j^2)' = \pm\frac{i(\zeta_j^2 P(\zeta_j))^{1/2}}{\prod_{k\neq j}(\zeta_j^2 - \zeta_k^2)}. \tag{1.10}$$

Similarly we have

$$(\dot{\zeta}_j^2) = \pm\frac{(-1)^{N-1} i\prod_{k\neq j}(\zeta_k^2)(\zeta_j^2 P(\zeta_j))^{1/2}}{\prod_{k\neq j}(\zeta_j^2 - \zeta_k^2)}. \tag{1.11}$$

§2. Sine-Gordon equation (2)

In this section we use notations $\zeta^2 = \lambda$, $\zeta_j^2 = \mu_j$, $P(\lambda) = \sum_{j=0}^{2N} p_j\lambda^j = \prod_{j=1}^{2N}(\lambda - \lambda_j)$. For simplicity we assume that the initial condition for the system (1.5) is given so that the eqations $P(\lambda) = 0$ and $\sum_{j=0}^{N} g_j(0,0)\lambda^j = 0$ have simple roots and $\lambda_j \neq 0$.

Let S be the Riemann surface of the hyperelliptic curve $\mu^2 = \lambda P(\lambda)$. The genus of S is N. We realize this surface S as a double covering of the Riemann sphere in a standard way and take a canonical homology basis α_j, β_j. For the theory of Riemann surfaces we refer to ref. 6) and references in it. We denote a basis of the abelian integrals of the first kind by

$$\omega_j = \sum_{\ell=0}^{N-1} c_{j\ell}\lambda^{\ell}(\lambda P(\lambda))^{-1/2}d\lambda, \qquad j = 1,\cdots, N$$

normalized by

$$\int_{\alpha_j}\omega_k = \delta_{jk}, \qquad j, k = 1,\cdots, N.$$

Put

$$\int_{\beta_j} \omega_k = \tau_{jk} \qquad j,\ k = 1,\cdots,\ N$$

and $T = (\tau_{jk})$. Let Γ be the lattice of C^N generated by the columns of the period matrix $(I_N,\ T)$, where I_N denotes the identity matrix of degree N.

We regard the differential equations (1.10) and (1.11) as that on S. The locations of $\mu_j(0,0)$ are determined so that the relations

$$\sum_{k=1}^{N} f_k(0,0)\mu_j^k(0,0) = (\mu_j(0,0)P(\mu_j(0,0))^{1/2}$$

hold. We rewrite differential equations for μ_j:

$$\mu_j' = \frac{i(\mu_j P(\mu_j))^{1/2}}{\prod_{k\neq j}(\mu_j - \mu_k)}$$

$$\dot{\mu}_j = \frac{(-1)^N i\prod_{k\neq j}\mu_k(\mu_j P(\mu_j))^{1/2}}{\prod_{k\neq j}(\mu_j - \mu_k)}.$$

By similar calculations as in ref. 6), we have

$$\left(\sum_{k=1}^{N} \int_{\mu_0}^{\mu_k(\xi,\eta)} \omega_j \right) \equiv \left(ic_{j,N-1}\xi - ip_0^{-1/2}c_{j,0}\eta + \sum_{k=1}^{N} \int_{\mu_0}^{\mu_k(0,0)} \omega_j \right) \mod. \Gamma \qquad (2.1)$$

where μ_0 is a fixed point on S.

Introducing the Riemann theta function

$$\Theta(u) = \sum_{m\in Z^N} \exp(2\pi i m^t u + \pi i m T^t m)$$

$$u = (u_1,\cdots,u_N)\in C^N, \quad m = (m_1,\cdots,m_N),$$

we can express symmetric functions of μ_j in terms of Θ by a standard method. As a result we have

$$\sum_{j=1}^{N}\log\mu_j(\xi,\eta) = 2\log\frac{\Theta(w(0) + \phi(\xi,\eta))}{\Theta(w(\infty) + \phi(\xi,\eta))} + \sum_{j=1}^{N}\int_{\alpha_j}\log\lambda\omega_j + 2\pi i m, \qquad m\in Z \qquad (2.2)$$

where

$$w(\mu) = (w_1(\mu),\cdots,w_N(\mu)), \quad w_j(\mu) = \int_{\mu_0}^{\mu}\omega_j \qquad \mu\in S$$

$$\phi(\xi,\eta) = (\phi_1(\xi,\eta),\cdots,\phi_N(\xi,\eta))$$

$$\phi_j(\xi,\eta) = -ic_{j,N-1}\xi + ip_0^{-1/2}c_{j,0}\eta - 2^{-1}\sum_{k=1}^{N}\tau_{jk} + 2^{-1}j - \sum_{k=1}^{N}w_j(\mu_k(0,0)).$$

Combining (2.2) and (1.8), we have the following formula for the solution of the sine-Gordon equation

$$u(\xi,\eta) = -2i\log\frac{\Theta(w(0)+\phi(\xi,\eta))}{\Theta(w(\infty)+\phi(\xi,\eta))} + C + 2\pi m,$$

$$C = -i\sum_{j=1}^{N}\int_{\alpha_j}\log\lambda\omega_j - i\log(-1)^N p_0^{-1/2}.$$

§3. The massive Thirring model

For the field equation of the massive Thirring model, we can give a formula of the solution in terms of the Riemann theta functions by a similar method as in sections 1 and 2. Here we descrive our result briefly. For detail, we refer to ref. 2).

We can show that the following system of differential equations is completely integrable

$$f_j' = -if_N^{-1}h_Ng_j + if_N^{-1}g_Nh_j \qquad j = 1,\cdots,N-1$$

$$g_j' = -2^{-1}if_N^{-2}g_Nh_Ng_j + 2if_N^{-1}g_Nf_{j-1} - 2ig_{j-1} \quad j = 1,\cdots,N$$

$$h_j' = 2^{-1}if_N^{-2}g_Nh_Nh_j - 2if_N^{-1}h_Nf_{j-1} + 2ih_{j-1} \quad j = 1,\cdots,N$$

$$\dot{f}_j = -if_0^{-1}h_1g_{j+1} + if_0^{-1}g_1h_{j+1} \qquad j = 1,\cdots,N-1 \qquad (3.1)$$
$$\dot{g}_j = -2^{-1}if_0^{-2}g_1h_1g_j + 2if_0^{-1}g_1f_j - 2ig_{j+1} \quad j = 1,\cdots,N$$
$$\dot{h}_j = 2^{-1}if_0^{-2}g_1h_1h_j - 2if_0^{-1}h_1f_j + 2ih_{j+1} \quad j = 1,\cdots,N$$

where f_0 and f_N are constants and N is any natural number. This system is derived from the following linear equations

$$i\phi_1' + |v|^2\phi_1 + 2i\zeta v^*\phi_2 - \zeta^2\phi_1 = 0$$
$$i\phi_2' - |v|^2\phi_2 + 2i\zeta v\phi_1 + \zeta^2\phi_2 = 0$$
$$i\dot{\phi}_1 + |u|^2\phi_1 + 2i\zeta^{-1}u^*\phi_2 - \zeta^{-2}\phi_1 = 0$$
$$i\dot{\phi}_2 - |u|^2\phi_2 + 2i\zeta^{-1}u\phi_1 + \zeta^{-2}\phi_2 = 0$$

which were given by Mikhairov for the massive Thirring model.

Then functions a, b, c, d defined by

$$a = -2^{-1}f_N^{-1}h_N, \quad b = 2^{-1}f_N^{-1}g_N$$
$$c = -2^{-1}f_0^{-1}h_1, \quad d = 2^{-1}f_0^{-1}g_1 \qquad (3.2)$$

with solutions satisfy the following differential equations

$$d' = 2iabd + 2ib$$
$$-c' = 2iabc + 2ia$$
$$\dot{b} = 2icdb + 2id \qquad (3.3)$$
$$-\dot{a} = 2icda + 2ic.$$

Define polynomials f, g, h in ζ by

$$f = \sum_{j=0}^{N} f_j\zeta^{2j}, \quad g = \sum_{j=1}^{N} g_j\zeta^{2j-1}, \quad h = \sum_{j=1}^{N} h_j\zeta^{2j-1}.$$

The polynomial

$$P(\zeta) = f^2 - gh = \sum_{j=0}^{2N} p_j\zeta^{2j}$$

is a polynomial with constant coefficients.

If we choose the initial condition $\{ f_j(0,0)\ (j = 1,\cdots,N-1),\ g_j(0,0),\ h_j(0,0)\ (j = 1,\cdots,N)\}$ so that the relations

$$f_j^* = f_j, \quad g_j^* = -h_j \qquad (3.4)$$

hold, we have

$$a = b^*, \quad c = d^*.$$

Therefore by (3.3), we have

$$-u' = 2i|v|^2 u + 2iv$$
$$-\dot{v} = 2i|u|^2 v + 2iu$$

with $u = -2^{-1}f_0^{-1}h_1$, $v = -2^{-1}f_N^{-1}h_N$, that is, a pair of functions u, v is a solution of the field equation of the massive Thirring model.

The system (3.1) is rewritten in terms of the roots $\pm\zeta_j(\xi,\eta)$ $(j = 1,\cdots,N-1)$ of the equation $h = 0$ and constants p_j, f_0, f_N (under the condition (3.4)). The result is the following:

$$\begin{aligned}
&(\partial/\partial\xi)(\log(g_N h_N)) = 4\mathrm{Im}\sum_{j=1}^{N-1}\zeta_j^2 \\
&(\partial/\partial\eta)(\log(g_N h_N)) = 4(-1)^{N-1}f_0^{-1}f_N \mathrm{Im}\prod_{j=1}^{N-1}\zeta_j^2 \\
&(\partial/\partial\xi)(\log(g_1 h_1)) = 4(-1)^{N-1}f_N^{-1}f_0 \mathrm{Im}\prod_{j=1}^{N-1}\zeta_j^{-2} \\
&(\partial/\partial\eta)(\log(g_1 h_1)) = 4\mathrm{Im}\sum_{j=1}^{N-1}\zeta_j^{-2}
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&(\partial/\partial\xi)(\log h_N) = -2^{-1}if_N^{-2}g_N h_N - 2i\sum_{j=1}^{N-1}\zeta_j^2 - if_N^{-2}p_{2N-1} \\
&(\partial/\partial\eta)(\log h_N) = 2^{-1}if_0^{-2}g_1 h_1 - 2i(-1)^{N-1}f_0^{-1}f_N\prod_{j=1}^{N-1}\zeta_j^2 \\
&(\partial/\partial\xi)(\log h_1) = 2^{-1}if_N^{-2}g_N h_N - 2i(-1)^{N-1}f_N^{-1}f_0\prod_{j=1}^{N-1}\zeta_j^{-2} \\
&(\partial/\partial\eta)(\log h_1) = -2^{-1}if_0^{-2}g_1 h_1 - 2i\sum_{j=1}^{N-1}\zeta_j^{-2} - if_0^{-2}p_1.
\end{aligned} \tag{3.6}$$

The roots $\mu_j = \zeta_j^2$ satisfy the following differential equations

$$\mu_j' = \pm\frac{2iP(\mu_j)^{1/2}}{f_N\prod_{k\neq j}(\mu_j - \mu_k)}$$

$$\dot{\mu}_j = \pm\frac{(-1)^{N-1}2i\prod_{k\neq j}\mu_k P(\mu_j)^{1/2}}{f_0\prod_{k\neq j}(\mu_j - \mu_k)}.$$

Introducing the Riemann surface of the hyperelliptic curve $\mu^2 = \sum_{j=0}^{2N}p_j\lambda^j$, we can integrate the above differential equations. Corresponding to (2.1), we have

$$\left(\sum_{k=1}^{N-1}\int_{\mu_0}^{\mu_k(\xi,\eta)}\omega_j\right) \equiv \left(2if_N^{-1}c_{j,N-2}\xi - 2if_0^{-1}c_{j,0}\eta + \right.$$

$$\left. +\sum_{k=1}^{N-1}\int_{\mu_0}^{\mu_k(0,0)}\omega_j\right).$$

Further we can express symmetric functions of μ_j in terms of the Riemann theta functions by using the above relation. A formula for the solution of (0.2) is obtained by the following procedure. The right hand side of (3.5) are expressed in terms of theta functions. Integrating those relations, we have formulas for $\log(g_N h_N)$ and $\log(g_1 h_1)$ in terms of theta functions and initial conditions $g_N(0,0)$, $h_N(0,0)$, $g_1(0,0)$, $h_1(0,0)$. Substituting these expressions into (3.6) and integrating the resulted relations, we have formulas for h_N and h_1. By (3.2), we have an expression for the solution of (0.2).

References

1) V.A. Kozel and V.P. Kotlyarov, Dokl. Akad. Nauk UkSSR., Ser. A. 10 (1976), 878.
2) E. Date, Prog. Theor. Phys. 59 (1978), 265.
3) M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, Phys. Rev. Lett. 30 (1973) 1262.
4) V.E. Zakharov, L.A. Takhtadzhian and L.D. Faddeev, Sov. Phys. Dokl. 19 (1975), 824.
5) A.V. Mikhairov, JETP. Lett. 23 (1976), 320.
6) E. Date and S. Tanaka, Prog. Theor. Phys. 55 (1976), 457; Suppl. Prog. Theor. Phys. No.59 (1976), 107.