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Finite Toda Lattice

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The Hamiltonian of our system is:

$$H = \sum_{j=1}^{N} \frac{1}{2}p_j^{\,2} + \sum_{j=1}^{N-1} e^{-(q_j - q_{j+1})} .$$

Introducing $a_j = e^{-(q_j - q_{j+1})}$ and $b_j = -p_j$, the equations of Motion are

$$\dot{a}_j = a_j(b_j - b_{j+1})$$

$$\dot{b}_j = a_{j-1} - a_j \quad ,$$

where we assume that $a_0 = a_N = 0$. Then "Hamiltonian operator" $\tilde{H}$ is given by

$$\tilde{H} = \sum_{j=1}^{N-1} ((b_j - b_{j+1})a_j \frac{\partial}{\partial a_j} - a_j(\frac{\partial}{\partial b_j} - \frac{\partial}{\partial b_{j+1}})) .$$

Using this we can prove that[1) 2)]

$$\frac{d}{dt} I_n = \tilde{H} I_n = 0,$$

where

$$I_n = \frac{\left(\sum_{j=1}^{N} \frac{\partial}{\partial b_j}\right)^{N-n}}{(N-1)!} \exp\left\{-\sum a_k \frac{\partial^2}{\partial b_k\, b_{k+1}}\right\} \prod_{l=1}^{N} b_l \quad .$$

These are the integrals of Henon's type. Defining f_k by

$$f_k = e^{-\sum_{j=k}^{N-1} a_j \frac{\partial^2}{\partial b_j \partial b_{j+1}}} \prod_{l=k}^{N} (b_l - \lambda)$$

with

$$f_1(\lambda) = \sum_{n=0}^{N} (-\lambda)^{N-n} I_n$$

$$= \exp\left\{-\sum_{j=1}^{N-1} a_j \frac{\partial^2}{\partial b_j \partial b_{j+1}}\right\} \prod_{l=1}^{N} (b_l - \lambda) \quad ,$$

we have

$$f_k(\lambda) = e^{-\sum_{j=k+1}^{N-n} a_j \frac{\partial^2}{\partial b_j \partial b_{j+1}}} \left(1 - a_k \frac{\partial^2}{\partial b_k \partial b_{k+1}}\right) \prod_{l=k}^{N} (b_l - \lambda)$$

$$= (b_k - \lambda) f_{k+1}(\lambda) - a_k f_{k+2}(\lambda) \quad .$$

Thus we have

$$-\frac{f_2(\lambda)}{f_1(\lambda)} = \frac{1|}{|\lambda - b_1} - \frac{a_1|}{|\lambda - b_2} - \cdots\cdots - \frac{a_{N-1}}{\lambda - b_N} \quad .$$

As $f_k(\lambda)$ do not depend on $b_1, b_2, \cdots, b_{k-1}$, the recursion formula for $f_k(\lambda)$'s give

$$\frac{\partial f_1(\lambda)}{\partial b_1} = f_2(\lambda) \quad \text{and} \quad \frac{\partial f_2(\lambda)}{\partial b_2} = f_3(\lambda) \qquad (**)$$

and

$$0 = f_1(\lambda_j) = (b_1 - \lambda_j) f_2(\lambda_j) - a_1 f_3(\lambda_j) \qquad (*)$$

where $\{\lambda_j\}$ are the roots of $f_1(\lambda) = 0$. On the other hand

$$\begin{aligned} \frac{d}{dt} f_2(\lambda) &= \tilde{H} \frac{\partial}{\partial b_1} f_1(\lambda) \\ &= [\tilde{H}, \frac{\partial}{\partial b_1}] f_1(\lambda) + \frac{\partial}{\partial b_1} \tilde{H} f_1(\lambda) \\ &= - a_1 \frac{\partial}{\partial a_1} f_1(\lambda) \end{aligned}$$

where we have used the fact that $[\tilde{H}, \frac{\partial}{\partial b_1}] = - a_1 \frac{\partial}{\partial a_1}$ and $\tilde{H} f_1(\lambda) = 0$. Further as $f_k(\lambda)$ do not depend on $a_1, a_2, \cdots, a_{k-1}$ we have

$$\frac{d}{dt} f_2(\lambda) = a_1 f_3(\lambda).$$

Using (*) we have

$$\frac{d}{dt} f_2(\lambda_j) = (b_1 - \lambda_j) f_2(\lambda_j)$$

or

$$\frac{d}{dt} \ln f_2(\lambda_j) = b_1 - \lambda_j .$$

Thus for every pairs of the roots of $f_1(\lambda) = 0$ we have

$$\frac{d}{dt} \ln \left(\frac{f_2(\lambda_j)}{f_2(\lambda_k)} \right) = \lambda_k - \lambda_j .$$

From this we have

$$f_2(\lambda_j) = c_j \, F(t) \, e^{-\lambda_j t}$$

where F(t) is a function independent of j. Because

$$f_1(\lambda) = \prod_{j=1}^{N} (\lambda_j - \lambda)$$

and (**),

$$f_2(\lambda) = \frac{\partial}{\partial b_1} f_1(\lambda) = \frac{\partial}{\partial b_1} \prod_{j=1}^{N} (\lambda_j - \lambda) = \sum_{k=1}^{N} \frac{\partial \lambda_k}{\partial b_1} \prod^{(k)} (\lambda_j - \lambda).$$

This gives

$$\frac{\partial \lambda_j}{\partial b_1} = \frac{f_2(\lambda_j)}{\prod^{(j)} (\lambda_k - \lambda_j)}$$

and

$$-\frac{f_2(\lambda)}{f_1(\lambda)} = \sum_{j=1}^{N} \frac{\frac{\partial}{\partial b_1} \lambda_j}{\lambda - \lambda_j} = \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \left(\frac{f_2(\lambda_j)}{\prod^{(j)} (\lambda_k - \lambda_j)} \right).$$

Using these, the fact that the sum of the residues of $-f_2(\lambda)/f_1(\lambda)$ is equal to unity can be written in the following from:

$$\sum_{j=1}^{N} \frac{f_2(\lambda_j)}{\prod^{(j)}(\lambda_k-\lambda_j)} = 1$$

Thus we have

$$\frac{\partial\lambda_j}{\partial\beta_1} = \frac{c_j e^{-\lambda_j t}}{\prod^{(j)}(\lambda_k-\lambda_j)} \Bigg/ \sum_{s=1}^{N} \frac{c_s e^{-\lambda_s t}}{\prod_k^{(s)}(\lambda_k-\lambda_s)}$$

$$= R_j(t)$$

Here we will use the classical method of Stieltjes[3) 4)].

For that purpose we write

$$-\frac{f_2(\lambda)}{f_1(\lambda)} = \sum_{j=0}^{\infty} (-1)^j \frac{\gamma_j}{\lambda^{j+1}}$$

with

$$\gamma_j = (-1)^j \sum_{k=1}^{N} \lambda_k^j R_k$$

According to Stieltjes we have

$$a_j = \frac{\begin{vmatrix} \gamma_0 & \cdots & \gamma_{j-2} \\ \vdots & & \vdots \\ \gamma_{j-2} & \cdots & \gamma_{2j-4} \end{vmatrix} \begin{vmatrix} \gamma_0 & \cdots & \gamma_j \\ \vdots & & \vdots \\ \gamma_j & \cdots & \gamma_{2j} \end{vmatrix}}{\begin{vmatrix} \gamma_0 & \cdots & \gamma_{j-1} \\ \vdots & & \vdots \\ \gamma_{j-1} & \cdots & \gamma_{2j-2} \end{vmatrix}^2}$$

$$-b_j = \frac{\begin{vmatrix} \gamma_0 & \cdots & \gamma_{j-1} \\ \vdots & & \vdots \\ \gamma_{j-1} & \cdots & \gamma_{2j-2} \end{vmatrix} \begin{vmatrix} \gamma_1 & \cdots & \gamma_{j-2} \\ \vdots & & \vdots \\ \gamma_{j-2} & \cdots & \gamma_{2j-5} \end{vmatrix}}{\begin{vmatrix} \gamma_0 & \cdots & \gamma_{j-2} \\ \vdots & & \vdots \\ \gamma_{j-2} & \cdots & \gamma_{2j-4} \end{vmatrix} \begin{vmatrix} \gamma_1 & \cdots & \gamma_{j-1} \\ \vdots & & \vdots \\ \gamma_{j-1} & \cdots & \gamma_{2j-3} \end{vmatrix}} + \frac{\begin{vmatrix} \gamma_0 & \cdots & \gamma_{j-2} \\ \vdots & & \vdots \\ \gamma_{j-2} & \cdots & \gamma_{2j-4} \end{vmatrix} \begin{vmatrix} \gamma_1 & \cdots & \gamma_{j} \\ \vdots & & \vdots \\ \gamma_{j} & \cdots & \gamma_{2j-1} \end{vmatrix}}{\begin{vmatrix} \gamma_0 & \cdots & \gamma_{j-1} \\ \vdots & & \vdots \\ \gamma_{j-1} & \cdots & \gamma_{2j-2} \end{vmatrix} \begin{vmatrix} \gamma_1 & \cdots & \gamma_{j-1} \\ \vdots & & \vdots \\ \gamma_{j-1} & \cdots & \gamma_{2j-3} \end{vmatrix}}$$

In this formalism, λ_j's correspond to the square of the eigenvalues of Lax's operator of Kac - van Moerbeko system[3). But here λ_j's are not always positive. These a_j's and b_j's have the same form as the results of the reference 3 (c.f. eq.(4.1)) except the signe of b_j's.

Reference

1) K. Sawada and T. Kotera: J. Phys. Soc. Japan 44 (1977) 659.

2) K. Sawada and T. Kotera: Progr. Theor. Phys. Suppl. 59 (1976) 101.

3) T. Kotera and S. Yamazaki: J. Phys. Soc. Japan 43 (1977) 1797.

4) T. J. Stieltjes: Ann. Fac. de Toulouse, VIII (1894) J1.