数理解析研究所講究録
第 333 巻 1978 年 77-88

Principle of Inclusion-Exclusion on Partially Ordered Sets

by

Hiroshi Narushima

Department of Mathematical Sciences, Faculty of Science, Tokai University,

Hiratsuka, Kanagawa, Japan

Abstract

The principle of inclusion-exclusion on semilattices is extended on partially ordered sets as follows. Let Ω be a nonempty set and P be a finite partially ordered set with the unique maximal element. Let $f:P \longrightarrow \mathfrak{p}(\Omega)$ be a map satisfying $f(x) \cap f(y) \subseteq f(z)$ for each x and y in P and for some minimal element z in the set of all upper bounds of $\{x, y\}$. Then for any measure m on $\mathfrak{p}(\Omega)$ the following identity holds.

$$m(\bigcup_{x \in P} f(x)) = \sum_{c \in C} (-1)^{\ell(c)} m(\bigcap_{x \in c} f(x))$$

where C is the set of all chains in P and $\ell(c)$ denotes the length of a chain c. Also the theorem can be dualized, which results in other three cases. Furthermore, the theorem can be restated in terms of valuations on distributive lattices instead of measures on $\mathfrak{p}(\Omega)$. Herewith, by a slight application of the theorem we obtain some identities on the number of chains and unchains contained in a partially ordered set with the unique maximal element.

Introduction

The following theorem was introduced in order to enumerate reducible or irreducible types of finite systems such as mappings, finite automata and sequential machines [3]([4, Corollary]).

Theorem (Inclusion-Exclusion on a Partition Lattice). *Let S be a finite set. Let $\mathcal{F}(S)$ be the set of mappings from S into itself and $(PL(S), \vee, \wedge)$ be a partition lattice of S. Let $\mathcal{S}$ be a relation between $PL(S)$ and $\mathcal{F}(S)$ defined by $\pi \mathcal{S} f$ for each π in $PL(S)$ and f in $\mathcal{F}(S)$ if and only if for each s and t in S $f(s)$ and $f(t)$ are contained in a same block of π whenever s and t are in a same block of π. Let $\tilde{\mathcal{S}}$ be the map induced by the relation and L be any subsemilattice in $PL(S)$. Then for any measure m on $\wp(\mathcal{F}(S))$ the following identity holds.*

$$m\left(\bigcup_{\pi \in L} \tilde{\mathcal{S}}(\pi)\right) = \sum_{c \in C} (-1)^{\ell(c)}\, m\left(\bigcap_{\pi \in c} \tilde{\mathcal{S}}(\pi)\right),$$

where C is the set of chains in L and $\ell(c)$ denotes the length of a chain c.

In the proof it is essential that for each π and τ in $PL(S)$

$$\tilde{\mathcal{S}}(\pi) \cap \tilde{\mathcal{S}}(\tau) \subseteq \tilde{\mathcal{S}}(\pi \wedge \tau) \text{ and } \tilde{\mathcal{S}}(\pi) \cap \tilde{\mathcal{S}}(\tau) \subseteq \tilde{\mathcal{S}}(\pi \vee \tau)$$

hold. Therefore the theorem has been extended on semilattices as follows [4, Theorem 1].

Theorem (Inclusion-Exclusion on Semilattices). *Let Ω be a nonempty set and $(L, \vee)$ be a finite join-semilattice. Let*

$f: L \longrightarrow \mathfrak{p}(\Omega)$ be a map satisfying $f(x) \cap f(y) \subseteq f(x \vee y)$ for each x and y in L. Then for any measure m on $\mathfrak{p}(\Omega)$ the following identity holds.

$$m\left(\bigcup_{x \in L} f(x)\right) = \sum_{c \in C} (-1)^{\ell(c)} m\left(\bigcap_{x \in c} f(x)\right),$$

where C is the set of chains in L and $\ell(c)$ denotes the length of a chain c. The theorem can be dualized.

Furthermore, the theorem was applied to a Boolean lattice and a product partition lattice [4, Proposition 1, Theorem 2]. Also the theorem has been restated in terms of valuations on distributive lattices instead of measures on $\mathfrak{p}(\Omega)$ [5]. In the proofs, the following Rota's theorem obtained from [1, Theorem 1] plays an important role.

Theorem (Möbius Functions on Closure Relations). Let $x \longrightarrow \bar{x}$ be a closure relation in a partially ordered set P having a unique minimal element 0, with the property that $\bar{x} = 0$ only if $x = 0$. Let Q be the partially ordered subset of all closed elements in P. Then for each y in Q

$$\mu_q(0, y) = \sum_{x : \bar{x} = y} \mu_p(0, x),$$

where μ_p and μ_q are the Möbius functions of P and Q.

A map $x \longrightarrow \bar{x}$ of a partially ordered set P into itself with the following properties is called a closure relation in P:

(1) $\bar{x} \geq x$, (2) $\bar{\bar{x}} = \bar{x}$, (3) $x \geq y$ implies $\bar{x} \geq \bar{y}$.

If $\bar{\underline{x}} = \underline{x}$ then $\underline{x}$ is called a closed element. Let $(\underline{L}, \vee)$ be a finite join-semilattice. A map $\underline{X} \longrightarrow \bar{\underline{X}}$ of $\wp(\underline{L})$ into itself defined by $\bar{\underline{X}}$ = subsemilattice of $\underline{L}$ generated by $\underline{X}$ ($\bar{\phi} = \phi$) is a closure relation in $\wp(\underline{L})$. The set of closed elements results in the lattice of all subsemilattices of $\underline{L}$, written $\underline{L}^*$. Then Rota's theorem is applied to the Möbius functions μ and μ^* of $\wp(\underline{L})$ and $\underline{L}^*$, which leads to the following identity. For each $\underline{Y}$ in L^*,

$$\mu^*(\phi, \underline{Y}) = \sum_{\underline{X}:\bar{\underline{X}}=\underline{Y}} \mu(\phi, \underline{X}) = \sum_{\underline{X}:\bar{\underline{X}}=\underline{Y}} (-1)^{|\underline{X}|}.$$

Furthermore, for each $\underline{Y}$ in $\underline{L}^*$ let $\underline{Y}_0$ be the set of join-irreducibles of $\underline{Y}$. Then $\underline{Y}_0$ is the unique minimal element of $\{\underline{X} \in \wp(\underline{L}) \mid \bar{\underline{X}} = \underline{Y}\}$. Therefore, it follows that

$$\sum_{\underline{X}:\bar{\underline{X}}=\underline{Y}} (-1)^{|\underline{X}|} = (-1)^{|\underline{Y}_0|}(1-1)^{|\underline{Y}|-|\underline{Y}_0|}$$

$$= \begin{cases} (-1)^{|\underline{Y}|} & \text{for } \underline{Y}_0 = \underline{Y}\text{, i.e., } \underline{Y} \text{ is a chain} \\ 0 & \text{otherwise.} \end{cases}$$

The identities are showed in Proposition of [4, p.198]. On the other hand, since $\underline{f}(\underline{x}) \cap \underline{f}(\underline{y}) \subseteq \underline{f}(\underline{x} \vee \underline{y})$ for each $\underline{x}$ and $\underline{y}$ in $\underline{L}$, it follows that for each $\underline{X}$ in $\wp(\underline{L})$

$$\bigcap_{\underline{x} \in \underline{X}} \underline{f}(\underline{x}) = \bigcap_{\underline{x} \in \bar{\underline{X}}} \underline{f}(\underline{x}).$$

The inclusion-exclusion on semilattices is proved from these identities. Thus Rota's theorem is a guiding principle in our subject, herewith it is also worth notice that some elemental proofs without the use of the closure relation are possible. In this paper it is shown that the inclusion-exclusion on semilattices can be extended on partially ordered sets. The two different proofs are given, one in which the closure relation is used, the other is elemental. Then the relationships between the results obtained before and the present theorem are described. Finally, a new slight application to counting the chains and unchains contained in a partially ordered set is given.

The Theorem

A partially ordered set is also called a poset. The theorem is as follows.

Theorem (Inclusion-Exclusion on Posets). *Let Ω be a nonempty set and P be a finite partially ordered set with the unique maximal element. Let $f: P \longrightarrow \mathfrak{p}(\Omega)$ be a map satisfying $f(x) \cap f(y) \subseteq f(z)$ for each x and y in P and for some minimal element z in the subposet (of P) of all upper bounds of $\{x, y\}$. Then for any measure m on $\mathfrak{p}(\Omega)$ the following identity holds.*

$$m\Big(\bigcup_{x \in P} f(x)\Big) = \sum_{c \in C} (-1)^{\ell(c)} m\Big(\bigcap_{x \in c} f(x)\Big),$$

where C is the set of all chains in P and $\ell(c)$ denotes the length of a chain c. Also the theorem can be dualized, which results in other three cases. Furthermore, the theorem can be restated in terms of valuations on distributive lattices instead of measures on $\mathfrak{p}(\Omega)$.

The two proofs are now given. Let P be a finite poset and L be a finite semilattice. The first proof is carried out by an extention of the closure relation on $\mathfrak{p}(L)$ in the proof of

[4, Theorem 1] described in Introducton to the closure relation on $\wp(P)$. The second proof is elemental without the use of the closure relation, in which $\wp(P)$ is classified by the pre-ordered incomparable pair contained in each subset of P and a bijection on the set of unchains in P is defined. For each unordered pair $p = \{x, y\}$ in P let $\langle p \rangle$ denote the minimal element z in the subposet (of P) of all upper bounds of p which satisfies $f(x) \cap f(y) \subseteq f(z)$. $\langle \{x, y\} \rangle$ is abbreviated to $\langle x, y \rangle$. Note that $\langle x, y \rangle = y$ for $x < y$ in P and $\langle x, y \rangle = x \vee y$ if there exists the least upper bound $x \vee y$ of $\{x, y\}$. For each X in $\wp(P)$ let $Icp(X)$ denote the set of all incomparable unordered pairs in X. Then the proofs are as follows.

First Proof. For each X in $\wp(P)$ $\bar{X}$ is inductively defined in the following way. Let $X_0 = X$ and for $i \geq 0$

$$X_{i+1} = X_i \cup \{\langle p \rangle \mid p \text{ in } Icp(X_i)\}.$$

Then $\bar{X}$ is defined by $\bar{X} = \bigcup_{0 \leq i} X_i$ which is equal to X_n for the least integer n such that $X_{n+1} = X_n$. Therefore the map $X \longrightarrow \bar{X}$ of $\wp(P)$ into itself is a closure relation in $\wp(P)$. The set of closed elements is denoted by Q. Now we show that the following identity holds. For each Y in Q

$$\sum_{X : \bar{X} = Y} (-1)^{|X|} = \begin{cases} (-1)^{|Y|} & \text{when } Y \text{ is a chain or } \phi \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where the sum of the left side ranges over all X satisfying $\bar{X} = Y$. Considering that for each Y in Q $Y - \{\langle p \rangle \mid p \text{ in } Icp(Y)\}$, written Y_0, is the unique minimal element in $\{X \text{ in } \wp(P) \mid \bar{X} = Y\}$,

$$\sum_{X:\bar{X}=Y} (-1)^{|X|} = (-1)^{|Y_0|}(1-1)^{|Y|-|Y_0|}$$

$$= \begin{cases} (-1)^{|Y|} & \text{for } Y_0 = Y \\ 0 & \text{otherwise.} \end{cases}$$

$Y_0 = Y$ means that Y is a chain, completing the proof of the identity (1). Also, since $f(\langle x, y\rangle) \cap f(x) \cap f(y) = f(x) \cap f(y)$ for each x and y in P, it follows that

$$\bigcap_{x \in \bar{X}} f(x) = \bigcap_{x \in X} f(x). \tag{2}$$

On the other hand, by the principle of inclusion-exclusion,

$$m(\bigcup_{x \in P} f(x)) = \sum_{X \in \wp(P)} (-1)^{|X|-1} m(\bigcap_{x \in X} f(x)).$$

Then from (2) the following identity is obtained.

$$m(\bigcup_{x \in P} f(x)) = \sum_{Y \in Q} \{ \sum_{X:\bar{X}=Y} (-1)^{|X|-1} \} m(\bigcap_{x \in Y} f(x)).$$

Now the theorem follows from the identity (1).

Second Proof. Let M be the set of minimal elements in P. Let's arrange elements in M and then minimal elements in the subposet $P-M$ and so on up to the unique maximal element. Thus P is totally ordered $x_1 \leq_e x_2 \leq_e \cdots \leq_e x_s$ in the arrangement. Note $x <_e y$ for $x < y$ in P. For each i $(1 \leq i \leq s)$ let P_i denote

$$\{ p \text{ in } Icp(P) \mid \langle p\rangle = x_i \}.$$

For some i P_i may be ϕ. Then $\sum_{i=1}^{s} P_i = Icp(P)$. Now number elements in P_1 and then P_2 and so on till P_s. Thus the elements of $Icp(P)$ is numbered $p_1, p_2, \cdots, p_t$. Then for each i $(1 \leq i \leq k \leq t)$ p_i

contains no $<p_k>$. Because, if for some i $(1 \leq i \leq k)$ p_i contains $<p_k>$ then $<p_k> < <p_i>$, that is, $<p_k> <_e <p_i>$, which is contrary to $<p_i> \leq_e <p_k>$. For p_i $(1 \leq i \leq t)$ in $Icp(P)$ $U(p_i)$ is inductively defined in the following way. Let

$$U(p_1) = \{ X \text{ in } \wp(P) \mid X \supseteq p_1 \}$$

and for k $(2 \leq k \leq t)$

$$U(p_k) = \{ X \text{ in } (\wp(P) - \bigcup_{i=1}^{k-1} U(p_i)) \mid X \supseteq p_k \}.$$

Then $\wp(P) = \sum_{i=1}^{t} U(p_i) + C$, where C is the set of chains in P.
Let for k $(1 \leq k \leq t)$

$$U^{(0)}(p_k) = \{ X \in U(p_k) \mid <p_k> \notin X \}$$

$$U^{(1)}(p_k) = \{ X \in U(p_k) \mid <p_k> \in X \}.$$

Then $U^{(1)}(p_k)$ contains $X + \{<p_k>\}$ if $U^{(0)}(p_k)$ contains X. Because, for each X in $U^{(0)}(p_k)$ $Icp(X + \{<p_k>\})$ contains p_k but not p_i $(1 \leq i \leq k-1)$. Therefore, a map

$$\alpha_k : U^{(0)}(p_k) \longrightarrow U^{(1)}(p_k)$$

is well defined by $\alpha_k(X) = X + \{<p_k>\}$ and it is easily shown that α_k $(1 \leq k \leq t)$ is a bijection. Let $U^{(i)} = \sum_{k=1}^{t} U^{(i)}(p_k)$ for each i in $\{0, 1\}$. Then a bijection

$$\alpha : U^{(0)} \longrightarrow U^{(1)}$$

is defined by $\alpha(X) = \alpha_k(X)$ for each X in $U^{(0)}(p_k)$. Now, by the principle of inclusion-exclusion

$$\underline{m}(\bigcup_{\underline{x}\in \underline{P}} \underline{f}(\underline{x})) = \sum_{\underline{X}\in \wp(\underline{P})} (-1)^{|\underline{X}|-1} \underline{m}(\bigcap_{\underline{x}\in \underline{X}} \underline{f}(\underline{x}))$$

$$= \sum_{\underline{X}\in \underline{U}} (-1)^{|\underline{X}|-1} \underline{m}(\bigcap_{\underline{x}\in \underline{X}} \underline{f}(\underline{x})) + \sum_{\underline{c}\in \underline{C}} (-1)^{\ell(\underline{c})} \underline{m}(\bigcap_{\underline{x}\in \underline{c}} \underline{f}(\underline{x})) \qquad (3)$$

where $\underline{U}$ is the set of unchains $(\underline{U}^{(0)} + \underline{U}^{(1)})$ and $\underline{C}$ is the set of chains. Also, since $\underline{f}(\underline{x}) \cap \underline{f}(\underline{y}) \subseteq \underline{f}(\langle \underline{x}, \underline{y}\rangle)$, for each $\underline{X}$ in $\underline{U}$

$$\bigcap_{\underline{x}\in \underline{X}} \underline{f}(\underline{x}) = \bigcap_{\underline{x}\in \alpha(\underline{X})} \underline{f}(\underline{x}), \text{ and } |\alpha(\underline{X})| - 1 = |\underline{X}|.$$

Therefore, the first term of the right side of the identity (3) is equal to 0, completing the proof.

If necessary, a map $\underline{f}:\underline{P} \longrightarrow \wp(\Omega)$ satisfying $\underline{f}(\underline{x}) \cap \underline{f}(\underline{y}) \subseteq \underline{f}(\underline{z})$ for each $\underline{x}$ and $\underline{y}$ in $\underline{P}$ and for some minimal element $\underline{z}$ in the subposet (of $\underline{P}$) of all upper bounds of $\{\underline{x}, \underline{y}\}$ is called a <u>weak morphism</u> on $\underline{P}$ (contains other three dual cases). For a given poset $\underline{P}$ and map $\underline{f}:\underline{P} \longrightarrow \wp(\Omega)$, it is of interest whether $\underline{f}$ is a weak morphism or not.

<u>Remark</u> 1. " To dualize the theorem " means that $\cup$ and $\cap$ are interchangeable by setting $\underline{f}(\underline{x}) \cup \underline{f}(\underline{y}) \supseteq \underline{f}(\underline{z})$ for $\underline{f}(\underline{x}) \cap \underline{f}(\underline{y}) \subseteq \underline{f}(\underline{z})$ and that for any finite poset with the unique minimal element the theorem holds.

<u>Remark</u> 2. Let $(\underline{D}, \vee, \wedge)$ be a distributive lattice and $(\underline{A}, +)$ be a commutative ring with identity. A map $\underline{v}:\underline{D} \longrightarrow \underline{A}$ satisfying

$$\underline{v}(\underline{x} \vee \underline{y}) + \underline{v}(\underline{x} \wedge \underline{y}) = \underline{v}(\underline{x}) + \underline{v}(\underline{y})$$

for each $\underline{x}$ and $\underline{y}$ in $\underline{D}$ is called a valuation on $\underline{D}$. Then it is easily shown that the theorem can be restated in terms of valuations on distributive lattices instead of measures on $\mathcal{P}(\Omega)$.

Remark 3. Let $\underline{P}$ be a finite chain. Then a map $\underline{f}:\underline{P} \longrightarrow \mathcal{P}(\Omega)$ is a weak morphism on $\underline{P}$ and $\mathcal{P}(\underline{P}) = \underline{C}$. Thus the principle of inclusion-exclusion is derived. Let $\underline{P}$ be a finite semilattice. Then a map $\underline{f}:\underline{P} \longrightarrow \mathcal{P}(\Omega)$ satisfying $\underline{f}(\underline{x}) \cap \underline{f}(\underline{y}) \subseteq \underline{f}(\underline{x} \vee \underline{y})$ for each $\underline{x}$ and $\underline{y}$ in $\underline{P}$ is a weak morphism on $\underline{P}$. Thus the principle of inclusion-exclusion on semilattices is derived.

Remark 4. Let $\underline{P}$ be a finite poset and $\underline{Q}$ be the set of closed elements ($\bar{\phi} = \phi$) in the first proof. Let μ and μ^* be the Möbius functions of $\mathcal{P}(\underline{P})$ and $\underline{Q}$. Then for each $\underline{Y}$ in $\underline{Q}$

$$\mu^*(\phi, \underline{Y}) = \begin{cases} (-1)^{|\underline{Y}|} & \text{if } \underline{Y} \text{ is a chain} \\ 0 & \text{otherwise.} \end{cases}$$

Because, $\mu^*(\phi, \underline{Y}) = \sum_{\underline{X}:\bar{\underline{X}}=\underline{Y}} \mu(\phi, \underline{X}) = \sum_{\underline{X}:\bar{\underline{X}}=\underline{Y}} (-1)^{|\underline{X}|}$, which is just the identity (1). This formula is an extention of Proposition in [4, p. 198] described in Introduction.

Finally a slight application is shown.

Proposition. Let $\underline{P}$ be a finite poset of the cardinality $\underline{n}$ with the unique maximal element. Let c_i be the number of chains of length $\underline{i}$ in $\underline{P}$ and u_i be the number of unchains of size $\underline{i}$ in $\underline{P}$. Then the following identities hold.

$$(1)\ \sum_{0\leq i} (-1)^{i} c_i = 1 \qquad (2)\ \sum_{2\leq i} (-1)^{i-1} u_i = 0.$$

For non-negative integers k and l such that $k + l = 2^{n-1}-1$

$$(3)\ \sum_{0\leq i} c_i = 2k + 1 \qquad (4)\ \sum_{2\leq i} u_i = 2l$$

$$(5)\ \sum_{0\leq i} c_{2i} = k + 1 \qquad (6)\ \sum_{1\leq i} u_{2i} = l$$

$$(7)\ \sum_{0\leq i} c_{2i+1} = k \qquad (8)\ \sum_{1\leq i} u_{2i+1} = l.$$

Proof. In the sequel, for each X in $\mathcal{P}(P)$ let $m(X)$ be the cardinality of X. (1) Let $f: P \longrightarrow \mathcal{P}(\Omega)$ be a constant map defined by $f(x) = \{a\}$ for all x in P. Then clearly f is a weak morphism and the identity is easily derived. (2) Let $\tilde{P}$ denote a totally ordered P. Let $f: \tilde{P} \longrightarrow \mathcal{P}(\Omega)$ be a constant map defined by $f(x) = \{a\}$. Then by the principle of inclusion-exclusion,

$$\sum_{c\in C} (-1)^{\ell(c)} + \sum_{u\in U} (-1)^{|u|-1} = 1$$

where C is the set of elements in $\mathcal{P}(\tilde{P})$ which are chains in P and U is the set of elements in $\mathcal{P}(\tilde{P})$ which are unchains in P. Therefore the identity follows from (1). The identities (3)–(8) are easily derived.

References

1. G.-C. Rota, On the foundation of combinatorial theory I, theory of Mobius functions, Z. Wahrsheinlichkeitstheorie und Verw. Gebiete **2** (1964), 340-368.

2. G.-C. Rota, On the combinatorics of Euler characteristics, in: Studies in Pure Mathematics; (to R. Rado), Ed. Mirsky, (Academic Press, 1971), 221-233.

3. H. Narushima, Partitions and principle of inclusion-exclusion, Master's thesis, Tokyo Univ. of Education, Feb. 1969.

4. H. Narushima, Principle of inclusion-exclusion on semilattices, *J. Combinatorial Theory, Ser. A*, **17** (1974), 196-203.

5. H. Narushima and H. Era, A variant of inclusion-exclusion on semilattices, *Discrete Math.*, to appear.

6. M. Aigner, "Kombinatorik I, Grundlagen und Zahltheorie", Springer-Verlag, Berlin, 1975.