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Some Generalized Toeplitz Operators

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1. Introduction

Let L^2 be the Hilbert space of all square Lebesgue integrable unctions defined on the unit circle, and L^∞ the Banach algebra of all ssentially bounded functions defined on the unit circle. We denote he Hardy spaces corresponding to L^2 and L^∞ by H^2 and H^∞ respecti-ely. If h is a function in some Hardy space, then by the Poisson ntegral formula h is extended to an analytic function in the open nit disc.

Given ϕ in L^∞, let M_ϕ be the operator on L^2 defined by

$$M_\phi h = \phi h \quad \text{for } h \text{ in } L^2.$$

et P' be the projection from L^2 onto H^2. Then an operator on H^2 efined by

$$T_\phi = P' M_\phi |H^2$$

s called a Toeplitz operator with symbol ϕ. In particular, if ϕ is he identity function, then T_ϕ is called the shift operator. From eurling's theorem, each invariant subspace for the shift operator s of the form ψH^2 with some inner function ψ (i.e., ψ is in H^∞ nd $|\psi(e^{it})| = 1$ a.e.)

For an inner function well known Hilbert space $H(\psi)$ is determined y

$$H(\psi) = H^2 \ominus \psi H^2.$$

rom now we fix an inner function ψ and hence Hilbert space $H(\psi)$.

We denote the projection from H^2 onto $H(\psi)$ by P.

Definition. For ϕ in L^{∞}, we define the general Toeplitz operator $\phi(S(\psi))$ in the sense of [1] by $\phi(S(\psi))=PT_{\phi}|H(\psi)$.

If ϕ belongs to H^{∞}, then $\phi(S(\psi))$ was defined in [3] and [4], and its properties are well known. Sarason showed that if ϕ is in H^{∞}, then $\phi(S(\psi))$ is compact if and only if $\overline{\psi}\phi$ belongs to $H^{\infty}+C$, where C is the Banach algebra of all continuous functions defined on the unit circle. In this paper we extend this result to ϕ in $H^{\infty}+C$.

2. Trivial Results

We denote the inner product in $H(\psi)$, H^2 and L^2 by $(\,,\,)$, $(\,,\,)'$ and $(\,,\,)''$ respectively. Let $B(H(\psi))$ be the Banach algebra of all bounded operators on $H(\psi)$, and x a mapping from L^{∞} to $B(H(\psi))$ defined by $x(\phi)=\phi(S(\psi))$.

Proposition 1. x is a contractive star linear mapping.

Proof. For f and g in $H(\psi)$, we have

$$(\overline{\phi}(S(\psi)f,g)=(PT_{\overline{\phi}}f,g)=(T_{\overline{\phi}}f,g)'=(f,T_{\phi}g)'=(f,PT_{\phi}g)'=(f,\phi(S(\psi))g).$$

Thus $x(\overline{\phi})=\phi(S(\psi))^*$. The rest is trivial.

Proposition 2. If ϕ is an invertible function in L^{∞} whose essential range is contained in the open right half plane, then $\phi(S(\psi))$ is invertible.

Proof. There exists an $\varepsilon>0$ such that $\|\varepsilon\phi-1\|<1$ (c.f.[2]). From proposition 1, it follows that

$$\|\varepsilon\phi(S(\psi))-I\|<1,$$

which implies that $\phi(S(\psi))$ is invertible.

From this proposition, we can obtain the next proposition by the same techniques as the proof of 7.19 of [2].Therefore we omit the proof.

Proposition 3. $\sigma(\phi(S(\psi)))$ is included in the closed covex hull of essential range of ϕ.

Proposition 4. If ϕ is a real valued function in L^∞, then

$$\sigma(\phi(S(\psi))) \subset \sigma(T_\phi).$$

Proof. Hartmann-Wintner showed that

$$\sigma(T_\phi)=[\text{ess inf } \phi, \text{ ess sup } \phi],$$

which is a closed convex hull of the essential range of ϕ. Thus the assertion follows from proposition 3.

3. Main Results

We denote the identity operators on $H(\psi)$, H^2 and L^2 by I, I' and I''.

Lemma 1. For ϕ in $H^\infty + C$, $(I''- P')M_\phi P'$ is a compact operator on L^2.

Proof. Let $\phi = \phi^1+\phi^2$ be a decomposition of ϕ such that ϕ^1 is in H^∞ and ϕ^2 is in C. Then it follows that

$$(I''-P')M_\phi P' = (I''-P')M_{\phi^2}P'.$$

Take trigonometric polynomials $q_n (n=1,2,\ldots)$ whose sequence uniformly converges to ϕ^2. Then ,since

$$\|(I''-P')M_{q_n}P' - (I''-P')M_{\phi^2}P'\| \leq \|M_{q_n} - M_{\phi^2}\| \leq \|q_n - \phi^2\| \to 0 \ (n\to\infty),$$

finiteness of the rank of $(I''-P')M_{q_n}P'$ implies that $(I'-P')M_{\phi^2}P'$ is compact.

Lemma 2. For ϕ in $H^\infty + C$, $PT_\phi(I'-P)$ is a compact operator.

Proof. This lemma follows from Lemma 1 and next relations;

$$PT_\phi(I'-P)=PP'M_\phi(I'-P)=PP'M_\phi M_\psi M_{\overline{\psi}}(I'-P)=PP'M_\psi M_\phi M_{\overline{\psi}}(I'-P)=$$
$$=PP'M_\psi(I''-P')M_\phi P'M_{\overline{\psi}}(I'-P).$$

Proposition 5. If ϕ is in C and η is in L^∞, then

$$\phi(S(\psi))\eta(S(\psi)) - (\phi\eta)(S(\psi))$$
$$\text{and} \quad \eta(S(\psi))\phi(S(\psi)) - (\phi\eta)(S(\psi))$$

are compact.

Proof. Since $T_\phi T_\eta - T_{\phi\eta}$ is compact, we have

$$PT_\phi PT_\eta P - PT_{\phi\eta}P = PT_\phi PT_\eta P - PT_\phi T_\eta P + \text{compact} =$$
$$= PT_\phi(I'-P)T_\eta P + \text{compact}.$$

Thus ,by Lemma 2, $\phi(S(\psi))\eta(S(\psi)) - (\phi\eta)(S(\psi))$ is compact. Since

$$[\eta(S(\psi))\phi(S(\psi)) - (\eta\phi)(S(\psi))]^*=\overline{\phi}(S(\psi))\overline{\eta}(S(\psi))-(\overline{\phi\eta})(S(\psi)),$$

we can conclude the proof.

Set $H_0^p = \{ f \in H^p;\ f(0)=0 \}$. It is well known that if f belongs to H_0^1 then there exist f_1 in H^2 and f_2 in H_0^2 such that $f=f_1f_2$ and $|f| = |f_1|^2 = |f_2|^2$ a.e. .

Lemma 3. If ϕ is in $H^\infty + C$, then there exists a compact operator K from H^2 to $\overline{H}_0^2$ (conjugate space of H_0^2) such that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi\overline{\psi}\, f\, dt = (Kf_1,f_2)'' + (\phi(S(\psi))Pf_1,P'\psi\overline{f}_2)$$

for every f in H_0^1 , f_1 in H^2 and f_2 in H_0^2 such that $f = f_1f_2$.

Remark. It is not assumed that $|f| = |f_1|^2 = |f_2|^2$.

Proof. Since $\psi\overline{f}_2$ is orthogonal to ψH^2, $P'\psi\overline{f}_2$ belongs to $H(\psi)$.

$$\frac{1}{2\pi}\int_0^{2\pi} \phi\overline{\psi}\, f\, dt = (\phi f_1, \psi\overline{f}_2)" = (P'\phi P f_1, \psi\overline{f}_2)" +$$

$$+(P'\phi(I'-P)f_1,\psi\overline{f}_2)" + ((I"-P')\phi f_1,\psi\overline{f}_2)"$$

$$= (P'\phi P f_1, P'\psi\overline{f}_2)" + (\overline{\psi}PP'\phi(I'-P)f_1,\overline{f}_2)" + (\overline{\psi}(I"-P')\phi f_1,\overline{f}_2)"$$

$$= (\phi(S(\psi))Pf_1, P'\psi\overline{f}_2) + (\overline{\psi}PT_\phi(I'-P)f_1,\overline{f}_2)" + (\overline{\psi}(I"-P')M_\phi f_1,\overline{f}_2)"$$

Thus $K = M_{\overline{\psi}}PT_\phi(I'-P) + M_{\overline{\psi}}(I"-P')M_\phi\ |H^2$ satisfies the conditions of this lemma.

The proof of next theorem is deeply depend to [3].

Theorem 1. Let ϕ be a function in $H^\infty + C$. Then $\phi(S(\psi))$ is compact if and only if $\overline{\psi}\phi$ belongs to $H^\infty + C$.

Proof. Suppose first that $\overline{\psi}\phi$ is in $H^\infty + C$. Then there exist η in H^∞ and ζ in C such that $\phi = \psi(\eta+\zeta)$. Since $(\psi\eta)(S(\psi))=0$, it follows that $\phi(S(\psi)) = (\psi\zeta)(S(\psi))$, which is compact [3].

Suppose next that $\phi(S(\psi))$ is compact. We wish to show that the kernel of functional of $\overline{\psi}\phi + H^\infty$ on H_0^1 is sequentially weak star closed. Let f_n be a sequence in the kernel of it which converges weak star to f. Let $f_n = f_{1n}f_{2n}$ be the factorization of f_n such that f_{1n} and f_{2n} belong to H^2 and H_0^2 ,respectively, and that $|f_n| = |f_{1n}|^2 = |f_{2n}|^2$. Then , since each of sequences of $\{f_{1n}\}$ and $\{f_{2n}\}$ is bounded in L^2, we may assume that each of sequences above weakly converges to f_1 and f_2 respectively in L^2, and $f = f_1 f_2$ [3]. It is clear that f_1 is in H^2 and f_2 is in H_0^2. From lemma 3, there is a compact operator K such that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi\overline{\psi} f_n\, dt = (Kf_{1n},\overline{f}_{2n})" + (\phi(S(\psi))Pf_{1n}, P'\psi\overline{f}_{2n})$$

and $$\frac{1}{2\pi}\int_0^{2\pi} \phi\overline{\psi} f\, dt = (Kf_1,\overline{f}_2)" + (\phi(S(\psi))Pf_1, P'\psi\overline{f}_2).$$

Since both K and $\phi(S(\psi))$ are compact,it follows that

$$(Kf_{1n},\overline{f}_{2n})" \longrightarrow (Kf_1,\overline{f}_2)" \quad \text{as} \quad n \longrightarrow \infty$$

and $(\phi(S(\psi))Pf_{1n}, P'\psi\bar{f}_{2n}) \longrightarrow (\phi(S(\psi))Pf_1, P'\psi\bar{f}_2)$ as $n \to \infty$.
Thus we have

$$\frac{1}{2\pi}\int_0^{2\pi} \phi\bar{\psi}\, f\, dt = 0.$$

Thus we can conclude the proof .

4.Miscellaneous Results

Let $\underline{K}$ be the ideal of all compact operators of $B(H(\psi))$.

Corollary 1. $\{\phi(S(\psi)) + \underline{K}\ ;\ \phi \text{ in } H^\infty+C\}$ is an algebra and the natural mapping from it onto $\{\phi + \psi(H^\infty+C);\ \phi \text{ in } H^\infty+C\}$ is an isomorphism.

Proof. From Proposition 5, $\{\phi(S(\psi)) + \underline{K}\}$ is an algebra. From Theorem 1, the natural mapping is well defined and one to one.

Corollary 2. If ϕ is in $H^\infty+C$ and $\phi(H^\infty+C)+\psi(H^\infty+C)=H^\infty+C$, then $\phi(S(\psi))$ is a Fredholm operator.

Proof. There exists η in $H^\infty+C$ such that $\phi\eta + \psi(H^\infty+C)=1+\psi(H^\infty+C)$. Therefore we have

$$\phi(S(\psi))\eta(S(\psi))+\underline{K} = (\phi\eta)(S(\psi)) + \underline{K} = I + \underline{K}.$$

Consequently $\phi(S(\psi))$ is a Fredholm operator.

Corollary 3. If ϕ is in $H^\infty+C$ and T_ϕ is a Fredholm operator,then $\phi(S(\psi))$ is a Fredholm operator.

Proof. Since ϕ is invertible in $H^\infty+C$ [2], this corollary follows from Corollary 2.

If ϕ is in H^∞, then we showd that $\phi(S(\psi))$ is a Fredholm operator if and only if there is a factorization $\phi = \phi^1\phi^2$, where $\phi^1(S(\psi))$ is invertible and ϕ^2 is a finite Blaschke function [5].

Theorem 2. If ϕ is in H^∞, then next conditions are equivalent ;

(a) $\phi(S(\psi))$ is a Fredholm operator,

(b) there are $\varepsilon>0$ and $1>\delta\geqq 0$ such that $|\phi(z)|+|\psi(z)|\geqq\varepsilon$ for $1>|z|\geqq\delta$,

(c) $\phi(H^\infty+C) + \psi(H^\infty+C) = H^\infty+C$.

Proof. First assume (a). Let $\phi=\phi^1\phi^2$ be the factorization given above. Then there is an $\varepsilon>0$ such that

$$|\phi^1(z)| + |\psi(z)|\geqq\varepsilon \quad \text{for} \quad |z|<1 ,$$

because $\phi^1(S(\psi))$ is invertible. And there are $\varepsilon'>0$ and $1>\delta\geqq 0$ such that $|\phi^2(z)| \geqq \varepsilon'$ for $1> |z|\geqq\delta$,

because ϕ^2 is a finite Blaschke function. Therefore we have

$$|\phi(z)| + |\psi(z)|\geqq|\phi^2(z)|\{|\phi^1(z)| + |\psi(z)|\}\geqq\varepsilon\varepsilon' \text{ for } 1>|z|\geqq\delta .$$

Thus we have (b).

Next assume (b). Let η be the greatest common inner divisor of ϕ and ψ.Then it is obvious that there is an $\varepsilon'\geqq 0$ such that

$$|\eta(z)|\geqq\varepsilon' \quad \text{for} \quad 1> |z|\geqq\delta.$$

Consequently $1/\eta$ belongs to $H^\infty+C$ [2]. Set $\phi'=\phi/\eta$ and $\psi'=\psi/\eta$.Then it is clear that there is an $\varepsilon''>0$ such that

$$|\phi'(z)| + |\psi'(z)|\geqq\varepsilon'' \quad \text{for} \quad |z|<1.$$

Hence ,by corona theorem ,we have $\phi'H^\infty + \psi'H^\infty= H^\infty$, which yields

$$\phi(H^\infty+C)+\psi(H^\infty+C)= H^\infty+C.$$

Last, by Corollary 2, (c) implies (a).

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