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Riemann-Hilbert Problem and its Application to Analytic Functions of Several Complex Variables

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0. In this note, we shall give a brief sketch of the proof of the local existence of holomorphic functions in an analytic cover $\pi: Y \longrightarrow X$ by using a solution of Riemann-Hilbert problem. The existence of such functions was earlier proved in 1958 by H. Grauert and R. Remmert [2] and in 1960 by R. Kawai [3] by different methods. The complete proof of the note will be found in [4].

1. Let us recall the definitions of analytic covers (ramified Riemann domains) and holomorphic functions on them: let Y be a locally compact, Hausdorff space and let X be a complex manifold. An analytic cover is a triple $\pi : Y \longrightarrow X$ such that 1) π is a proper, continuous mapping of Y onto X with discrete fibers. 2) There are a divisor D of X and a positive integer $q \in N$ such that π is a q-sheeted topological covering map from $Y - \pi^{-1}(D)$ onto $X - D$. 3) $Y - \pi^{-1}(D)$ is dense in Y. 4) For any point $y \in \pi^{-1}(D)$ and any connected open negihbourhood U of y, there exists an open neighbourhood $U' \subset U$ such that $U' - \pi^{-1}(D) \cap U'$ is connected.

D is called the critical locus of an analytic cover $\pi : Y \longrightarrow X$ and q is called the sheet number of it. A continuous function $f(y)$ on an open subset U of Y is, by def., holomorphic on U if the restriction of $f(y)$ to $U - U \cap \pi^{-1}(D)$ is holomorphic in the usual sense. Roughly, speaking, analytic space in the sense of Behnke-Stein is a $\mathbb{C}$-local ringed space $(X, \mathcal{O})$ such that locally it is isomorphic to an analytic cover. H. Grauert and R. Remmer [2] and R. Kawai [3] proved that $(X, \mathcal{O})$ is a normal analytic space in the sense of Cartan-Serre. Our aim is to prove this theorem by using the Riemann-Hilbert problem.

Let $\pi : Y \longrightarrow X$ be an analytic cover with critical locus D whose sheet number is q. By the def. of analytic space in the sense of Behnke-Stein, the problem is local, i.e., we can assume X to be a polydisc in $\mathbb{C}^n$ and it is sufficient to show the existence of a holomorphic function f(y) on Y separating any two points in $\pi^{-1}(x_o)$, $x_o \in X - D$.

2. Later on, we suppose that X is a polydisc in $\mathbb{C}^n$. We write $Y^* := Y - \pi^{-1}(D)$ and $X^* := X - D$. A holomorphic function on Y* can be considered as a many-valued function on X*. Using this fact, we obtain the relation between holomorphic functions on Y* and linear representation of $\pi_1(X^*, x_o)$. We state this more detail: let $\pi^{-1}(x_o) = \{y_1, \cdots, y_q\}$ and fix this numbering. Since Y* is a Stein manifold, there exists a holomorphic funciton g(y) on Y* such that $g(y_i) = i$ for $i = 1, \cdots, q$. Choose a sufficiently small polydisc $U \subset X$ centered at x_o and let $g_i(x)$ be the branch of g(y) on U such that $g_i(x_o) = i$. It follows that $g_i(x)$ can be continued analytically on X*, but in general, it is not single-valued. Consider the vector-valued function $\vec{g}(x) = (g_1(x), \cdots, g_q(x))$ on U which can be continued analytically on X* and is many-

valued on X^*. We shall show that $\vec{g}(x)$ gives a linear representation of $\pi_1(X^*, x_o)$; let γ be a closed curve in X^* issuing from x_o. Since $\pi : Y^* \longrightarrow X^*$ is a topological covering, there are the paths γ_i in Y^* starting from y_i such that $\pi(\gamma_i) = \gamma$. Let us denote by $x_{\gamma_*(i)}$ the end point of γ_i; then $\begin{pmatrix} 1, & \cdots, & q \\ \gamma_*(1), & \cdots, & \gamma_*(q) \end{pmatrix}$ is a permutation of q letters $\{1, \cdots, q\}$.
It follows that the result of analytic continuation of $g_i(x)$ along γ is the element $g_{\gamma_*(i)}(x)$. Let S_q be the symmetric group of q letters $\{1, \cdots, q\}$ and let $e_i = (0, \cdots, \overset{i\text{-th place}}{1}, \cdots, 0)$ $(i = 1, \cdots, q)$ be the standard basis of $\mathbb{C}^q$. We denote by $J : S_q \longrightarrow GL(q, \mathbb{C})$ the following standard faithful representation: for $\sigma \in S_q$, $j(\sigma)(\sum_i u_i e_i) = \sum_i u_i e_{\sigma(i)}$.
Let γ be a closed curve in X^* issuing from x_o, and we denote by $\gamma_*(g) = (g_{\gamma_*(1)}, \cdots, g_{\gamma_*(q)})$ the result of analytic continuation of $g = \vec{g}(g_1, \cdots, g_q)$ along γ. It follows that

$$(g_{\gamma_*(1)}, \cdots, g_{\gamma_*(q)}) = (g_1, \cdots, g_q) \; ([\gamma])$$

if we write $\rho([\gamma]) = j\left(\begin{pmatrix} 1, & \cdots, & q \\ \gamma_*(1), & \cdots, & \gamma_*(q) \end{pmatrix}\right)$.

Lemma 1. Let $\rho : \pi_1(X^*, x_o) \longrightarrow GL(q, \mathbb{C})$ be as above. Then ρ is a finite representation of $\pi_1(X^*, x_o)$.

We call ρ the monodromy representation associated with the analytic cover $\pi : Y \longrightarrow X$. Conversely, we consider a many-valued holomorphic function $\vec{h}(x) = (h_1(x), \cdots, h_q(x))$ on X^* satisfying $\gamma_*\vec{h}(x) = \vec{h}(x)\rho([\gamma])$ for any closed curve γ in X^* issuing from x_o. Then we obtain the following:

Lemma 2. Let $\vec{h}(x)$ be as above and suppose that Y^* is connected. Write $h(y) := h_1(\pi(y))$ in a small polydisc in Y^* centered at y_1. Then $h(y)$ can be continued analytically along any path in Y^* starting from y_1; moreover it determines a single-valued holomorphic function $\tilde{h}(y)$ on Y^* whose function element at y_i coincides with $h_i(\pi(y))$ for $i = 1, \cdots, q$.

Suppose that $\tilde{h}(y)$ is locally bounded at every point of $\pi^{-1}(D - \text{sing } D)$. Then from well-known facts about function theory on analytic covers, it follows that $\tilde{h}(y)$ can be extended to the unique holomorphic function on Y.

Summarizing, we obtain the following:

Proposition 1. Let $\pi : Y \longrightarrow X$ be an analytic cover and let $\rho : \pi_1(X, x_o) \longrightarrow GL(q, \mathbb{C})$ be the monodromy representation associated with the analytic cover Y. Suppose that there exists a many-valued holomorphic function $\vec{h}(x) = (h_1(x), \cdots, h_q(y))$ on X^* such that $\gamma_*(\vec{h}) = \vec{h}\rho([\gamma])$ for any $[\gamma] \in \pi_1(X^*, x_o)$ and that $h_1(x_o) \neq h_j(x_o)$ for any $i \neq j$. Let $\tilde{h}(y)$ be the single-valued function on $Y - \pi^{-1}(D)$ defined in Lemma 2. If $\tilde{h}(y)$ is locally bounded at every point of $\pi^{-1}(D - \text{sing } D)$, then $\tilde{h}(y)$ can be extended to the unique holomorphic function on Y which is desired at the end of n° 1.

3. Therefore we must construct the many-valued function whose behaviour is the given one. For this purpose, we solve Riemann-Hilbert problem. Let X be a connected complex manifold and let D be a divisor of X. Let $X^* = X - D$ and $x_o \in X$. We consider a completely integrable total differential eq. on X^*

$$(3.1) \qquad d\begin{pmatrix} y_1 \\ \vdots \\ y_q \end{pmatrix} + \begin{pmatrix} \Omega_{11} & \cdots & \Omega_{1q} \\ & \cdots & \\ \Omega_{q1} & \cdots & \Omega_{qq} \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_q \end{pmatrix} = 0$$

We choose q linearly independent solutions $f_1, \cdots, f_q$ of (3.1) at x_o, and let γ be any closed curve in X* issuing from x_o. We denote by $\gamma_*[f_1, \cdots, f_q]$ the result of analytic continuation of the function element $[f_1, \cdots, f_q]$ along the curve γ. Since $[\gamma_*(f_1), \cdots, \gamma_*(f_q)]$ are also q linearly independent solutions of (3.1), we have

$$\gamma_*[f_1, \cdots, f_q] = [f_1, \cdots, f_q]M(\gamma)$$

for some $M(\gamma) \in GL(q, \mathbb{C})$. It is well known that the mapping $\rho : [\gamma] \in \pi_1(X^*, x_o) \to M(\gamma) \in GL(q, \mathbb{C})$ is a homomorphism and ρ is called the monodromy representation of (3.1) with respect to $[f_1, \cdots, f_q]$.

Conversely, given a linear representation $\rho : \pi_1(X^*, x_o) \to GL(q, \mathbb{C})$. We shall attempt to construct a completely integrable total diff. eq. (3.1) on X* which satisfies the following two conditions:

1) (3.1) is regualr singular along D, moreover there exists a divisor A in X along which (3.1) may have apparent singularity.

2) the monodromy representation of (3.1) with respect to certain independent solutions coincides with the given ρ.

We shall call the Riemann-Hilbert problem the problem of constructing the eq. (3.1) which satisfies the above two conditions.

In order to construct the many-valued function stated in Proposition 1, using the results of P. Deligne [1], we solve Riemann-Hilbert problem in the following situation: let X be a Stein manifold and let D be a divisor of X (not necessarily normal crossing). In solving the problem, we use essentially the extension theorem of coherent analytic sheaves of J.-P. Serre [7] and Y.-T. Siu [8].

Our main results is the following:

Theorem 1. Let X and D be as above. Suppose that a linear representation $\rho : \pi_1(X^*, x_o) \longrightarrow GL(q, \mathbb{C})$ is given, where $X^* = X - D$. Then we can construct a total diff. eq. (3.1) as follows:

1) there exists a divisor A of X such that A does not contain any irreducible component of D.

2) the eq. (3.1) is completely integrable on $X - A \cup D$; moreover A is the apparent singularity of (3.1).

3) the monodromy representation of (3.1) coincides with the given ρ.

Using the Oka principle and results of F. Peterson [5] and J.-P. Serre [7], we can study more detail the case of $\dim X = 2$ than that of $\dim X \geq 3$.

Theorem 2. Let X be a connected Stein manifold of $\dim X = 2$. If $H^2(X, Z) = 0$, then for any divisor D and representation $\rho : \pi_1(X - D, x_o) \longrightarrow GL(q, \mathbb{C})$, we can always find a solution of the Riemann-Hilbert problem without apparent singularity.

Remark. In the case of Theorem 2, let $\Omega = (\Omega_{ij})$ be the connection matrix of the eq. (3.1). From the construction of the eq. (3.1), we see that each Ω_{ij} is a meromorphic form with <u>generically logarithmic poles</u> along D. This notion was introduced by K. Saito, [6].

4. Let $\pi : Y \longrightarrow X$ be an analytic cover where X is a polydisc in $\mathbb{C}^n$, and let q be the sheet number of Y. <u>We shall solve the problem proposed at the end of n^o 1</u>. Since the problem is local, we can suppose that the critical locus D of Y has <u>finite irreducible components</u>: $D = \bigcup_{i=1}^{m} D_i$ and that $Y - \pi^{-1}(D)$ is connected by 4) of the def. of analytic cover (See n^o 1). Let ρ be the monodromy representation associated with Y. Since X is a Stein manifold, there exists, by Theorem 1, total diff. eq. (3.1) as follows:

1) there exists a divisor A of X such that $x_o \notin A$, $D_i \not\subset A$ and (3.1) is regular singular along $A \cup D$; moreover A is the apparent singularity of (3.1).

2) If we choose q linearly independent solutions $f_1, \cdots, f_q$ of (3.1) at x_o properly, we have

$$\gamma_*[f_1, \cdots, f_q] = [f_1, \cdots, f_q]\rho([\gamma])$$

for any closed curve γ in X - D issuing from x_o. Put $f_i(x) = {}^t(f_{1i}(x), \cdots, f_{qi}(x))$, and we define $g_j(x) = (f_{j1}(x), \cdots, f_{jq}(x))$; thus we have

$$\gamma_*(g_j) = g_j\rho([\gamma]) \quad \text{for any } [\gamma] \in \pi_1(X - D, x_o).$$

Since $f_1, \cdots, f_q$ are linearly independent solutions of (3.1) at x_o, there are constants $c_i \in \mathbb{C}$ $(i = 1, \cdots, q)$ such that, putting $\vec{h} = \sum_{i=1}^{q} c_i g_i$, we have $\vec{h}(x_o) = (1, \cdots, q)$ and $\gamma_*(\vec{h}) = \vec{h}\rho([\gamma])$ for any $[\gamma] \in \pi_1(X - D, x_o)$. By lemma 2, there exists a holomorphic function $\tilde{h}(y)$ on Y^* such that $\tilde{h}(y_i) = i$ for $i = 1, \cdots, q$. Since the eq. (3.1) is regular singular along $A \cup D$ and since $\pi : Y - \pi^{-1}(\text{Sing } D) \longrightarrow X - \text{sing } D$ is a finite holomorphic map between complex manifolds, $\tilde{h}(y)$ has at most pole along $[Y - \pi^{-1}(\text{Sing } D)] \cap \pi^{-1}(A \cup D)$. Since the problem is local, by shrinking X slightly if necessary, we can suppose that the number of irreducible components of A is finite: $A = \bigcup_{j=1}^{\ell} A_j$. Since the Cousin's second problem has always a solution on X, we can write A_j and D_i in the form $A_j = \{a_j(x) = 0\}$ and $D_i = \{d_i(x) = 0\}$ for any i and j where $a_j, d_i \in \Gamma(X, \mathcal{O}_X)$. Since $\tilde{h}(y)$ has at most pole along $[Y - \pi^{-1}(\text{Sing } D)] \cap \pi^{-1}(A \cup D)$, there are positive integers μ_j and ν_i such that, putting $c(x) = \prod_{j=1}^{\ell} a_j(x)^{\mu_j} \prod_{i=1}^{m} d_i(x)^{\nu_i}$, $c(\pi(y))\tilde{h}(y)$ is holomorphic on $Y - \pi^{-1}(\text{Sing } D)$; hence by proposition 1 $c(\pi(y))\tilde{h}(y)$ can be extended to the unique holomorphic function $H(y)$ on Y. Since $c(x_o) \neq 0$, we have $H(y_i) \neq H(y_j)$ for any $i \neq j$. This is the function which we want to construct.

Summarizing, we obtain the following:

Theorem 3. Let $\pi : Y \longrightarrow X$ be an analytic cover whose critical locus is D, where X is a polydisc in $\mathbb{C}^n$. Let $x_o \in X - D$ and suppose that $\rho : \pi_1(X - D, x_o) \longrightarrow GL(q, \mathbb{C})$ is the monodromy representation associated with the analytic cover Y. Then, using a solution of the Riemann-Hilbert problem for the representation ρ, by shrinking X slightly if necessary, we can construct a holomorphic function on Y which separates any two points in $\pi^{-1}(x_o)$.

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