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Proper maps modulo CE-maps between ANR's

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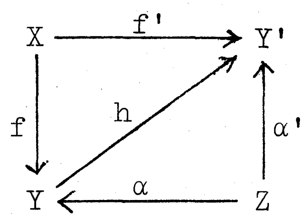
1. Introduction and notations. In 1970, L.C.Siebenmann extended the simple homotopy theory for locally finite simplicial complexes. On the other hand, T.A.Chapman defined simple homotopy type for compact ANR in 1977. We can analogously extend the Siebenman's theory to locally compact separable metric ANR.

All spaces here are locally compact separable metric ANR's, and maps are proper maps if otherewise are not stated. Hence homotopic, homotopy equivalence etc. are all in the category of proper maps. Subspaces are closed subspaces. Hence inclusions are in fact (proper) maps. If f and g are homotopic, then we denote it by $f \simeq g$. A CE-map $f: X \longrightarrow Y$ is a map f such that $f^{-1}(y)$ has trivial shape for each y in Y. Note that a CE-map is onto and the composition of two CE-maps is also a CE-map. Hilbert cube is denoted by Q, $Q = \prod_{i=1}^{\infty} I_i$, where $I_i = [0, 1]$.

2. Semi-group P(X). Let $f: X \longrightarrow Y$ and $f': X \longrightarrow Y'$ be maps. We identify f and f' if there is a homeomorphism $h: Y \longrightarrow Y'$ such that $f' = h \circ f$. f and f' are equivalent modulo CE-maps if there are CE-maps $\alpha: Z \longrightarrow Y$ and $\alpha': Z \longrightarrow Y'$ for some space Z and there is a map $h: Y \longrightarrow Y'$ such that $h \circ f \simeq f'$ and $h \circ \alpha \simeq \alpha'$.



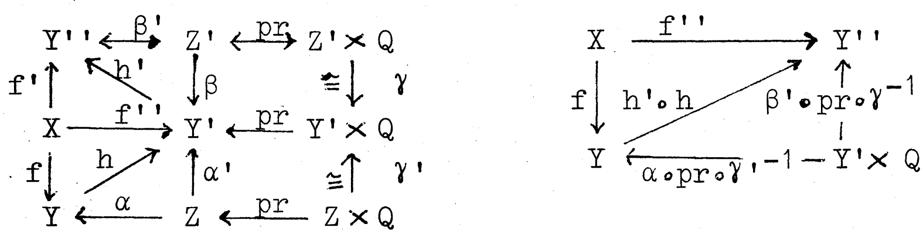
In this cace, h is necessarily a homotopy equivalence. We can see, by using the following Edwards and Chapman's theorems, that this relation is an equivalence relation.

Theorem(R.D.Edwards). A locally compact separable metric ANR is a Q-manifold factor.

X is a Q-manifold factor if $X \times Q$ is a Q-manifold, where Q-manifold is a separable metric manifold modeled on the Hilbert cube Q.

Theorem(T.A.Chapman). Let $f: M \to N$ be a CE-map between Q-manifolds. Then f is a near homeomorphism. Hence in particularly, f is homotopic to a homeomorphism.

The following diagram show that the relation modulo CE-maps is transitive;



where γ, γ' are homeomorphism homotopic to $\beta \times \mathrm{id}$, $\alpha' \times \mathrm{id}$ resp.

For any map $f: X \to Y$, let $\langle f \rangle$ be the equivalence class of f modulo CE-maps.

(i) Let f and f' be maps from X to Y. If $f \simeq f'$, then $\langle f \rangle = \langle f' \rangle$. The converse is not true.

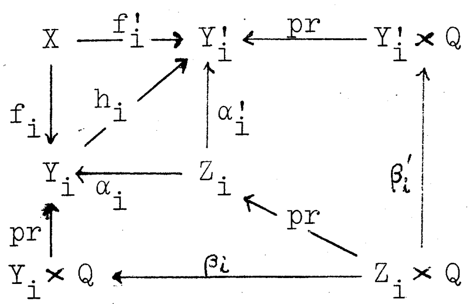
Let P(X) be the set of the equivalence classes of maps from X. For any $f: X \to Y$, let $f': X \to Y \times Q \times I$ be a map defined by $f'(x) = (f(x), \alpha(x), 0)$, where $\alpha: X \to Q$ is any embedding (not necessary proper). Then f' is a Z-embedding equivalent to f.

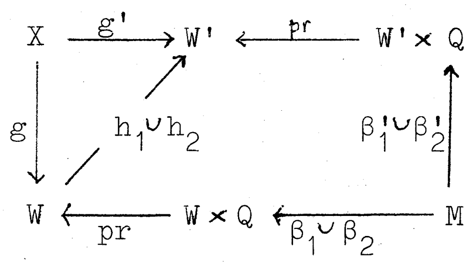
(ii) For any $\langle f \rangle$, there is a Z-embedding f' in $\langle f \rangle$.

Addition in P(X). If X is a Z-set in a space Y, then $X \times Q$ is also a Z-set in Q-manifold $Y \times Q$.

Theorem(Anderson-Chapman). Let f_0, f_1: $A \to M$ be a proper homotopic Z-embeddings from a locally compact separable metric space A to a Q-manifold M. Then there is an isotopy h_t: $M \to M$ such that $h_0 = \mathrm{id}$ and $h_1 \circ f_0 = f_1$.

We use this theorem to define an addition in P(X) as follows. Let f_i: $X \to Y_i$ be Z-embeddings, i = 1, 2. Then the quotient space $W = Y_1 \cup_X Y_2$, identifying $f_1(X) = X = f_2(X)$, is an ANR, and the inclusion map g: $X \to W$ is proper. Define $\langle f_1 \rangle + \langle f_2 \rangle = \langle g \rangle$. This addition is well defined(see the diagrams below).



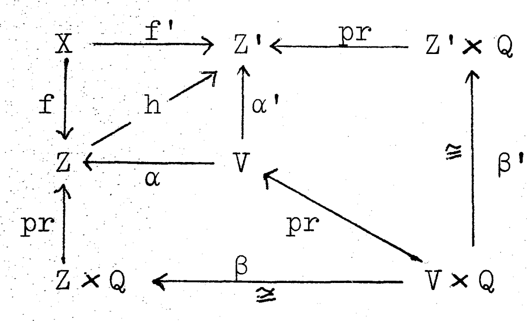


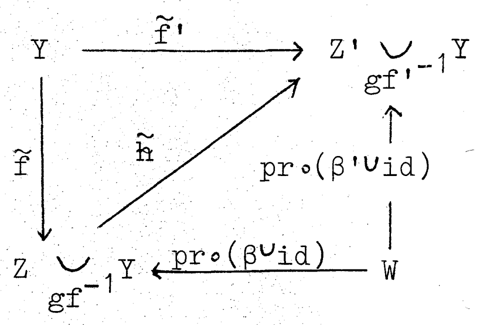
where $\beta_i' \circ \beta_i^{-1} = \mathrm{id}$ on $f_i(X) \times Q = X \times Q = f_i'(X) \times Q$, β_i and β_i' are homeomorphisms, $M = (Z_1 \times Q) \cup (Z_2 \times Q)$ identifying $\beta_1^{-1}(X \times Q)$ and $\beta_2^{-1}(X \times Q)$.

(iii) P(X) is an abelian semi-group with unit.

Homomorphism. Let f: $X \to Z$ be a Z-embedding and let g: $X \to Y$ be a map. The quotient space $Z \cup_{gf^{-1}} Y$ containes Y as a closed subspace. Define $g_*\langle f \rangle = \langle \tilde{f} \rangle$, where $\tilde{f}$: $Y \to Z \cup_{gf^{-1}} Y$ is the inclusion map. Then g_* is well defined. In the following diagrams, $\beta' \circ \beta^{-1} = \mathrm{id}$ on $X \times Q = f(X) \times Q$, $h = \mathrm{id}$ on $f(X)$ (change h if necessary by using proper homotopy extention theorem) and W is a quotient space of $V \times Q \cup Y \times Q$ identifying

$\beta^{-1}(x, q)$ to $(g(x), q)$ for x in X.





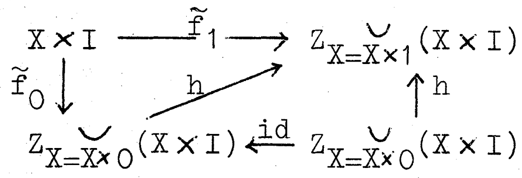
The following proposition is easy to prove;

(iv) $g_*\langle id_X\rangle = \langle id_Y\rangle$, $(id_X)_* = id_{P(X)}$, $(g\circ f)_* = g_*f_*$,

$g_*(\langle f_1\rangle + \langle f_2\rangle) = g_*\langle f_1\rangle + g_*\langle f_2\rangle$,

$g_0 \simeq g_1$ implies $g_{0*} = g_{1*}$.

The last statement is proved as follows. Let $k_i\colon X \to X\times I$ be the maps defined by $k_i(x) = (x, i)$ for all x in X, $i = 0, 1$, and let $f\colon X \to Z$ be a Z-embedding. In the diagram, h is defined by $h(z) = z$ for z in Z and $h(x, t) = (x, 1-t)$ for (x, t) in $X\times I$.



Then the diagram shows that $(k_0)_*\langle f\rangle = \langle \tilde{f}_0\rangle = \langle \tilde{f}_1\rangle = (k_1)_*\langle f\rangle$.

From the proposition (iv), we obtain;

(v) If g is a homotopy equivalence, then g_* is an isomorphism between semi-groups.

3. <u>Group PE(X)</u>. We will show that the subset $\{\langle f\rangle$; f is a homotopy equivalence$\}$ of $P(X)$ is a set of all invertible elements and hence is an abelian group.

Teorem 1. If $f: X \to Y$ has a homotopy right inverse, then for any map $g: Y \to Z$, $f_*\langle g \circ f \rangle = f_*\langle f \rangle + \langle g \rangle$.

Proof. We may assume that f and g are Z-embeddings (replace Y and Z by $Y \times Q$ and $Z \times Q$ if necessary). Let Z' and Y' be copies of Z and Y respectively and let $k: Z \to Z'$ and $l: Y \to Y'$ be homeomorphisms. Let W be the quotient space of $Z' \cup Y'$ identifying $k(x) = l(x)$ for all x in X. Then the composition $i: Y \xrightarrow{l} Y' \hookrightarrow W$ represents $f_*\langle g \circ f \rangle$ and $j: Y \hookrightarrow Z \xrightarrow{k} Z' \hookrightarrow W$ represents $f_*\langle f \rangle + \langle g \rangle$. Since Y deforms onto a subset of X in Y, i and j are homotopic. This proves that $f_*\langle g \circ f \rangle = \langle i \rangle = \langle j \rangle = f_*\langle f \rangle + \langle g \rangle$.

Note that the map dose not necessary be a homotopy equivalence. But the condition for f to have a homotopy right inverse cannot be dropped.

If $f: X \to Y$ is a homotopy equivalence with a homotopy inverse g, then $\langle f \rangle + g_*\langle g \rangle = \langle id_X \rangle$. Hence we have;

Theorem 2. $PE(X) = \{ \langle f \rangle \in P(X)$; f is a homotopy equivalence$\}$ is an abelian group.

The following proposition is clear;

(vi) PE(X) is the maximal group contained in P(X). If $f: X \to Y$, then f_* maps PE(X) into PE(Y). Furthermore if f is a homotopy equivalence, f_* restrict to PE(X) is an isomorphism from PE(X) onto PE(Y).

The following theorem is clear from definitions;

Theorem 3. Let $Y = Y_1 \cup Y_2$, $X = Y_1 \cap Y_2$ and let $j_i: X \to Y_i$, $j: Y_2 \to Y$ be the inclusions. Then $j_{2*}\langle j_1 \rangle = \langle j \rangle$.

4. Product and sum theorems in P(X).

Theorem 4. Let X be a subset of $Y = Y_1 \cup Y_2$ and let $Y_0 = Y_1 \cap Y_2$, $X_i = X \cap Y_i$, $i = 0, 1, 2$ (recall that subsets are always closed). Let $f: X \to Y$ and $f_i: X_i \to Y_i$, $i = 0, 1, 2$ be the inclusion maps. If the inclusion maps $X_2 \to X_2 \cup Y_0$ and $X \to X \cup Y_1$ have homotopy right inverses, then

$$f_*\langle f \rangle = j'_{1*}\langle f_1 \rangle + j'_{2*}\langle f_2 \rangle - j'_{0*}\langle f_0 \rangle.$$

Furthermore, if f is a homotopy equivalence, then

$$\langle f \rangle = j_{1*}\langle f_1 \rangle + j_{2*}\langle f_2 \rangle - j_{0*}\langle f_0 \rangle,$$

where j_i, j'_i are appropriate inclusions.

Proof. We may assume that f and f_i are Z-embeddings. For any inclusion $i: B \to A$, we write $i = (A, B)$, $\langle i \rangle = \langle A, B \rangle$ and $i_* = (A, B)_*$. Then

$$\begin{aligned}
f_*\langle f \rangle &= (Y, X)_*\langle Y, X \rangle \\
&= (Y, X \cup Y_1)_*(X \cup Y_1, X)_*\langle (Y, X \cup Y_1) \circ (X \cup Y_1, X) \rangle \\
&= (Y, X \cup Y_1)_*\{\langle Y, X \cup Y_1 \rangle + (X \cup Y_1, X)_*\langle X \cup Y_1, X \rangle\} \\
&= (Y, X \cup Y_1)_*(X \cup Y_1, X_2 \cup Y_0)_*\langle Y_2, X_2 \cup Y_0 \rangle \\
&\quad + (Y, X \cup Y_1)_*(X \cup Y_1, X)_*(X, X_1)_*\langle Y_1, X_1 \rangle \\
&= (Y, X_2 \cup Y_0)_*\langle Y_2, X_2 \cup Y_0 \rangle + j'_{1*}\langle f_1 \rangle.
\end{aligned}$$

$$\begin{aligned}
(X_2 \cup Y_0, X_2)_*\langle Y_2, X_2 \rangle &= (X_2 \cup Y_0, X_2)_*\langle (Y_2, X_2 \cup Y_0)(X_2 \cup Y_0, X_2) \rangle \\
&= \langle Y_2, X_2 \cup Y_0 \rangle + (X_2 \cup Y_0, X_2)_*\langle X_2 \cup Y_0, X_2 \rangle \\
&= \langle Y_2, X_2 \cup Y_0 \rangle + (X_2 \cup Y_0, X_2)_*(X_2, X_0)_*\langle Y_0, X_0 \rangle
\end{aligned}$$

$$= \langle Y_2, X_2 \cup Y_0 \rangle + (X_2 \cup Y_0, X_0)_* \langle Y_0, X_0 \rangle.$$

Hence we have

$$f_* \langle f \rangle = j'_{1*} \langle f_1 \rangle + (Y, X_2 \cup Y_0)_* (X_2 \cup Y_0, X_2)_* \langle f_2 \rangle$$
$$- (Y, X_2 \cup Y_0)_* (X_2 \cup Y_0, X_0)_* \langle f_0 \rangle$$
$$= j'_{1*} \langle f_1 \rangle + j'_{2*} \langle f_2 \rangle - j'_{0*} \langle f_0 \rangle.$$

If f has a homotopy left inverse g, then

$$\langle f \rangle = g_* f_* \langle f \rangle$$
$$= g_* j'_{1*} \langle f_1 \rangle + g_* j'_{2*} \langle f_2 \rangle - g_* j'_{0*} \langle f_0 \rangle$$
$$= j_{1*} \langle f_1 \rangle + j_{2*} \langle f_2 \rangle - j_{0*} \langle f_0 \rangle.$$

Theorem. If the inclusion map $f: X \longrightarrow Y$ is a homotopy equvalence, then for any finite connected complex K,

$$\langle id_K \times f \rangle = \chi(K) j_* \langle f \rangle,$$

where $j: X \longrightarrow K \times X$ be defined by $j(x) = (k_0, x)$ for a fixed point k_0 in K.

Proof. The proof is a double induction about dim K and number of simplexes of K.

Let s be a maximal dimensional simplex of K such that the complex $L = K - \{s\}$ is connected (if such a simplex dose not exist, then K collapse to a subcomplex with lower number of simplexes, and this case is easy to prove). We may assume that k_0 is in the boundary $\dot{s}$ of s. Then

$$\langle id_K \times f \rangle = j_{1*} \langle id_L \times f \rangle + j_{2*} \langle id_s \times f \rangle - j_{0*} \langle id_{\dot{s}} \times f \rangle$$
$$= \chi(L) j_* \langle f \rangle + \chi(s) j_* \langle f \rangle - \chi(\dot{s}) j_* \langle f \rangle$$
$$= \chi(K) j_* \langle f \rangle.$$

By the Q-manifold factor theorem of Edwards and the above theorem, we have

Corollary. If $f: X \to Y$ is a homotopy equivalence and Z is compact ANR, then $\langle id_Z \times f \rangle = \chi(Z) j_* \langle f \rangle$.

Corollary. Let f, X, Y and Z be as above. Then if $\chi(Z) = 0$, $X \times Z \times Q$ is homeomorphic to $Y \times Z \times Q$.

Note that Chapman proved that if X and Y are (not proper) homotopy equivalent, then $X \times [0, 1) \times Q$ is homeomorphic to $Y \times [0, 1) \times Q$.