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SIMPLE BUCKLINGS — A GROUP THEORETICAL INTRODUCTION

Hiroshi FUJII
Inst. of Computer Sciences
Kyoto Sangyo University

Masaya YAMAGUTI
Department of Mathematics
Kyoto University

Abstract : In this note, we discuss the structure of singularities in non-linear elasticity theory in the light of the symmetry group G and the *class* (L or N) of the problem. The emphasis is on the discussion of *structural* stability of simple bifurcation points with respect to small changes of the equation. Sec.1 is of introductory nature, where we give formal classification of simple critical points. In Sec.2, we study the structure of those singularities.

1.1 CLASSIFICATION OF SIMPLE CRITICAL POINTS

Let V be a real Hilbert space with inner product $<\ ,\ >$ and norm $\| \cdot \|_V$. We consider the equation:

$$\text{(P)} \qquad F(\mu,w) = 0 \qquad (1.1)$$

where F is a continuous mapping $\mathbb{R}^1 \times V \to V$.

Envisaging applications to numerical analysis, our object is, for a known solution $0 \equiv (\mu_0, w_0) \in \mathbb{R}^1 \times V$, to obtain all the *paths* in $\mathbb{R}^1 \times V$ which contain 0. By a *path*, we mean a connected component of S, or its subcomponent where S denotes the closure of the solution of (P) in $\mathbb{R}^1 \times V$.

Notice that in Eq. (1.1), $F(\mu,0) = 0$ $(\forall \mu \in \mathbb{R}^1)$ is *not* assumed, implying that the problem (P) may not have a trivial path $(\mu,0) \in \mathbb{R}^1 \times V$.

We assume to F the following:

+ This note is a short version of Chapter I of Fujii and Yamaguti [2].

(A)$_1$ F : $\mathbb{R}^1 \times V \to V$ of class C^p, $p \geq 3$,

(A)$_2$ F : Fredholm mapping of index 0, namely,
$\dim \ker F'(\mu,w) = \dim \operatorname{coker} F'(\mu,w) = d < +\infty$.

(A)$_3$ $F'(\mu,w) \in B(V)$ *) is self-adjoint.

Here, $F'(\mu,w)$ denotes the Fréchet derivative of F with respect to w at (μ,w):

$$F'(\mu,w) \overset{def.}{=} \frac{\partial F}{\partial w}(\mu,w) \tag{1.2}$$

We shall also denote by $\dot{F}(\mu,w)$ the Fréchet derivative of F with respect to μ at (μ,w):

$$\dot{F}(\mu,w) \overset{def.}{=} \frac{\partial F}{\partial \mu}(\mu,w). \tag{1.3}$$

Higher order derivatives will be also denoted by, for example, $F''(\mu,w)$, $F'''(\mu,w)$ and so on.

It in noted that every result in this section is applicable to non-self-adjoint cases (with obvious modifications). The assumption (A)$_3$ may characterize the nonlinear elasticity theory, and since our main object is the application to nonlinear elasticity in numerical analysis aspects, we assume (A)$_3$ in the whole of subsequent discussions

Definition 1.1 Let $0 \equiv (\mu,w) \in \mathbb{R}^1 \times V$ be a solution of $F(\mu,w) = 0$. Then, 0 is called an *ordinary* (regular) point of (P), if $F'(\mu,w)$ has a bounded inverse, i.e., $F'(\mu,w)^{-1} \in B(V)$, and a *critical* (singular) point if not.

The following lemma is an immediate consequence of the implicit function theorem (see, e.g., Nirenberg [9]).

Lemma 1.2 Suppose $0 \equiv (\mu_0,w_0) \in \mathbb{R}^1 \times V$ is an ordinary point of (P). Then, there exist an interval $I_\delta = \{\mu; |\mu-\mu_0| < \delta\}$ and a unique C^p function $w(\mu): I_\delta \to V$ such that $F(\mu,w(\mu)) = 0$, $(\mu \in I_\delta)$.

Suppose now $C \equiv (\mu_c,w_c) \in \mathbb{R}^1 \times V$ is critical. We consider the problem in the particular case that the kernel and the cokernel are one dimensional (which we shall call the *simple* case). Denote by F_c', $\dot{F}_c$, ... the Fréchet derivatives of F at C.

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*) $B(X,Y)$ denotes the set of bounded linear maps $X \to Y$. $B(X) \overset{def.}{=} B(X,X)$.

Let $L_c \equiv F'_c$. Let $\{\phi_c\} = \ker L_c$, and denote by Π'_c the functional $\Pi'_c u = \langle u,\phi_c\rangle$, $u \in V$. Let R_c = range $L_c = \{\ker L_c\}^\perp$ and denote by ω_c the orthogonal projection $V \to R_c$. We let denote by $L_c^\dagger$ the bounded map $L_c^\dagger : V \to V$ such that $L_c^\dagger \cdot L_c = \omega_c$.*)

Let:

$$\left.\begin{aligned} A_c &\equiv \Pi'_c\, F''_c\, (\phi_c, \phi_c), \\ B_c &\equiv \Pi'_c\, F''_c\, (\phi_c, g_c) + \Pi'_c\, \dot{F}'_c\, \phi_c \\ C_c &\equiv \Pi'_c\, F''_c\, (g_c, g_c) + 2\Pi'_c\, \dot{F}'\, g_c + \Pi'_c\, \ddot{F}_c \\ D_c &\equiv \Pi'_c\, F'''_c (\phi_c, \phi_c, \phi_c) - 3\Pi'_c\, F''_c\, (\phi_c, L_c^\dagger\, \omega_c\, F''_c(\phi_c, \phi_c)), \\ f_c &\equiv \dot{F}_c, \end{aligned}\right\} \quad (1.4)$$

where

$$g_c \equiv -L_c^\dagger\, \omega_c\, f_c.$$

Definition 1.3 A simple, critical point $C \equiv (\mu_c, w_c) \in \mathbb{R}^1\times V$ is called a *snap* point if $\Pi'_c f_c \neq 0$. Moreover, if $A_c \neq 0$, C is a *non-degenerate* snap point.

Note 1.4 *A snap point* (a snapping point, a snap-through point) may also be called as *a limit point* (a limiting point) or *a turning point*. See, e.g., [6], [7], [19] and [20].

Definition 1.5 A simple critical point $C \equiv (\mu_c, w_c) \in \mathbb{R}^1\times V$ is called a *non-degenerate point of bifurcation* if $\Pi' f_c = 0$ and $B_c^2 - A_c C_c > 0$. Moreover, if $A_c \neq 0$, C is called a non-degenerate, *asymmetric* point of bifurcation, and if $A_c = 0$, $D_c \neq 0$, a non-degenerate *symmetric* point of bifurcation.

Note 1.6 The term "symmetric or asymmetric point of bifurcation" often appears in engineering literatures, e.g., [19]. However, as we shall introduce the concept of *group symmetry* to nonlinear singularities, we prefer to call the symmetric and asymmetric points of bifurcations as the *fold* and *cusp* bifurcations, respectively, to avoid possible confusions in terminology. Our terminology corresponds to the first two elementary catastrophes in the theory of universal unfoldings of singularities due to R. Thom [18].

*) $L_c^\dagger = (L_c|R_c)^{-1}\omega_c$

Classification of Simple Critical Points

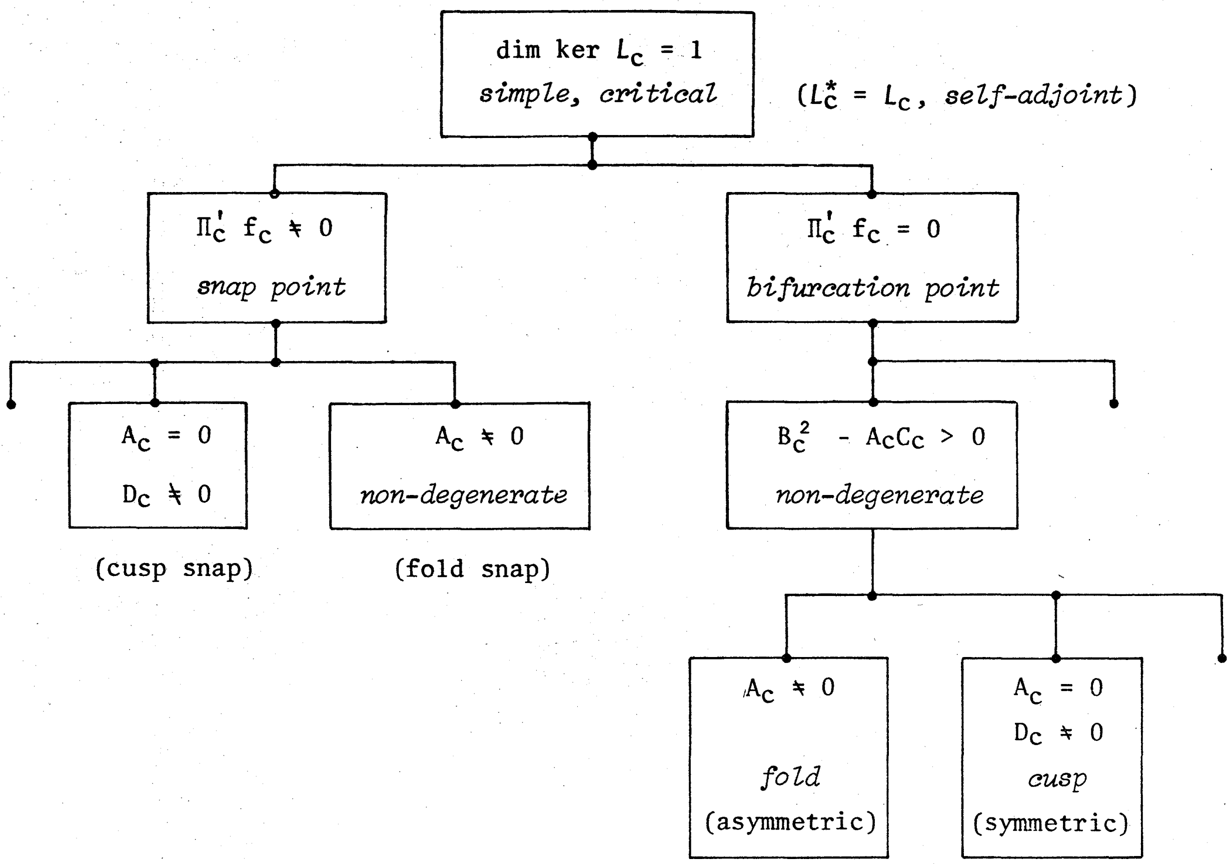

Fig. 1.1

We shall see, however, that the appearance of symmetric or asymmetric points of bifurcation has a crucial relation with the existence or non-existence of symmetry groups.

Remark 1.7 Suppose (P) has a trivial path $(\mu,0) \in \mathbb{R}^1 \times V$. Then, a simple critical point can never be a snap point, since $f_c = \frac{\partial F}{\partial \mu}(\mu,0) \equiv 0$ for all $\mu \in \mathbb{R}^1$.

1.2 BEHAVIORS OF SOLUTIONS IN A NEIGHBORHOOD OF SIMPLE CRITICAL POINTS

We now summarize results on local behaviors of solutions of (P) in the vicinity of simple critical points. The knowledge about the critical eigenvalues on the paths will be indispensable in the discussion of numerical solutions about those critical points. Hence, we state the lemmas as well as brief proofs of them.

Firstly:

Proposition 1.8 *(Snap point)*

Suppose $C \equiv (\mu_c, w_c) \in \mathbb{R}^1 \times V$ is a simple, non-degenerate snap point of (P). Then,

(i) in a neighborhood of C, there is a unique path, say α-path, which meets C. In other words, there exist an interval $I_\delta = \{\alpha; |\alpha|<\delta\} \subset \mathbb{R}^1$ (δ: sufficiently small), and two C^p functions $\mu(\alpha)$: $I_\delta \to \mathbb{R}^1$ and $w(\alpha)$: $I_\delta \to V$, such that

$$F\ (\mu(\alpha), w(\alpha)) = 0,$$

and $\quad \mu(0) = \mu_c,\ w(0) = w_c.$

(ii) For $\alpha \in I_\delta$, $\mu(\alpha)$ and $w(\alpha)$ satisfy

$$|\mu(\alpha) - \mu(0)| \le C\,\alpha^2, \tag{1.5}$$

and

$$\|w(\alpha) - w_c\|_V \le C'\,\alpha.^{*)} \tag{1.6}$$

In fact, they take the form

$$\mu(\alpha) = \mu_c + \frac{A_c}{2\Pi_c' f_c}\ \alpha^2 + O(\alpha^3) \tag{1.7}$$

and

$$w(\alpha) = w_c + \alpha \cdot \phi_c + \left[\frac{A_c}{2\Pi_c' f_c}\ L_c^{\dagger}\ \omega_c\ f_c\right] \alpha^2 + O(\alpha^3) \tag{1.8}$$

(iii) Furthermore, the linearized eigenproblem on the α-path:

$$(E)_\alpha \qquad F'\ (\mu(\alpha), w(\alpha)) \cdot \phi(\alpha) = \zeta(\alpha) \cdot \phi(\alpha),\ \alpha \in I_\delta \tag{1.9}$$

has a pair of C^{p-1} functions $\zeta_c(\alpha)$: $I_\delta \to \mathbb{R}^1$ and $\phi_c(\alpha)$: $I_\delta \to V$ such that

$$\left.\begin{aligned} &\zeta_c(0) = 0,\ \frac{d\zeta_c}{d\alpha}(0) \ne 0 \\ \text{and}\quad &\phi_c(0) = \phi_c. \end{aligned}\right\} \tag{1.10}$$

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*) Here and in the sequel, C, C' or C" denotes a positive generic constant, which may take different values when it appears in different places.

Remark 1.8' The (iii) of the above proposition means that an eigenvalue changes its sign when it crosses a non-degenerate snap point of (P). See, Fig. 1.2.

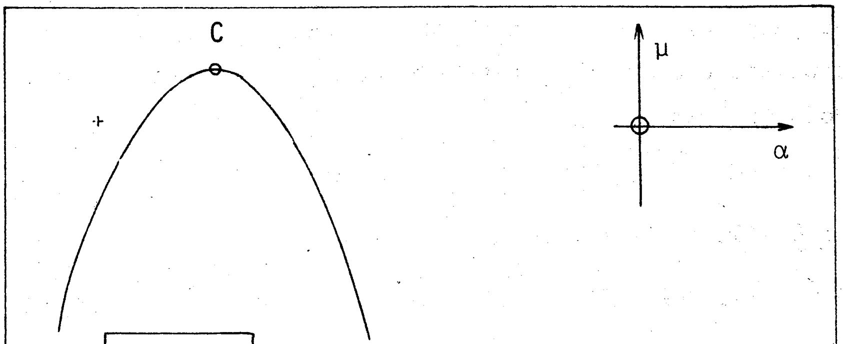


Fig. 1.2 *"A non-degenerate snap point C"*

We turn to the bifurcation cases.

Proposition 1.9 *(Fold bifurcation)*

Suppose $C \equiv (\mu_c, w_c) \in \mathbb{R}^1 \times V$ is a fold bifurcation of (P). Then, (i) there exist two paths, μ_+ and μ_- paths, in a neighborhood of C, which intersect at C. In other words, there is an interval $I_\delta = \{\nu; |\nu|<\delta\} \subset \mathbb{R}^1$ ($^\exists\delta$: sufficiently small), and two C^{p-2} functions $w_\pm(\nu)$: $I_\delta \to V$ such that

$$F\ (\mu_c+\nu,\ w_\pm(\nu)) = 0,\ \nu \in I_\delta$$

and $w_+(0) = w_-(0) = w_c$.

(ii) For $\nu \in I_\delta$, $w_\pm(\nu)$ are such that

$$\|w_\pm(\nu) - w_c\|_V \le C\cdot|\nu|. \tag{1.11}$$

In fact, they have the form

$$w_\pm(\nu) = w_c - \nu\ L_{c}^{\dagger}\omega_c f_c + \alpha_\pm(\nu)\phi_c + O(\nu^2) \tag{1.12}$$

where $\alpha_\pm(\nu)$ are C^{p-2} functions $I_\delta \to \mathbb{R}^1$ such that

$$\alpha_\pm(\nu) = \frac{-B_c \pm \sqrt{B_c^2 - A_c C_c}}{A_c}\,\nu + O(\nu^2). \tag{1.13}$$

(iii) Furthermore, each of the linearized operators on the μ_+ and μ_- paths, $L_\pm(\nu) \equiv F'\ (\mu_c+\nu, w_\pm(\nu))$, $\nu \in I_\delta$, has critical pairs of C^{p-2} functions $(\zeta_c^+(\nu), \phi_c^+(\nu))$: $I_\delta \to \mathbb{R}^1 \times V$ and $(\zeta_c^-(\nu), \phi_c^-(\nu))$: $I_\delta \to \mathbb{R}^1 \times V$, respectively, such that

$$\zeta_c^+(0) = \zeta_c^-(0) = 0 \text{ and } \phi_c^+(0) = \phi_c^-(0) = \phi_c,$$

and $$\frac{d\zeta_c^+}{d\nu}(0) \cdot \frac{d\zeta_c^-}{d\nu}(0) < 0. \qquad (1.14)$$

Remark 1.9' Assertion (iii) implies that at any point of fold bifurcations, the stability is exchanged from one path to the other path. This is an example of the famous *exchange of stability* of Poincaré. A fold bifurcation may be called a *transcritical* bifurcation by this reason.

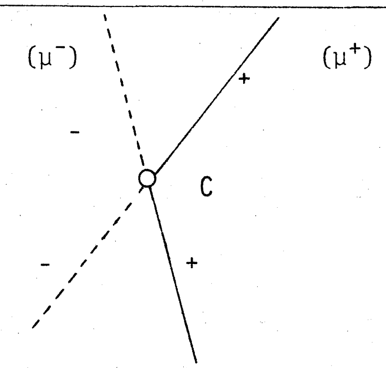


Fig. 1.3 "fold bifurcation and the exchange of stability"

With regards to the cusp bifurcation, we have the following

Proposition 1.10 *(Cusp bifurcation)*

Suppose $C \equiv (\mu_c, w_c) \in \mathbb{R}^1 \times V$ is a cusp bifurcation point of (P). Then, (i) these exist two paths, μ- and α-paths, in a neighborhood of C, which intersect at C. The μ-path is parametrized by $\nu \in I_\delta = \{\nu; |\nu| < \delta\} \subset \mathbb{R}^1$, and is expressed as $(\mu_c + \nu, w^\mu(\nu)) \in \mathbb{R}^1 \times V$, $\nu \in I_\delta$, while the α-path is parametrized by $\alpha \in J_{\delta'} = \{\alpha; |\alpha| < \delta'\} \subset \mathbb{R}^1$, being expressed as $(\mu_c + \nu(\alpha), w^\alpha(\alpha)) \in \mathbb{R}^1 \times V$, $\alpha \in J_{\delta'}$. The functions $w^\mu(\nu)$, $\nu(\alpha)$, $w^\alpha(\alpha)$ are all of C^{p-2} class, and satisfy the relations $w^\mu(0) = w^\alpha(0) = w_c$ and $\nu(0) = 0$.

(ii) $w^\mu(\nu)$ is such that for $\nu \in I_\delta$

$$\|w^\mu(\nu) - w_c\|_V \leq C\,|\nu| \qquad (1.15)$$

and has the form

$$w^{\mu}(\nu) = w_c - \nu\ [L_c^{\dagger}\omega_c f_c + \frac{C_c}{2B_c}\phi_c] + O(\nu^2) \tag{1.16}$$

On the otherhand, $\nu(\alpha)$ and $w^{\alpha}(\alpha)$ satisfy for $\alpha \in J_{\delta'}$,

$$|\nu(\alpha)| \leq C\ \alpha^2 \tag{1.17}$$

$$\|w^{\alpha}(\alpha) - w_c\|_V \leq C\ |\alpha| \tag{1.18}$$

and take the forms:

$$\nu(\alpha) = -\frac{D_c}{6B_c}\alpha^2 + O(\alpha^3) \tag{1.19}$$

and

$$w^{\alpha}(\alpha) = w_c + \alpha\phi_c + \alpha^2\ [\frac{D_c}{6B_c} L_c^{\dagger}\omega_c f_c - \frac{1}{2} L_c^{\dagger}\omega_c F_c''(\phi_c,\phi_c)\ ] + O(\alpha^3) \tag{1.20}$$

(iii) Furthermore, let $L^{\mu}(\nu) \equiv F'(\mu_c+\nu, w^{\mu}(\nu))$ and $L^{\alpha}(\alpha) \equiv F'(\mu_c+\nu(\alpha), w^{\alpha}(\alpha))$ be the linearized operators of F on the μ- and α-paths, respectively. Then, L^{μ} and L^{α} have, respectively, the oritical pairs $(\zeta_c^{\mu}(\nu),\phi_c^{\mu}(\nu)) \in \mathbb{R}^1\times V$, $\mu \in I_{\delta}$, and $(\zeta_c^{\alpha}(\alpha), \phi_c^{\alpha}(\alpha)) \in \mathbb{R}^1\times V$, $\alpha \in J_{\delta'}$, such that

$$\zeta_c^{\mu}(0) = \zeta_c^{\alpha}(0) = 0,$$

$$\phi_c^{\mu}(0) = \phi_c^{\alpha}(0) = \phi_c.$$

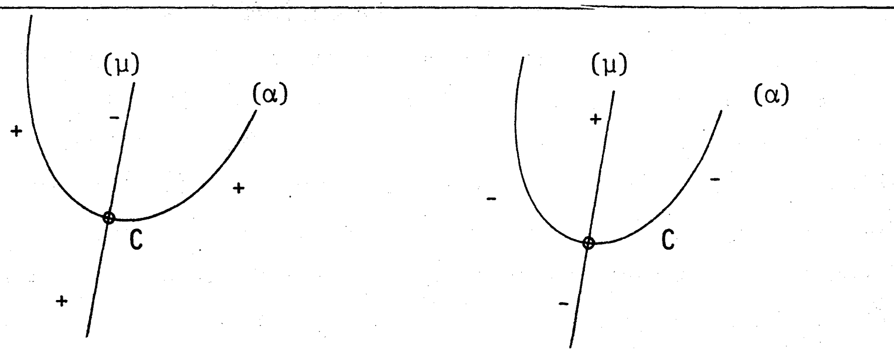
They satisfy the relations:

$$\frac{d\zeta_c^{\mu}}{d\nu}(0) = B_c \neq 0, \quad \frac{d\zeta_c^{\alpha}}{d\alpha}(0) = A_c = 0 \tag{1.21$_a$}$$

and moreover, if $p \geq 4$,

$$\frac{d^2\zeta_c^{\alpha}}{d\alpha^2}(0) = \frac{2}{3} D_c = -2\frac{d\zeta_c^{\mu}}{d\nu}(0)\ \frac{d^2\nu}{d\alpha^2}(0) \neq 0. \tag{1.21$_b$}$$

<u>Remark 1.10'</u> The relations (1.21) show the stability behavior on the two paths near **C**. See, Fig. 1.5. If $D_c > 0$ ($D_c < 0$), **C** is called a *stable (unstable)* cusp bifurcation point. It is noted that in both cases, the critical eigenvalue $\zeta_c^{\mu}(\nu)$ on the μ-path changes sign when it crosses $\nu = 0$, while on the α-path $\zeta_c^{\alpha}(\alpha)$ does *not* change sign at $\alpha = 0$.



"stable cusp bifurcation" *"unstable cusp bifurcation"*

Fig. 1.5

Remark 1.11 As may be clearly seen from the proof, there exist two paths, μ- and α-paths, which intersect at C, *whether or not D_c vanishes*, provided C is simple and non-degenerate.

For proofs of the above propositions, we refer to Fujii and Yamaguti [2].

2.0 SIMPLE BUCKLINGS IN THE PRESENCE OR NON-PRESENCE OF SYMMETRY GROUPS

So far, we have concentrated on the *formal* classification of simple critical points. In this section, we go further into the *mechanism* of simple bucklings. In other words, we want to know how and when those simple critical points appear *stably*. Two concepts will be introduced for this purpose: the concept of *symmetry group* and that of *class* <L or N> of the problem. The class of the problem is a *path-dependent* concept, which essentially implies that the bifurcation problem is considered on *either* a *linear* (with respect to the bifurcation parameter μ) or a *nonlinear* path. (For example, even (P) has a linear fundamental path, the secondary bifurcation from the firstly bifurcated path should be considered as a <class N> problem.) We shall clarify the relation of the type (fold, cusp or etc.) of critical points and the presence or non-presence of a symmetry group. We shall show, for example, that a fold bifurcation is, if exists, symmetry preserving.

An important result in this section is the uniform existence of symmetry breaking bifurcation points with respect to small changes (=perturbations) of the equation under the presence of a non-trivial symmetry group G (which we shall call the *structural* stability of the bifurcation points). As an obvious analogue, we have the structural stability of <class L> bifurcations under perturbations which do not destroy the <class L> property. These are obviously non-generic situations; however, it is this structural stability that guarantees the numerical realization of bifurcation points in the actual numerical computations.

The introduction of group theoretical arguments to nonlinear singularities is not indeed new, particularly in pattern formation problems in fluid mechanics. (See, e.g., Ruelle [13] and Sattinger [14-16]. Also, see [11] and [10] for other problems.) However, the emphasis here is on the discussion of structural stability in the sense described in the above; our main tool is the *standard decomposition* of the Hilbert space V associated with the symmetry group of the problem. A remark is that our arguments here exhibit a sharp contrast with the general theory of imperfection sensitivities, e.g., by Thompson and Hunt [19], Hangai and Kawamata [4] and Keener and Keller [5]. See, however, Rooda [12] for discussions of non-generic imperfections.

2.1 SYMMETRY GROUP OF F

Let $\Omega \subset \mathbb{R}^m$ $(1 \le m \le 3)$ be a bounded domain with a piecewise smooth boundary. Let V be a *complex* Hilbert space of functions defined on Ω. Let $\langle\ ,\ \rangle$ be the inner product of V.

Definition 2.1 G is the symmetry group of the domain Ω, if

$$G = \{g \in O(m);\ g\Omega = \Omega\} \tag{2.1}$$

where O(N) is the classical orthogonal group.

Let $T:G \to GL(V)$ be a unitary representation of G on V.*)

Example 2.2 Let u, v, $\in H_0^2(\Omega)$ with $\langle u,v\rangle = \int_\Omega \Delta u \cdot \overline{\Delta v}$. The operators T_g $(g \in G)$:

$$(T_g u)(x) = u\ (g^{-1}\ x) \tag{2.2}$$

define an (infinite dimensional) representation of G on V. $T_g:V \to V$ $(g \in G)$ are *unitary* since

$$\langle T_g u, T_g v\rangle = \langle u,v\rangle,\ u,v \in H_0^2(\Omega), \tag{2.3}$$

noting that the Jacobian of the coordinate transformation is +1.

We assume *for the present* that $G \subset O(m)$ is a *finite* group of order $n(G)$. Let $\chi_1, \chi_2, \ldots, \chi_q$ be the complete set of simple characters of non-equivalent irreducible representations $\tau_1, \tau_2, \ldots, \tau_q$. By n_k $(k = 1,2,\ldots, q)$ we denote the dimensions of τ_k $(k = 1,2,\ldots,q)$. Note that q is equal to the number of conjugacy classes of G. See, e.g., Serre [17] or Miller [8], for details.

We define a direct sum decomposition of V — *the standard decomposition* of V:

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*) A representation of G on V is a homomorphism $T:g \to T_g$ of G into GL(V), where GL(V) denotes the group of all non-singular linear transformations of V onto itself.

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_q. \qquad (2.4)$$

The standard decomposition (2.4) is uniquely defined, and indeed, there exists a set of projection operators P_k: $V \to V_k$:

$$P_k = \frac{n_k}{n(G)} \sum_{g \in G} \overline{\chi_k(g)}\ T_g, \quad k = 1,2,\ldots,q. \qquad (2.5)$$

P_k $(k = 1,2,\ldots,q)$ are self-adjoint and commute with T_g $(g \in G)$. It holds that

$$\sum_{k=1}^{q} P_k = I \quad \text{and} \quad P_k P_j = \delta_{kj} P_k, \qquad (2.6)$$

where δ_{kj} is the Kronecker delta.

We summarize some of elementary properties of G and its characters χ which will be used in later discussions.

(i) $\chi_k(e) = n_k$, $k \in \langle 1,2,\ldots,q\rangle$,

(especially, $\chi_k(e) = 1$ for $\forall k$ such that $n_k = 1$).

(ii) if $k \in \langle 1,2,\ldots,q\rangle$ such that $n_k = 1$,

$$|\chi_k(g)| = 1 \quad \text{for } \forall g \in G \qquad (2.7)$$

and $$T_g\phi = \chi_k(g)\phi \quad \text{for } \forall g \in G, \ \forall \phi \in V_k. \qquad (2.8)$$

(iii) $\chi_k(g) = 1$ $(\forall g \in G)$ if and only if $k = 1$.*) (2.9)

Note that the decomposition (2.4) is reducible, and in fact, each V_k (which is infinite dimensional, in general) can be decomposed into an (infinite number of) direct sum of W_k's which are all homomorphic to τ_k. For the present purpose, we need only the standard decomposition (2.4). The subspaces V_k $(k = 1,2,\ldots,q)$ may be characterized as: each $u \in V_k$ transforms according to τ_k. Also, with each V_k, one can associate the maximal subgroup $G_k \subset G$ under which every element of V_k is invariant, namely, $G_k = \{g \in G;\ T_g u = u,\ \forall u \in V_k\}$. G_k is the symmetry group of functions in V_k. We shall call G_k *the maximal symmetry group of* V_k. Obviously, G is

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*) Thus, $$P_1 = \frac{1}{n(G)} \sum_{g \in G} T_g. \qquad (2.9)^*$$

the maximal symmetry group of V_1, since $T_g P_1 = P_1$ for all $g \in G$(*see*, Eq. (2.9)*.) In this sense, we may call V_1 the *G-symmetric space*.

Example 2.3 (a); $C_s \cong C_{1h}$; *the reflection through a plane*

$G = \{e, s\}$, $s^2 = e$.

character table:

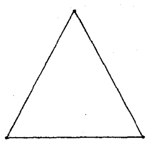
	{e}	{s}
χ_1	1	1
χ_2	1	-1

standard decomposition: $V = V^+ \oplus V^-$

$(P^{\pm} u) = \frac{1}{2} (I \pm T_s)u$, $(T_s u)(x) = u(-x)$

Example 2.3 (b); $C_{3v} \cong D_3$; *group of the equi-lateral triangle in a plane*

$G = \{e, g, g^2, s, gs, g^2s\}$



two generators g, s with $g^3 = s^2 = e$,

and $sgs = g^{-1}$, where g: counterclockwise rotation through 120°

s: a reflection across a median

character table:

	{e}	$\{g, g^2\}$	$\{s, gs, g^2s\}$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

standard decomposition $V = V_1 \oplus V_2 \oplus V_3$

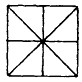
$$P_1 = \frac{1}{3} (I + T_g + T_g{}^2) \frac{1}{2} (I + T_s)$$

$$P_2 = \frac{1}{3} (I + T_g + T_g{}^2) \frac{1}{2} (I - T_s)$$

$$P_3 = I - \frac{1}{3} (I + T_g + T_g{}^2)$$

$G_1 = G$, $G_2 = \{e, g, g^2\}$, $G_3 = \{e\}$

Example 2.3 (c); $D_4 \cong C_{4v}$; *group of plane operations which sends a square into itself*



We omit the details.

We shall now define the notion of *symmetry group* of F, where F is a smooth (at least C^1) mapping of $\mathbb{R}^1 \times V$ into V. We shall generally assume that the mapping F is *real* in the sense that $\overline{F(\mu,w)} = F(\mu,\overline{w})$, for all $(\mu,w) \in \mathbb{R}^1 \times V$.

Definition 2.4 G is said to be the symmetry group of F, if G is the maximal symmetry group of Ω such that F is covariant under G.

Here, F *is covariant under* G means that

$$F(\mu, T_g w) = T_g F(\mu,w), \qquad (2.10)$$

for all $g \in G$, and $(\mu,w) \in \mathbb{R}^1 \times V$.

Example 2.5 (a) The Laplacian Δ is covariant under O(m), namely, $T_g \Delta = \Delta T_g$, $\forall g \in O(m)$.

Example 2.5 (b) The von Kármán-Donnell-Marguerre shell operator is covariant under G, where G is the symmetry group of the domain $\Omega \subset O(2)$, provided the initial deflection w_0 and the known Airy function corresponding the edge force are invariant under G. See, Appendix A of Fujii and Yamaguti [2].

In the sequel, we shall assume that G is the symmetry group of F. G may be either *trivial* $G = \{e\}$ or non-trivial. Note that if G is trivial (that is, if F has *no* group symmetry), the standard decomposition (2.4) is the trivial one $V = V_1$.

Definition 2.6 Suppose V is decomposed into a direct sum

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_q,$$

with $P_i: V \to V_i$ $(i = 1,2,\ldots,q)$ the associate projections.

We say that $F: \mathbb{R}^1 \times V \to V$ is *enclosed* in V_1, if

(i) $P_i\ F'(\mu,w_1)\ P_j = 0$ (2.11)

for $i,j = 1,2,\ldots,q;\ i \neq j$, and for $\forall(\mu,w_1) \in \mathbb{R}^1 \times V_1$, and

(ii) $P_j\ F(\mu,w_1) = 0$ (2.12)

for $j = 2,3,\ldots,q$, and for $\forall\ (\mu,w_1) \in \mathbb{R}^1 \times V_1$.

That F is enclosed in V_1 implies that the linearized operator of F at $(\mu,w_1) \in \mathbb{R}^1 \times V_1$ has a block diagonal form and that the problem $P_j\ F(\mu,(w_1, w_2,\ldots,w_q)) = 0$ $(j = 2,3,\ldots,q)$ has "a trivial solution" $w_2 = w_3 = \ldots = w_q \equiv 0$, for all $(\mu,w_1) \in \mathbb{R}^1 \times V_1$.

It is almost direct to show the following

Lemma 2.7 F is enclosed in the G-symmetric space V_1.

Proof. If $G = \{e\}$, the proposition is obvious. Hence, we assume $n(G) > 1$. Firstly, from Eq. (2.10), we find that

$$\frac{1}{n(G)} \sum_{g \in G} F(\mu, T_g w) = \frac{1}{n(G)} \sum_{g \in G} T_g F(\mu,w)$$

for all $(\mu,w) \in \mathbb{R}^1 \times V$. In view of the relation $T_g w = w$ for any $w \in V_1$, we have

$$F(\mu,w) = P_1 \cdot F(\mu,w),\ \forall\ (\mu,w) \in \mathbb{R}^1 \times V_1.$$

Therefore, Eq. (2.12) follows.

Secondly, differentiating Eq. (2.10) with respect to w,

$$F'(\mu,T_g w) \cdot T_g = T_g \cdot F'(\mu,w),\ \forall\ g \in G,\ \forall\ (\mu,w) \in \mathbb{R}^1 \times V.$$

Hence, for $w \in V_1$, $F'(\mu,w)$ commutes with T_g, i.e.,

$$F'(\mu,w)\ T_g = T_g\ F'(\mu,w). \qquad (2.13)$$

Multiplying the above relation by $\overline{\chi_i(g)}$ and summing all the $g \in G$, we have that

$$F'(\mu,w) \cdot P_i = P_i \cdot F'(\mu,w),\ (i = 1,2,\ldots,q)$$

for all $(\mu,w) \in \mathbb{R}^1 \times V_1$, which in turn implies Eq. (2.11)

2.2 SIMPLE BUCKLINGS IN THE PRESENCE/NON-PRESENCE OF A SYMMETRY GROUP

Under the existence of a symmetry group G, either trivial or non-trivial, in *any* simple critical points a further structure is built-in there; namely, (G-) *symmetry preserving* and (G-) *symmetry breaking* critical points. We shall see that a symmetry breaking critical point is necessarily a bifurcation point, and which cannot be a fold. (Thus, a fold bifurcation should be, if exists, symmetry preserving!) A symmetry preserving bifurcation *can* exist formally, however, the essential nature of such bifurcations will not become clear until at the next paragraph, where we shall consider them with the viewpoint of "structural" stability. We remark here that when G is trivial, only the symmetry preserving case can appear as a critical point. In this paragraph, we shall study such *symmetry structure* of simple critical points.

We begin by recalling that our problem is given by

$$(P) \quad F(\mu,w) = 0 \tag{1.1}$$

where $F: \mathbb{R}^1 \times V \to V$ is a C^p $(p \geq 3)$ mapping of Fredholm type. Assume that G is the symmetry group of F (not necessarily non-trivial). Assume also G is of finite order. For a compact Lie group e.g., $G = D_\infty$ case, see Remark 2.15. The standard decomposition of V, Eq. (2.4), is assumed, with the corresponding projections p_i: $V \to V_i$ $(i = 1,\ldots,q)$, given by Eq. (2.5).*) By Lemma 2.7, F is *enclosed* in V_1 – the G-symmetric space. We shall sometimes denote by V^+ the G-symmetric space V_1, and by V^- the G-asymmetric space $V_2 \oplus \ldots \oplus V_q$. Also, P^+ and P^- denote the corresponding projections.

The following lemma may explain why we say that F is enclosed in V^+.

Lemma 2.8 Suppose $0^+ \equiv (\mu_0, w_0^+) \in \mathbb{R}^1 \times V^+$ is an ordinary point of (P). Then, the ordinary path which contains 0^+ lies in $\mathbb{R}^1 \times V^+$. (*cf*. Lemma 1.2, §1.)

---

*) When $G = \{e\}$ (trivial), $q = 1$.

Proof. Restricting the problem (P) on V^+, we have an ordinary path which lies in $\mathbb{R}^1 \times V^+$ using Lemma 1.2 on the space V^+. Here, the properties (2.11) and (2.12) are essential. The uniqueness of the ordinary path in the whole space V guaranteed by Lemma 1.2 shows the proposition.

This lemma shows that a *G*-symmetric ordinary path continues to be *G*-symmetric *until* it arrives at a critical point C^+, *which itself is G-symmetric* by the completeness of the space V^+.

We now suppose $C^+ \equiv (\mu_c, w_c{}^+) \in \mathbb{R}^1 \times V^+$ is a *simple* non-degenerate critical point of (P) on a *G*-symmetric path. Let $\phi_c \in \ker L_c$, where $L_c \equiv F'(\mu_c, w_c{}^+)$. First, we note that since $F(\mu, w^+) \in V^+$ for all $(\mu, w^+) \in \mathbb{R}^1 \times V^+$ by Eq. (2.12), $\dot{F}_c = \frac{\partial}{\partial \mu} F(\mu, w_c{}^+)|_{\mu=\mu_c} \in V^+$. Next, since L_c commutes with T_g ($\forall\, g \in G$) by Eq. (2.13), if $\phi_c \in \ker L_c$, then $T_g \phi_c \in \ker L_c$. This fact together with the simpleness assumption of C^+ necessarily implies that ϕ_c belongs to such V_k ($\exists\, k \in \langle 1,2,..,q \rangle$) that the corresponding irreducible representation τ_k is one dimensional (i.e., $n_k = 1$).

In view of the classification theorems in §1, we have the following possibilities *formally*:

(i) *Symmetry preserving snap buckling* (k = 1):

$$\phi_c \in V^+ \text{ and } \langle \dot{F}_c, \phi_c \rangle \neq 0 \tag{2.14}$$

(ii) *Symmetry preserving bifurcation buckling* (k = 1):

$$\phi_c \in V^+ \text{ and } \langle \dot{F}_c, \phi_c \rangle = 0. \tag{2.15}$$

(iii) *Symmetry breaking bifurcation buckling* ($\exists\, k \in \langle 2,..,q \rangle$):

$$\phi_c \in V_k \subset V^- \text{ and hence, } \langle \dot{F}_c, \phi_c \rangle = 0. \tag{2.16}$$

It may be immediate to see the following

Lemma 2.9

(i) Suppose C^+ is a symmetry-preserving, simple non-degenerate snap point of (P). Then, the unique path emerging from C^+ lies in $\mathbb{R}^1 \times V^+$.

(ii) Suppose C^+ is a symmetry-preserving, simple, non-degenerate bifurcation point of (P). Then, both of two paths emerging from C^+ (*see*, Lemmae 1.9 and 1.10, §1) lie in $\mathbb{R}^1 \times V^+$.

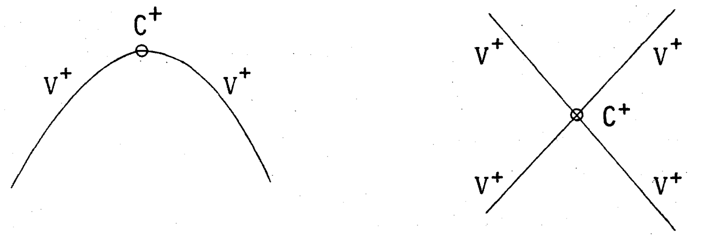


Fig. 2.1
"Symmetry-preserving critical points"

In case of the symmetry breaking bifurcations, we have the following two lemmas, which exhibit an interesting nature of simple, symmetry breaking bifurcations.

Lemma 2.10 Suppose $(C^+;\phi_c) \equiv (\mu_c, w_c{}^+;\phi_c) \in \mathbb{R}^1\times V_1\times V_k \subset \mathbb{R}^1\times V^+\times V^-$ is a simple, non-degenerate symmetry breaking bifurcation point of (P). Then,

(i) there imerges a G-symmetric path $(\mu, w^+(\mu)) \in \mathbb{R}^1\times V^+$ for $\mu-\mu_c \in I_\delta = \{\nu; |\nu| < \delta\}$ such that $w^+(\mu_c) = w_c{}^+$.

(ii) The other bifurcating path (*see*, Lemma 1.10 and Remark 1.10", §1, Chapter I.) $(\mu(\alpha), w^\alpha(\alpha)) \in \mathbb{R}^1\times V$ for $\alpha \in I_{\delta'} = \{\alpha; |\alpha| < \delta'\}$ is in the G_k-symmetric space $V^+_{(k)} \subset V$, which is defined by $V^+_{(k)} = P^+_{(k)} V$, where

$$P^+_{(k)} \overset{def}{=} \frac{1}{n(G_k)} \sum_{g \in G_k} T_g. \qquad (2.17)$$

Here, G_k is the maximal symmetry group of V_k in the sense of §2.1, V_k being the subspace of V to which ϕ_c belongs. See, Fig. 2.2.

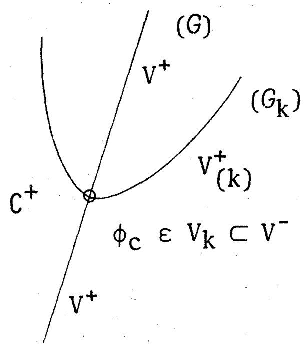


Fig. 2.2

This lemma shows a situation that *the symmetry group G on the fundamental (= G-symmetric) path breaks to a subgroup* G_k *on the bifurcating* (G_k*-symmetric*) *path.*

Proof. Restricting the problem (P) on V^+-space, the assertion (i) is easily checked using a similar reasoning as in Lemma 2.8. To show (ii), we return to the Lyapounov-Schmidt decomposition of F at $(\mu_c, w_c{}^+)$,

$$\omega_c G_c\ (\nu, \alpha\phi_c + \psi) = 0, \tag{2.18}$$

$$\Pi_c G_c\ (\nu, \alpha\phi_c + \psi) = 0, \tag{2.19}$$

where $\psi \in R_c$ = range $F_c' = \{\ker F_c'\}^{\perp}$. Π_c is the projection of V onto ker F_c', and $\omega_c = I - \Pi_c$. By Lemma 1.8, we know the unique existence of $\psi = \psi(\alpha, \nu)$ such that Eq. (2.18) is satisfied. We show that ψ is covariant under G, i.e.,

$$T_g\ \psi(\alpha,\nu) = \psi(T_g\alpha,\nu),\ \forall\ g \in G. \tag{2.20}$$

Here, we understand that if $u = \alpha\cdot\phi \in V_k$, $\alpha \in C^1$,

$$T_g\ u = T_g\ \alpha\phi = \alpha\ \chi_k(g)\ \phi$$

using the relation (2.8), so

$$T_g\ \alpha = \chi_k(g)\ \alpha. \tag{2.21}$$

We first note that Π_c and T_g, and hence ω_c and T_g commute. In fact, if we let $u = \alpha\phi_c + \psi$ ($\forall\ u \in V$), $T_g\ \Pi_c\ u = T_g\ \alpha\ \phi_c = \alpha\ \chi_k(g)\ \phi_c$, but $\Pi_c\ T_g\ u = \langle T_g u, \phi_c\rangle\ \phi_c = \langle u, T_g{}^*\phi_c\rangle\ \phi_c = \chi_k(g)\ \langle u,\phi_c\rangle\ \phi_c = \chi_k(g)\ \alpha\ \phi_c$. Next, the G-covariance of G_c, which follows obviously from Eq. (2.10), and $\omega_c\ T_g = T_g\ \omega_c$ yield

$$\begin{aligned} T_g\ \omega_c\ G_c\ &(\nu, \alpha\phi_c + \psi(\alpha,\nu)) \\ &= \omega_c\ G_c\ (\nu, \alpha T_g\phi_c + (T_g\psi)(\alpha,\nu)) \\ &= 0 \end{aligned} \tag{2.23}$$

The uniqueness of the solution of $\psi = \psi(\alpha,\nu)$ in Eq. (2.18) implies the relation (2.20).

Now, recalling that G_k is the maximal symmetry group of V_k (*see*, §2.1), we have that

$$T_g\alpha = \alpha \quad \text{for } \forall\ g \in G_k. \tag{2.24}$$

Accordingly, Eqs. (2.20) and (2.24) show that

$$T_g\ \psi(\alpha,\nu) = \psi(\alpha,\nu),\ \forall\ g \in G_k, \tag{2.25}$$

from which follows

$$P^+_{(k)}\ \psi(\alpha,\nu) = \psi(\alpha,\nu). \tag{2.26}$$

Thus, (ii) is proved.

Lemma 2.11 A simple, symmetry breaking bifurcation point $(C^+;\phi_c^-) \equiv (\mu_c, w_c^+;\phi_c^-) \in \mathbb{R}^1\times V^+\times V^-$ can *not* be a fold bifurcation. Namely, it holds that

$$A_c \equiv \langle F''\ (\mu_c,w_c^+)(\phi_c^-,\phi_c^-),\phi_c^-\rangle = 0. \tag{2.27}$$

Remark 2.12 Accordingly, a fold (= transcritical) bifurcation should be, if exists, symmetry preserving.

A proof of the above lemma may follow from the following observations. Firstly, the bilinear mapping $F''\ (\mu_c,w_c^+)(\cdot,\cdot)$ is covariant under G:

$$F''_c\ (T_g u, T_g v) = T_g\ F''_c\ (u,v),\ \forall\ g \in G,\ \forall\ u,\ v \in V, \tag{2.28}$$

here $F''_c\ (\cdot,\cdot) \equiv F''\ (\mu_c,w_c^+)(\cdot,\cdot)$. Indeed, from the G-covariance of F, Eq. (2.10),

$$F''\ (\mu,T_g w)(T_g\cdot,T_g\cdot) = T_g\ F''\ (\mu,w)(\cdot,\cdot).$$

Using the relation $T_g w = w$ for $\forall\ w \in V_1 \equiv V^+$, Eq. (2.28) is immediate. Now, since T_g is unitary, the form

$$A_c(\phi) \overset{def}{\equiv} \langle F''_c\ (\phi,\phi),\phi\rangle,\ \phi \in V_k \tag{2.29}$$

is invariant under G in the sense that

$$A_c\ (T_g\phi) = A_c(\phi),\ \forall\ g \in G. \tag{2.30}$$

On the other hand, Eqs. (2.7) and (2.8) yield

$$\begin{aligned} A_c\ (T_g\phi) &= \chi_k(g)\,|\chi_k(g)|^2\ A_c(\phi), \\ &= \chi_k(g)\ A_c(\phi), \end{aligned} \tag{2.31}$$

Therefore,

$$(\chi_k(g) - 1)\ A_c(\phi) = 0 \quad \text{for } \forall\ g \in G. \qquad (2.32)$$

It is however only for k = 1 that $\chi_k(g) = 1$ for all $g \in G$ (*see*, Eq (2.9)). The symmetry breaking assumption $\phi_c \in V_k \subset V^-$, i.e., $k \in \langle 2,3,..,q \rangle$ implies $A_c\ (\phi_c) = 0$. This completes the proof.

We can perform similar arguments to know whether and when the other coefficients of the bifurcation equation, for instance D_c, vanish. However, this is a reflection of a more general situation that the G-covariance of the problem is inherited by the bifucation equation as was shown by Sattinger [14].

Lemma 2.13 (D. Sattinger) The bifurcation equation $\Gamma\ (\alpha,\nu)$ is covariant under G:

$$T_g\ \Gamma\ (\alpha,\nu) = \Gamma\ (T_g\alpha,\nu),\ g \in G, \qquad (2.33)$$

where $T_g\ \Gamma$ is understood in the sense of Eq. (2.21).

For completeness, we sketch the proof for our *simple* case. From the G-covariance of G_c and of ψ, we find that

$$\begin{aligned}\Gamma\ (\alpha,\nu) &= \langle G_c\ (\nu,\alpha\phi_c + \psi(\alpha,\nu)),\ \phi_c\rangle \\ &= \langle T_g\ G_c\ (\nu,\alpha\phi_c + \psi(\alpha,\nu)),\ T_g\ \phi_c\rangle \\ &= \langle G_c\ (\nu,\alpha T_g\phi_c + \psi(T_g\alpha,\nu)),\ T_g\ \phi_c\rangle \\ &= \overline{\chi_k(g)}\ \cdot\Gamma(T_g\alpha,\nu)\end{aligned}$$

which is nothing but the relation (2.33).

Remark 2.14 We return to the question: whether and/or when the coefficient D_c vanishes. We have similarly that $(1 - \chi_k(g)^2)\ D_c = 0$ for all $g \in G$. We may have to check whether/when $\chi_k(g)^2 = 1$ for all $g \in G$. In every case in Example 2.1, $\chi_k(g) = \pm 1$ for all $g \in G$ provided $\chi_k(e) = 1$ (i.e., $n_k = 1$), inplying thus D_c does not vanish (at least, not by group theoretical reasonings). We can say that a simple symmetry breaking cusp bifurcation actually realizes.

However, there are cases where D_c does vanish identically even in a simple, symmetry breaking bifurcation. For example, a problem with symmetry group C_3 — the cyclic group of order 3 consisting of a rotation through 120° and its powers, which may correspond to, e.g., a shell of revolution with C_3-loadings. The character table of C_3 is given by

C_3	ε	C_3	C_3^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

$, \omega = \exp(\frac{2\pi}{3} i).$

For k = 2 or 3 (i.e., simple symmetry breaking case), it is *not* true that $\chi_k(g)^2 = 1$ for $\forall g \in C_3$. Note, however, that the coefficient of α^4 in the bifurcation equation vanishes identically by the group theoretical reasoning.

Remark 2.15 *(A Remark on Shells of Revolution. D_∞ — a compact Lie group case)* So far, we have assumed that *G* is a finite group. An important case arises in non-linear elasticity in which *G* is not a finite group, but a compact Lie group. Shells of revolution or any other shells with rotational symmetry are such instances. Most of the techniques we have used so far are, up to modifications, applicable to classical Lie groups.*) However, it should be noted that in, e.g., D_∞ — the group of rotations and reflections that sends a plane into itself — the irreducible representations are two dimensional except two representations including the identity. This may lead to a bifurcation problem with *double* singularities. However, this *group-theoretical double eigenvalues* are *in a sense* only in appearance, as was pointed out by Sattinger in [14]. There bifurcates a one parameter *sheet* of solutions, which is merely a sheet obtained by rotating *a* one parameter path bifurcating from the double critical points C. Thus, in conclusion, we have only to restrict the problem to the subspace $V^{(c)} = \frac{1}{2}(I + T_s)V$ of V, where s is a reflection, reducing the problem to a simple critical case. For more discussions, refer Fujii-Yamaguti [3].

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*) The standard decomposition (2.4) equally holds with $q = +\infty$. The projection operators P_ℓ are defined with the aid of Haar measure of D_∞. See, Serre [17] for these materials.

2.3 STABILITY OF CRITICAL POINTS UNDER THE PRESENCE OF A SYMMETRY GROUP

At this final paragraph of Chapter I, we would like to discuss the stability of critical points, in particular that of bifurcation points, with respect to small changes of the equation (P).

Suppose we have a ε-family ($\varepsilon \in E \subset \mathbb{R}^1$) of perturbed problems:

$$(P)_\varepsilon \quad F(\varepsilon;\mu,w) = 0, \; E\times\mathbb{R}^1\times V \to V \tag{2.34}$$

with the condition that

$$F(0;\mu,w) \equiv F(\mu,w), \; \forall (\mu,w) \in \mathbb{R}^1\times V. \tag{2.35}$$

$F(\varepsilon;\mu,w)$ is assumed to be sufficiently smooth in each variable.

We want to discuss in what class of problem $(P)_\varepsilon$ or, under what kind of perturbations, a bifurcation point appears *stably*, or more precisely appears *uniformly in* $|\varepsilon| \in [0, \varepsilon_o[$, *for some* $\varepsilon_o > 0$.

We shall introduce *two* classes of $(P)_\varepsilon$, in which bifurcation points appears stably. Firstly,

Theorem 2.16 Suppose $F(\varepsilon;\mu,w)$ is covariant under a non-trivial symmetry group G *uniformly* in $\varepsilon \in E$. Suppose $F(0;\mu,w)$ possesses a simple, symmetry breaking bifurcation point $(C^+;\phi_c) \equiv ((\mu_c,w_c{}^+);\phi_c) \in \mathbb{R}^1\times V_1\times V_k$, for some $k \in \langle 2,3,..,q\rangle$. Then, there exits a constant $\varepsilon_o > 0$, such that a ε-family of simple, symmetry breaking bifurcations $(C^+(\varepsilon);\phi_c(\varepsilon)) \equiv ((\mu_c(\varepsilon),w_c{}^+(\varepsilon));\phi_c(\varepsilon)) \in \mathbb{R}^1\times V_1\times V_k$ exists in $(P)_\varepsilon$ uniformly in $|\varepsilon| \in [0, \varepsilon_o[$.

Proof. The standard decomposition (2.4) being taken in mind, we have as the symmetric component:

$$P_1 F(\varepsilon;\mu,w_1) = 0, \; (\mu,w_1) \in \mathbb{R}^1\times V_1. \tag{2.36}$$

When $\varepsilon = 0$, there exists a G-symmetric path $(\mu,w_1(\mu)) \in \mathbb{R}^1\times V_1$ for $\mu \in I_\delta$, such that $P_1 F(0;\mu,w_1(\mu)) = 0$ and $P_k F(0;\mu,w_1(\mu)) = 0$ $(k = 2,3,..,q)$. For each $\mu \in I_\delta$ (fixed), there exists a unique function $w_1 = w_1(\varepsilon;\mu) \in V_1$, for $|\varepsilon| < {}^\exists \varepsilon_o$, such that $w_1(0;\mu) = w_1(\mu)$ and that $\|w_1(\varepsilon;\mu) - w_1(\mu)\|_V \le C\,|\varepsilon| (\forall\, |\varepsilon| < \varepsilon_o)$, since $P_1 F'(0;\mu,w_1(\mu))$ is invertible on the space V_1.

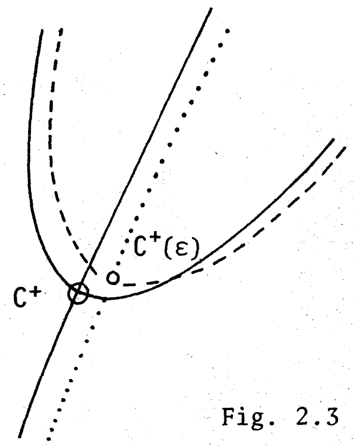

Fig. 2.3

"stability of symmetry breaking bifurcation under symmetry preserving perturbations"

The pair $(\mu, w_1(\varepsilon;\mu))$ satisfies Eq. (2.36), and consequently Eq. (2.34) since $(P)_\varepsilon$ is enclosed in V_1. The next stage is to study a (ε,μ)-family of eigenproblems on V_k: for $|\varepsilon| \in [0, \varepsilon_0]$, $\mu \in I_\delta$,

$$L_k\ (\varepsilon;\mu)\ \phi_c(\varepsilon;\mu) = \zeta_c\ (\varepsilon;\mu)\ \phi_c\ (\varepsilon;\mu),\ \phi_c\ (\varepsilon;\mu) \in V_k, \qquad (2.37)$$

where

$$L_k\ (\varepsilon;\mu) = P_k\ F'\ (\varepsilon;\mu, w_1(\varepsilon,\mu)). \qquad (2.38)$$

By hypothesis, $\zeta_c\ (0;\mu)$ vanishes at $\mu = \mu_c$, and

$$\frac{\partial}{\partial\mu} \zeta_c\ (0;\mu_c) \neq 0, \qquad (2.39)$$

$$\text{ker dim } L_k\ (0;\mu_c) = 1. \qquad (2.40)$$

See, Lemma 1.10. Here, $\zeta_c(\varepsilon;\mu)$ is the continuation of $\zeta_c\ (0;\mu)$. We want to seek $\mu = \mu_c(\varepsilon)$ such that

$$\zeta_c\ (\varepsilon;\mu) = 0 \qquad (2.41)$$

holds for each $|\varepsilon| \in [0, \varepsilon_1[$, for some $0 < \varepsilon_1 \leq \varepsilon_0$. By virtue of the relations $\zeta_c\ (0;\mu_c) = 0$ and (2.39), we have the unique existence of $\mu = \mu_c\ (\varepsilon)$ such that Eq. (2.41) satisfied and that $|\mu_c(\varepsilon) - \mu_c| \leq C\ |\varepsilon|$ for $|\varepsilon| < {}^\exists\ \varepsilon_1$. $(0 < {}^\exists\ \varepsilon_1 \leq \varepsilon_0$: sufficiently small).

Thus, we have again for each $\varepsilon \in [0, \varepsilon_1[$, a symmetric breaking bifurcation on a *G*-symmetric path. Especially, the bifurcation buckling load

$\mu_c(\varepsilon)$ is in an ε-neighborhood of that of unperturbed problem.

Suppose now $(C^+;\phi_c)$ is symmetry preserving, where G may or may not be trivial. There is a class of problems in which symmetry-preserving bifurcations may occur stably.

Definition 2.17 *A linear path of* F is a pair $(\mu,\mu\cdot w_o) \in \mathbb{R}^1\times V$, $\mu \in^\exists I \subset \mathbb{R}^1$, such that $F(\mu,\mu\cdot w_o) = 0$ for $\mu \in I$, where I is an open interval $\subset \mathbb{R}^1$, and $w_o \in V$ is a fixed function. In particular, if $w_o \equiv 0$, the pair $(\mu,0)$ is the *trivial* path. *A bifurcation problem* (P) *from a linear (trivial) path is called a problem of class* L (O).

If (P) is neither of class L nor O, it is called of class N (i.e., nonlinear path).*)

We remark that class L (class O) problems appear in many engineering and mathematical literatures.

For class L-problems, we have an almost trivial analogy of the previous proposition.

Proposition 2.18 Suppose (P) is of class L, and that F is simple critical and non-degenerate at $(C;\phi_c) \equiv (\mu_c,\mu_c\cdot w_o;\phi_c) \in \mathbb{R}^1\times V\times V$, $(\mu_c \in I)$. Then, $(C;\phi_c)$ is a bifurcation point. Moreover, this bifurcation is stable under any small change of equations, provided it does not destroy the class L-property of F.

Proof. Since

$$F(\mu,\mu\cdot w_o) = 0, \ \forall \mu \in I, \tag{2.42}$$

we can differentiate (2.42) on the path:

$$\begin{aligned} 0 &= \frac{d}{d\mu} F(\mu,\mu\cdot w_o) \\ &= \frac{\partial F}{\partial \mu}(\mu,\mu\cdot w_o) + \frac{\partial F}{\partial w}(\mu,\mu\cdot w_o)\cdot w_o \end{aligned} \tag{2.43}$$

*) Note that "the class of (P)" is a path-dependent notion. See, also, remarks at the introduction of §2.

Thus, at $\mu = \mu_c$, using the self-adjointness of F',

$$\begin{aligned} \langle \dot{F}_c, \phi_c \rangle &= - \langle F'_c \cdot w_o, \phi_c \rangle \\ &= 0 \end{aligned} \tag{2.44}$$

which shows the first assertion.

Suppose now the perturbed problem

$(P)_\varepsilon \quad F(\varepsilon;\mu,w) = 0$

is still of class L uniformly in $\varepsilon \in E$. Namely, we assume that for each $\varepsilon \in E$, there exists a function $w_o(\varepsilon) \in V$ such that $w_o(0) = w_o$, $\|w_o(\varepsilon) - w_o\|_V \le C\,|\varepsilon|$ and that $F(\varepsilon;\mu,\mu\cdot w_o(\varepsilon)) = 0$ for $\mu \in I$, $\varepsilon \in E$.
Letting

$$L(\varepsilon;\mu) \overset{def}{=} F'(\varepsilon;\mu,\mu\cdot w_o(\varepsilon)), \tag{2.45}$$

we consider a family of eigenproblems in V:

$$L(\varepsilon;\mu)\,\phi_c(\varepsilon;\mu) = \zeta_c(\varepsilon;\mu)\,\phi_c(\varepsilon;\mu),\ \phi_c(\varepsilon;\mu) \in V. \tag{2.46}$$

At $\varepsilon = 0$, $\zeta_c(0;\mu) = 0$ (simple) and $\frac{\partial \zeta_c}{\partial \mu}(0;\mu_c) \neq 0$ by Lemma 1.9.*) Here, $\zeta_c(\varepsilon;\mu)$ is the continuation of $\zeta_c(0;\mu)$. Hence, the implicit function theorem applies to $\zeta_c(\varepsilon;\mu) = 0$ at $(\varepsilon,\mu) = (0,\mu_c)$, obtaining a unique $\mu = \mu_c(\varepsilon)$ for each ε, $|\varepsilon| \in [0, \varepsilon_1]$ (ε_1: sufficiently small.) Accordingly, we have again a bifurcation for each small ε.

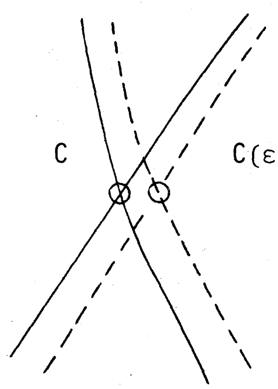

Fig. 2.4

"stability of (symmetry preserving) bifurcation under perturbations which does not destroy the class L-property".

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*) Also, by Lemma 1.10 for cusp case.

Remark 2.19 As was stated in Remark 2.12, a fold bifurcation is necessarily symmetry preserving, and such *fold* may appear stably if $(P)_\varepsilon$ preserve the class L-property. However, a symmetry preserving bifurcation is *not* necessarily a fold. A cusp or more degenerate bifurcation may appear by virtue of the degeneracy of F itself. For discussions including *non-simple* cases, see Fujii and Yamaguti [3].

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