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平板の大たわみ解析に対する離散キルヒホッフ仮定

(The Discrete Kirchhoff Assumption for Large Deflection Analysis of Plates)

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1. Introduction

The discrete Kirchhoff assumption is a brilliant idea available for finite element analysis of thin plates and shell, see Gallagher [1]. We will present a theoretical analysis of the method applied to large deflection problems of thin elastic plates.

2. Preliminaries

Let Ω be a bounded domain in R^2 occupied by the middle surface of a clamped plate. If necessary, its boundary $\partial\Omega$ is assumed to be smooth enough. We will use the following notations: $x = (x_1,x_2)$ = Cartesian coordinates of a point in Ω; $u = (u_1,u_2,w)$ = displacement of the plate; E = Young's modulus; ν = Poisson's ratio; t = thickness of the plate. The quantities E, ν, and t are assumed to be constant, for simplicity, and F and D are defined by $F = Et/(1-\nu^2)$ and $D = Ft^2/12$, respectively.

As real function spaces related to the domain Ω, we will use the Sobolev spaces $H^n(\Omega)$ and $H^n_0(\Omega)$ for n = 0,1,2, both being equipped with the same norm $\|\ \|_n$. In particular, We will use $(\ ,\ )$ and $\|\ \|$ as the inner product and the norm of $L_2(\Omega)=H^0(\Omega)$, respectively. We will also make implicit use of the space $L_p(\Omega)$.

3. Continuous Problem

Let us consider the space $X = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega)$ equipped with the structure of the Hilbert space in the usual way. This is a natural Hilbert space for the problems of clamped plates.

Define the following forms for $u = (u_1, u_2, w) \in X$ and $\bar{u} = (\bar{u}_1, \bar{u}_2, \bar{w}) \in X$:

$$B_1(u,\bar{u}) = F[\sum_{i=1}^{2}(\partial_i u_i + \tfrac{1}{2}(\partial_i w)^2, \partial_i \bar{u}_i + \partial_i w \cdot \partial_i \bar{w}) + \nu\{(\partial_1 u_1 + \tfrac{1}{2}(\partial_1 w)^2, \partial_2 u_2 + \partial_2 w \cdot \partial_2 \bar{w}) + (\partial_2 u_2 + \tfrac{1}{2}(\partial_2 w)^2, \partial_1 \bar{u}_1 + \partial_1 w \cdot \partial_1 \bar{w}) + \frac{1-\nu}{2}(\partial_2 u_1 + \partial_1 u_2 + \partial_1 w \cdot \partial_2 w, \partial_2 \bar{u}_1 + \partial_1 \bar{u}_2 + \partial_1 w \cdot \partial_2 \bar{w} + \partial_2 w \cdot \partial_1 \bar{w})] \quad , \qquad (1)$$

$$B_2(w,\bar{w}) = D[\sum_{i=1}^{2}(\partial_i^2 w, \partial_i^2 \bar{w}) + \nu\{(\partial_1^2 w, \partial_2^2 \bar{w}) + (\partial_2^2 w, \partial_1^2 \bar{w})\} + 2(1-\nu)(\partial_1\partial_2 w, \partial_1\partial_2 \bar{w})] \quad , \qquad (2)$$

$$B(u,\bar{u}) = B_1(u,\bar{u}) + B_2(w,\bar{w}) \quad , \qquad (3)$$

$$N_1(u_1,u_2) = [F\{\sum_{i=1}^{2}\|\partial_i u_i\|^2 + 2\nu(\partial_1 u_1, \partial_2 u_2) + \frac{1-\nu}{2}\|\partial_2 u_1 + \partial_1 u_2\|^2\}]^{1/2} \quad , \qquad (4)$$

where $\partial_i = \partial/\partial x_i$ and $\partial_i^2 = \partial_i \partial_i$.

Our problem is to find $u \in X$ such that

$$B(u,\bar{u}) = \sum_{i=1}^{2} (f_i,\bar{u}_i) + (f_3,\bar{w}) \quad \text{for all} \quad \bar{u} = (\bar{u}_1,\bar{u}_2,\bar{w}) \in X . \tag{5}$$

In the above, $f = (f_1,f_2,f_3)$ is the distributed force applied to the plate, and is taken from $H^{-1} \times H^{-1} \times H^{-1}$, where H^{-1} is the dual space of $H_0^1(\Omega)$. (We can of course take f from $H^{-1} \times H^{-1} \times H^{-2}$ for the continuous problem itself. However, this choice is essential for the description of our discretized problem.)

The existence of the solutions to (5) can be proved by the use of the fixed point theorem, cf. [3], but the uniqueness cannot be expected in general. The readers can also imagine the process of the existence proof from the analysis of the corresponding discrete problem to be presented later.

4. Finite Element Method

Let $X^h = X_1^h \times X_1^h \times X_2^h$ be an appropriate (finite dimensional) subspace of $H_0^1(\Omega)$. The label h (> 0) denotes the discretization parameter, and the case when h tends to zero will be considered. (Therefore, we will actually consider a series or a family of finite element spaces.)

Introduce the following two operators:

$$\partial_{hi} : X_2^h \to H_0^1(\Omega) \quad , \qquad i = 1,2 . \tag{6}$$

In the discrete Kirchhoff approach, we will use $\partial_{hi}w_h$ instead of $\partial_i w_h$ for $w_h \in X_2^h$. It is to be noted that $\partial_i w_h$ is not differentiable for the present type of non-conforming approximation (notice that X_2^h is merely included into $H_0^1(\Omega)$), but that $\partial_{hi}w_h$ is differentiable as may be seen from (6).

Define the following forms for $u_h = (u_{h1}, u_{h2}, w_h) \in X^h$ and $\bar{u}_h = (\bar{u}_{h1}, \bar{u}_{h2}, \bar{w}_h) \in X^h$:

$$\begin{aligned} B_{h1}(u_h,\bar{u}_h) = F[&\sum_{i=1}^{2}(\partial_i u_{hi} + \tfrac{1}{2}(\partial_{hi}w_h)^2, \partial_i\bar{u}_{hi} + \partial_{hi}w_h\cdot\partial_{hi}\bar{w}_h) \\ &+ \nu\{(\partial_1 u_{h1} + \tfrac{1}{2}(\partial_{h1}w_h)^2, \partial_2\bar{u}_{h2} + \partial_{h2}w_h\cdot\partial_{h2}\bar{w}_h) \\ &+ (\partial_2 u_{h2} + \tfrac{1}{2}(\partial_{h2}w_h)^2, \partial_1\bar{u}_{h1} + \partial_{h1}w_h\cdot\partial_{h1}\bar{w}_h)\} \\ &+ \tfrac{1-\nu}{2}(\partial_2 u_{h1} + \partial_1 u_{h2} + \partial_{h1}w_h\cdot\partial_{h2}w_h, \partial_2\bar{u}_{h1} + \partial_1\bar{u}_{h2} \\ &+ \partial_{h1}w_h\cdot\partial_{h2}\bar{w}_h + \partial_{h2}w_h\cdot\partial_{h1}\bar{w}_h)] \quad , \end{aligned} \tag{7}$$

$$\begin{aligned} B_{h2}(w_h,\bar{w}_h) = D[&\sum_{i=1}^{2}(\partial_i\partial_{hi}w_h, \partial_i\partial_{hi}\bar{w}_h) + \nu\{(\partial_1\partial_{h1}w_h, \partial_2\partial_{h2}\bar{w}_h) \\ &+ (\partial_2\partial_{h2}w_h, \partial_1\partial_{h1}\bar{w}_h)\} + \tfrac{1-\nu}{2}(\partial_1\partial_{h2}w_h + \partial_2\partial_{h1}w_h, \\ &\partial_1\partial_{h2}\bar{w}_h + \partial_2\partial_{h1}\bar{w}_h)] \quad , \end{aligned} \tag{8}$$

$$B_h(u_h,\bar{u}_h) = B_{h1}(u_h,\bar{u}_h) + B_{h2}(w_h,\bar{w}_h) \quad , \tag{9}$$

$$N_{h2}(w_h) = [B_{h2}(w_h,w_h)]^{1/2} = t^2/12\cdot N_1(\partial_{h1}w_h, \partial_{h2}w_h) \quad , \tag{10}$$

The finite element approximation $u_h \in X^h$ to (5) is now defined by

$$B_h(u_h,\bar{u}_h) = \sum_{i=1}^{2}(f_i,\bar{u}_{hi}) + (f_3,\bar{w}_h) \text{ for all } \bar{u}_h \in X^h . \qquad (11)$$

The difference of the above formulation from the standard one lies in the use of ∂_{hi} instead of ∂_i for the approximation of $\partial_i w$ and $\partial_i \bar{w}$ in (5).

We will employ the following hypothesis to assure the convergence of the finite element solutions.

[H1] For any $w_h \in X_2^h$, we have

$$\| \partial_{hi} w_h - \partial_i w_h \| \leqq M(h) N_{h2}(w_h) , \quad i = 1, 2, \qquad (12)$$

with

$$\lim_{h\downarrow 0} M(h) = 0 . \qquad (13)$$

[H2] For any $u = \{u_1,u_2,w\} \in X$, we can choose $\hat{u}_h = \{\hat{u}_{h1},\hat{u}_{h2},\hat{w}_h\} \in X^h$ such that

$$\sum_{i=1}^{2} \| \hat{u}_{hi} - u_i \|_1 + \| \hat{w}_h - w \|_1 \to 0 ,$$

$$\| \partial_{hi}\hat{w}_h - \partial_i w \|_1 \to 0 \quad (i = 1, 2) \quad \text{as} \quad h \downarrow 0 . \qquad (14)$$

5. Existence and Uniform Boundedness of the Finite Element Solutions

Define χ_h by

$$\chi_h(u_h,f) = B_h(u_h,u_h^*) - \sum_{i=1}^{2}(f_i,u_{hi}) - \frac{1}{2}(f_3,w_h) \quad , \tag{15}$$

where $u_h = (u_{h1},u_{h2},w_h) \in X^h$, and $u_h^* = (u_{h1},u_{h2},w_h/2)$.

We will use the Brouwer fixed point theorem to show the existence and uniqueness of the finite element solutions, cf. Lions[2]. To this end, Lemma 2 given below plays an essential role.

Lemma 1 (Korn's inequality) The forms $N_1(u_1,u_2)$ and $\|u_1\|_1 + \|u_2\|_1$ are equivalent in $H_0^1(\Omega) \times H_0^1(\Omega)$.

Lemma 2 Let u_h be an arbitrary element of X^h such that

$$N_1^2(u_{h1},u_{h2}) + N_{h2}^2(w_h) = 1 \quad . \tag{16}$$

Consider a transformation in X^h :

$$\tilde{u}_{h1} = R^2 u_{h1} \quad , \quad \tilde{u}_{h2} = R^2 u_{h2} \quad , \quad \tilde{w}_h = R\, w_h \quad , \tag{17}$$

where R is a positive number. Choosing R appropriately, $\tilde{u}_h = (\tilde{u}_{h1},\tilde{u}_{h2},\tilde{w}_h)$ satisfies

$$\chi_h(u_h,f) \geqq 0 \quad , \tag{18}$$

where R is independent of h but may be dependent on f.

Proof Use the results of [3], but we can slightly simplify that proof.

Theorem 1 The discrete problem (11) has at least one solution $u_h = (u_{h1}, u_{h2}, w_h) \in X^h$ that satisfies

$$N_1^2(u_{h1}, u_{h2})/R^4 + N_{h2}(w_h)/R^2 \leqq 1 \quad , \tag{19}$$

where R is the same as is introduced in Lemma 2. The above also implies the uniform boundedness of the present finite element solutions for h small enough.

Proof The above directly follows from Lemma 2 with the aid of Lemma 4.3 of Lions [2].

6. Convergence of the finite element solutions

We will show the convergence of (a certain sub-sequence of) the finite element solutions to (a certain) solution of (5). The estimation (19) is essential for the proof of Lemma 3 given below, while some compactness properties are needed for the proof of Lemma 4 and Theorem 2. We will deal with a family or a series of finite element spaces $\{X^h\}$ with the property $h \downarrow 0$.

Lemma 3 We can find sub-sequence of finite element solutions $\{u_h \in X^h\}$ to (11), such that, when $h \downarrow 0$,

$$u_{hi} \to u_i \quad , \quad \partial_{hi} w_h \to \partial_i w \quad (i = 1,2) \quad \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L_2(\Omega), \tag{20}$$

$$w_h \to w \quad \text{strongly in } H_0^1(\Omega) \quad , \tag{21}$$

where $u = (u_1, u_2, w)$ is an element of X.

Proof From (19), (12), and Lemma 1, we have that the finite element solutions $\{u_h \in X^h\}$ defined in Theorem 1 satisfy

$$\| u_{h1} \|_1 + \| u_{h2} \|_1 + \| \partial_{h1} w_h \|_1 + \| \partial_{h2} w_h \|_1 + \| w_h \|_1 \leqq C$$

for h small enough. Thus we can show the existence of a certain subsequence of $\{u_h\}$ such that, when $h \downarrow 0$,

$$u_{hi} \to u_i, \quad \partial_{hi} w_h \to w_i, \quad w_h \to w \quad \text{weakly in } H_0^1(\Omega), \text{ and strongly in } L_2(\Omega).$$

Here $i = 1, 2$, and u_i, w_i, and w are certain elements of $H_0^1(\Omega)$. Using (12), we also have

$$w_i = \partial_i w, \quad \text{and} \quad w_h \to w \quad \text{strongly in } H_0^1(\Omega).$$

Thus $u = (u_1, u_2, w)$ belongs to X, and the proof is completed.

Lemma 4 The element $u \in X$ introduced in the preceding Lemma is a solution to (5).

Proof To show that $u = (u_1, u_2, w)$ is an solution to (5), we should take limit of (11) after substituting the u_h taken from the subsequence defined in Lemma 3. On the other hand, $\bar{u}_h$ in (11) should be equated to $\hat{u}_h$ defined for each $\bar{u} \in X$ in [H2]. In the limiting process, the most critical step is to show

$$((\partial_{hi} w_h)^2, \partial_{hi} w_h \cdot \partial_{hi} \bar{w}_h) \to ((\partial_i w)^2, \partial_i w \cdot \partial_i \bar{w}) \quad \text{etc.}$$

But this follows from the fact

$$\partial_{hi} w_h \to \partial_i w \quad \text{strongly in } L_4(\Omega) \quad ,$$

and hence we have

$$B(u,\bar{u}) = \sum_{i=1}^{2} (f_i,\bar{u}_i) + (f_3,\bar{w}) \quad ,$$

which is nothing but the relation (5). This completes the proof.

We have now established the convergence of a certain sub-sequence of the present finite element solutions, and also the existence of the exact solution of (5). Actually, we can show convergence in a stronger sense:

Theorem 2 The sub-sequence of finite element solutions $\{u_h \in X^h\}$ introduced in Lemma 3 converges to a certain solution u of (5) in the following sense:

$$\lim_{h\downarrow 0} [\| u_{h1} - u_1 \|_1 + \| u_{h2} - u_2 \|_1 + \| w_h - w \|_1] = 0 \quad , \tag{22}$$

$$\lim_{h\downarrow 0} \| \partial_{hi} w_h - \partial_i w \|_1 = 0 \quad , \quad i = 1, 2 . \tag{23}$$

Remark Convergence of the finite element solutions as a whole cannot be expected in general, since uniqueness of the solution of (5) does not hold.

Proof For the proof, it is sufficient to show

$$N_1^2(u_{h1}-\hat{u}_{h1},u_{h2}-\hat{u}_{h2}) + N_{h2}^2(w_h-\hat{w}_h) \to 0 \quad (h \downarrow 0)$$

for the considered subsequence of $\{u_h \in X^h\}$ converging to the $u \in X$ defined in Lemma 3. Here $\hat{u}_h = (\hat{u}_{h1},\hat{u}_{h2},\hat{w}_h) \in X^h$ is the approximation of u constructed in accordance with [H2]. The details of the proof are essentially the same as are given in [5], and are omitted here. The essence is to utilize some compactness properties of nonlinear terms appearing in (7).

7. Example

We have assumed that the considered family $\{X^h\}$ of finite element spaces satisfies [H1] and [H2]. We have a few examples which actually satisfy these conditions, see Kikuchi [4]. We will illustrate two examples briefly. Since it is not difficult to find an appropriate X_1^h, we will restrict our attention to the construction of X_2^h, approximate space for w.

Example-1 shape of element = trinagle; nodes = vertices of triangle, $\{P_i\}_{i=1}^3$; subsidiary nodes = midpoints of sides, $\{Q_i\}_{i=1}^3$ (see Fig.1); element degrees of freedom = $\{w_h(P_i),\partial_1 w_h(P_i),$ $\partial_2 w_h(P_i)\}_{i=1}^3$ (w_h and its first order derivatives are forced to be continuous at least at nodes); X_2^h = Zienkiewicz's non-conforming space, or the reduced HCT space.

We define $\partial_{hj} w_h$ (j = 1,2) for each $w_h \in X_2^h$ as follows:

$\partial_{hj}w_h$ = second order polynomial of x_1 and x_2 such that

$$\partial_{hj}w_h(P_i) = \partial_j w_h(P_i) \qquad (1 \leq i \leq 3) \ ,$$

$$l_1\partial_{h1}w_h(Q_1) + m_1\partial_{h2}(Q_1)$$

$$= \frac{1}{2}\{l_1(\partial_1 w_h(P_2) + \partial_1 w_h(P_3)) + m_1(\partial_2 w_h(P_2) + \partial_2 w_h(P_3)\},$$

$$l_1\partial_{h1}w_h(Q_1) - m_1\partial_{h2}(Q_1)$$

$$= l_1\partial_1 w_h(Q_1) - m_1\partial_2(Q_1) \quad , \qquad \text{etc.,}$$

where (l_1, m_1) is the outward normal vector of the side P_2P_3. Similar relations are imposed on the other midpoints. Notice here that the tangential derivative of w_h on each side is continuous for the present finite element space. The present type of condition is called the discrete Kirchhoff assumption.

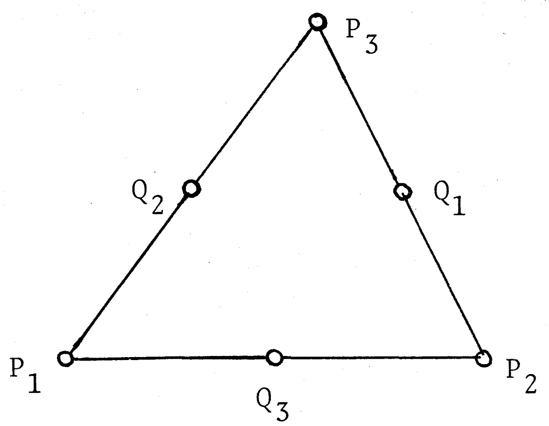

Fig. 1 Nodal configuration of the triangular element

Example 2 shape of element = triangle; nodes = $\{P_i\}_{i=1}^3$ and $\{Q_i\}_{i=1}^3$; element degree of freedom = $\{w_h(P_i), \partial_1 w_h(P_i), \partial_2 w_h(P_i)\}$ and $\{\partial w_h/\partial n(Q_i)\}$ for i = 1,2,3, where $\partial/\partial n$ implies the outward normal derivative; X_2^h = the HCT space, or the incomplete quartic space whose basis function in each element are

$$L_1^3,\ L_2^3,\ L_3^3,\ L_1^2L_2,\ L_1L_2^2,\ L_2^2L_3,\ L_2L_3^2,\ L_3^2L_1,\ L_3L_1^2,$$

$$L_1^2L_2L_3,\ L_1L_2^2L_3,\ L_1L_2L_3^2 \quad (\text{ L's are area coordinates}) .$$

In these choices, w_h and its tangential derivative are continuous slong sides of triangles, and,in particular, the HCT space is conforming. It is to be noted that the first order derivatives of w_h is continuous at nodes.

The mappings ∂_{hj} for j = 1,2 are defined as follows:

$\partial_{hj}w_h$ for each $w_h \in X_2^h$ = quadratic polynomial of x_1 and x_2 such that

$$\partial_{hj}w_h(P_i) = \partial_j w_h(P_i) \quad \text{and} \quad \partial_{hj}w_h(Q_i) = \partial_j w_h(Q_i)$$

for i = 1, 2, 3.

In the above two examples, we can prove [H1] and [H2]. We can consider similar finite element models in rectangular family.

8. Concluding remarks

We have performed convergence analysis of the discrete Kirchhoff assumption applied to large deflection analysis of flat plates. The present analysis is directly applicable to large deflection analysis of shallow shells. We can also obtain order estimates of errors of approximate solutions around normal (or non-critical) points of the original continuous problem, provided that certain regularity results are available.

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