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ELASTIC-PLASTIC VIBRATION OF A ROD

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INTRODUCTION. The following equation is a mathematical model to represent the elastic-plastic vibration of a straight uniform rod submitted to longitudinal impact.

$$\ddot{u} - \sigma_x = 0$$

$$\dot{\sigma} = \begin{cases} k\dot{u}_x & \text{in the elastic region} \\ (1-\xi)k\dot{u}_x & \text{in the plastic region,} \end{cases}$$

where ξ, k are positive constants and $0<\xi<1$ [2].

In this paper we prove that there is a unique weak solution to the initial-boundary value problem of this equation and it is obtained as a limit of the finite element solutions.

We derive a weak form of this problem and get the solution by a discretization technique.

We thus start from the vibration of a single masspoint system, then proceed to a multiple masspoint system and to the continuous case.

1. ELASTIC-PLASTIC VIBRATION OF A SINGLE MASSPOINT SYSTEM.

1.1 Equation of motion. Let us consider the vibration of a single masspoint system described by the following initial value problem.

$$(1.1) \qquad \rho\ddot{u} + \sigma = 0 \qquad \text{in } T,$$

where ρ:positive constant, $T=(0,T)$, $u(0)=0$, $\dot{u}(0)$:given and the yield displacement $\bar{u}^{(0)}$ (>0) is given.

The function σ is a continuous function and satisfies:

$$(1.1)_a \qquad \dot{\sigma} = k\dot{u} \qquad \text{if the system is elastic,}$$

$(1.1)_b$ $\qquad \dot{\sigma} = (1-\xi)k\dot{u}$ $\qquad$ if the system is plastic.

We define the states " elastic " and " plastic " as follows. First, put $\sigma = ku$. Assume that the solution of the equation (1.1) satisfies $|u(t)| = \bar{u}^{(0)}$ at $t=t_0$ for the first time. Then we define that the system is elastic for the time interval $[0,t_0)$. (Note that the definition of " elastic " is not independent of the solution of the intial value problem. These are determined at the same time.) If $\dot{u}(t_0) \neq 0$, then the system is defined to be plastic for $t \geqslant t_0$. If $\dot{u}(t_0) = 0$, the state for $t \geqslant t_0$ is determined by the following check. <u>(ABC)-check</u> :

(A) If $\dot{u}$ converges to 0 from above:

(1) if $\ddot{u}(t_0) \geqslant 0$ then plastic for $t \geqslant t_0$,

(2) if $\ddot{u}(t_0) < 0$ then elastic for $t \geqslant t_0$ (case(A)),

(B) If $\dot{u}$ converges to 0 from below:

(1) if $\ddot{u}(t_0) > 0$ then elastic for $t \geqslant t_0$ (case(B)),

(2) if $\ddot{u}(t_0) \leqq 0$ then plastic for $t \geqslant t_0$.

(C) If $\dot{u}$ converges to 0 vibrating. Plastic for $t \geqslant t_0$.

(REMARK: If $\dot{u}(t)=\ddot{u}(t)=0$ at $t=t_0$ then all $\overset{(k)}{u}$ $(k \geqslant 3)$ vanish at $t=t_0+0$ independently of the state for $t \geqslant t_0$, so that $u=u(t_0)$, $\sigma=0$ is the only possible solution for $t \geqslant t_0$. Observe that this is a formal classification for logical consistency.)

Subsequent states of the system is determined recursively as follows.

(I) The case when the present state is plastic. Assume that the present state began at $t=t_m$, and the solution of the equation (1.1) satisfies $\dot{u}(t)=0$ at $t=t_{m+1}(>t_m)$ for the first time. Then we define the system is plastic for the time interval $[t_m, t_{m+1})$. The state for $t \geqslant t_{m+1}$ is determined by the (ABC)-check.

(II) The case when the present state is elastic. Assume that the present state began at $t=t_m$ after the ABC-check.

(1) Case(A). Assume that the solution u satisfies

$$u(t) \leq u(t_m) - 2\bar{u}^{(0)},$$

or

$$u(t) = u(t_m)$$

at $t=t_{m+1}(>t_m)$ for the first time. Then we define the system is elastic for the time interval $[t_m, t_{m+1})$.

(2) Case(B). Assume that the solution u satisfies

$$u(t) \geqslant u(t_m) + 2\bar{u}^{(0)},$$

or

$$u(t) = u(t_m)$$

at $t=t_{m+1}(>t_m)$ for the first time. Then we define the system is elastic for the time interval $[t_m, t_{m+1})$.

The state for $t \geqslant t_{m+1}$ is determined as

$(II)_a$ if $\dot{u} \neq 0$, the system is plastic for $t \geqslant t_{m+1}$,

$(II)_b$ if $\dot{u} = 0$, the state for $t \geqslant t_{m+1}$ is determined by the (ABC)-check.

Our initial value problem is well defined by the above procedure and has a unique C^2-class solution. The hardening in the above rule corresponds to the kinematic hardening.

1.2 Energy of the single masspoint system. Let $u^{(j)}$ $(j=0,1,2,\ldots)$ be the displacement at which the (j+1)-th change of the state occurs. We say that the system is at stage(m) if the change of the state occured m+1 times in the past, so that $u^{(m)}$ is the displacement at which the (m+1)-th change occured.

The key to derive an energy equality for the present problem and also to develope our theory in this paper is the following theorem which represents the initial value problem by a single equation.

THEOREM 1.1. The equation (1.1) is represented as follows, if the system is at stage(m).

(1.2) $$\rho \ddot{u} + ku - \zeta k \sum_{0}^{m} (-1)^{j} (u - u^{(j)}) = 0.$$

PROOF Use induction on m.

From this equation we can easily derive a simple energy equality which represents the non-conservation of energy.

THEOREM 1.2. Let E_m be defined by

(1.4) $$E_m(t) = \frac{\rho}{2}(\dot{u})^2 + \frac{k}{2}u^2 - \frac{\zeta k}{2} \sum_{0}^{m} (-1)^{j} (u - u^{(j)})^2.$$

Then the following equality holds at stage(m).

$$E_m(t) = E_0,$$

where E_0 is the initial energy.

(REMARK: This theorem implies that the elastic-plastic vibration converges to an elastic vibration as $t \to \infty$. This is the case also for the multiple masspoint system considered later. See[4].)

1.3 A weak form of the equation of motion. The initial value problem formulated above can be represented simply as follows.

THEOREM 1.3 The initial value problem of the single masspoint system is equivalent to the following problem: Define

$$K = K_\alpha = \{ \tau \in C(T) ; \ |\tau - \alpha| \le \sigma_0 \ \text{ for any } t \in T \} ; \quad \sigma_0 = k\bar{u}^{(0)}.$$

Seek v, σ, α which are differentiable and satisfy, for all $t \in T$,

$$
(1.5)\quad \left\{
\begin{array}{ll}
(\dot{\sigma} - kv, \tau - \sigma) \geqslant 0 & \text{for all } \tau \in K \\
\dot{\alpha} = (1 - \frac{1}{\xi})(\dot{\sigma} - kv) & \\
\dot{v} + \sigma = 0, &
\end{array}
\right.
$$

and $\sigma \in K$, $\sigma(0)=0$, $\alpha(0)=0$, $v(0)=\dot{u}(0)$

REMARK: $(x,y)=xy$ in this case. This is for the generalization of our method to more complicated problems.

PROOF OF THE THEOREM. It is easy to see that α is the parameter representing the movement of the center of the yield surface. We thus prove the uniqueness of the solution for (1.5). Substitute the second equation into the first inequality. We then have

$$(\dot{\alpha}, \tau - \sigma) \quad 0 \quad \text{for any } \tau \in K .$$

Let θ be an arbitrary continuous function satisfying $|\theta| \leq 1$. Then the function of the form

$$\tau = \alpha + \sigma_0 \theta$$

is included in K. Therefore we have

$$(1.6)\qquad (\dot{\alpha}, \alpha + \sigma_0 \theta - \sigma) \leq 0 \qquad \text{for any such } \theta .$$

Assume that there is another solution $(v_*, \sigma_*, \alpha_*)$. obviously

$$(1.7)\qquad (\dot{\alpha}_*, \alpha_* + \sigma_0 \theta - \sigma_*) \leq 0 \qquad \text{for any such } \theta .$$

Put $\theta = (\sigma_* - \alpha_*)/\sigma_0$ in (1.6) and $\theta = (\sigma - \alpha)/\sigma_0$ in (1.7), and add the both inequalities. Then we have

$$(\dot{\alpha} - \dot{\alpha}_*, \alpha - \alpha_* - [\sigma - \sigma_*]) \leq 0.$$

By using two equations of (1.5), we have

$$(\dot{\alpha}-\dot{\alpha}_*,\sigma-\sigma_*) = (1-\frac{1}{\xi})[\ \frac{1}{2}\|\sigma-\sigma_*\|_t^2 + \frac{k}{2}\|v - v_*\|_t^2\ ].$$

Hence we have

$$\|\alpha-\alpha_*\|^2 - (1-\frac{1}{\xi})[\|\sigma-\sigma_*\|^2 + k\|v-v_*\|^2] \leq 0,$$

which implies $\alpha=\alpha_*$, $\sigma=\sigma_*$ and $v=v_*$. This completes the proof.

2. ELASTIC-PLASTIC VIBRATION OF A MULTIPLE MASSPOINT SYSTEM

2.1 Equation of motion. All results obtained in the preceding sections are extended formally to the multiple system.

Let $u_i (i=0,1,\ldots,N)$ be the displacement of the i-th masspoint (we assume $u_0=0$). Let ρ_i, k_i and ξ_i be the mass, stiffness and plasticity factor of the i-th masspoint. We introduce the quantity $U_i = u_i - u_{i-1}$ which corresponds to the strain at i-th point. Then the equation of motion of this system is written as

$$(2.1) \qquad \rho_i \ddot{u}_i + \sigma_i(U_i) - \sigma_{i+1}(U_{i+1}) = 0 \qquad i=1,2,\ldots,N,$$

where $\sigma_i(U_i)(i=1,2,\ldots,N+1)$ is a continuous function of t such that

$$(2.1)_a \qquad \dot{\sigma}_i(U_i) = k_i\dot{U}_i \qquad \text{if the i-th point is elastic,}$$

$$(2.1)_b \qquad \dot{\sigma}_i(U_i) = (1-\xi_i)k_i\dot{U}_i \qquad \text{if the i-th point is plastic,}$$

and $\sigma_{N+1}=0$. Definition of "elastic" and "plastic" is exactly the same as in the single system, except the case when both $\dot{U}_i$ and $\ddot{U}_i$ vanish at $t=t_0$ and we can not determine the state for $t \geq t_0$. In the single system, we do not have to bother about such problem since if it should happen, $u(t)=u(t_0), \sigma=0$ is the only possible solution for $t \geqslant t_0$.

The situation is,however,almost the same in the multiple system too. In fact,we have the following theorem.

THEOREM 2.1. Let U_i, σ_i be the solution of the intial value problem for (2.1). Assume that at $t=t_0$ the points $(U_i(t), \sigma_i(t))$ $(i=i_1,\ldots,i_r)$ reach to one of the lines

$$\sigma_i(U_i) = k_i U_i - \zeta_i k_i (U_i \pm \bar{U}_i^{(0)}) \tag{2.2}$$

in the $(U_i, \sigma_i(U_i))$ plane respectively,where $\bar{U}_i^{(0)}$ (>0) denotes the yield strain of the i-th masspoint, and moreover that

$$\overset{(\ell)}{U_i}(t_0+0) = 0 \qquad (1 \leq \ell \leq k) \qquad (k \geqslant 2)$$

for $i=i_1,\ldots,i_r$. Then $\overset{(k+1)}{U_i}(t_0+0)$ for such i is determined independently of the form of $\dot{\sigma}_i (i=i_1,\ldots,i_r)$ for $t \geqslant t_0$,provided that the other σ_i's are already decided for $t \geqslant t_0$.

PROOF Use induction on k, considering the facts that $\overset{(k-1)}{\sigma_j}(t_0+0)=$ constant . $\overset{(k-1)}{U_j}(t_0+0)=0$ if j is included in $(i_1,\ldots,i_r)$ and for all i

$$\overset{(k+1)}{U_i} = \frac{1}{\rho_i}\left(\overset{(k-1)}{\sigma}_{i+1} - \overset{(k-1)}{\sigma}_i\right) - \frac{1}{\rho_{i-1}}\left(\overset{(k-1)}{\sigma}_i - \overset{(k-1)}{\sigma}_{i-1}\right),$$

and that the sign of the non-vanishing lowest $\overset{(k)}{U_i}(t_0+0)$ can determine the state of the i-th point for $t \geqslant t_0$,since the point $(U_i(t), \sigma_i(t))$ in the $(U_i, \sigma_i(U_i))$ plane moves at $t=t_0$ to a definite direction determined by this sign.

Applying this theorem repeatedly,we can determine the state of each point for $t \geqslant t_0$,exept very special cases. If $\overset{(k)}{U_i}$ should vanish for any k at $t=t_0+0$,how should we determine the subsequent

state? In this case we can define free for $t \geqslant t_0$ (we thus define that the point is plastic).
The reason is this: The solution $\{U_i\}$ is analytic until some point j ($\neq$i) changes its state. Therefore, $U_i^{(k)}(t_0+0)=0$ (for any $k \geqslant 1$) implies that U_i=constant until that time, which implies the solution does not depend on if we define the i-th point to be plastic or elastic.

Our problem is hence well posed and has a unique C^2-class solution for any $t > 0$.

2.2 Energy of the multiple masspoint system. To derive an energy form for the multiple system, we define that the i-th masspoint is at stage(m_i) as the same way in the single system, replacing u and by U_i and $\sigma_i(U_i)$ (REMARK: There is a formal possibility that there exist infinitely many changes of the state in finite time interval. In this case the points $(U_i^{(j)}, \sigma_i(U_i^{(j)}))$ ($j=1,2,\ldots$) have an accumulation point on the line defined (2.2) ,without making any hysteresis loop. For the followings, however, we can assume without loss of generality that the number of the state change is finite in finite time interval, since, if such accumulation should happen, we can skip all stages near the accumulation point in numbering the stage. Note that these stages give no influence on both the equation and the energy form for $t \geqslant t_0$.

We say that the sysyem is at stage(m) ($m=(m_1,\ldots,m_N)$), if the i-th masspoint is at stage(m_i). Corresponding to Theorem 1.1 and Theorem 1.2, we have the following two theorems.

THEOREM 2.2 At stage(m) the equation of motion of the multiple masspoint system is represented as follows.

$$\rho_i \ddot{u}_i + [k_i U_i - \xi_i k_i \sum_{j=0}^{m_i} (-1)^j (U_i - U_i^{(j)})]$$

$$(2.3) \qquad - [k_{i+1} U_{i+1} - \xi_{i+1} k_{i+1} \sum_{j=0}^{m_{i+1}} (-1)^j (U_{i+1} - U_i^{(j)})] = 0, \quad (i=1,\dots,N)$$

where $U_i^{(j)}$ denotes the displacement of the i-th point when it enters in stage(m_i).

THEOREM 2.3 Let $E_m(t)$ be defined by

$$E_m(t) = \frac{1}{2} \sum_{1}^{N} [\rho_i (\dot{u}_i)^2 + k_i U_i^2 - \xi_i k_i \sum_{j=0}^{m_i} (-1)^j (U_i - U_i^{(j)})^2].$$

Then the following equality holds at stage(m).

$$E_m(t) = E_0,$$

where E_0 is the initial energy.

2.3 A weak form of the equation of motion. Hereafter we assume that k_i, ξ_i and $\bar{U}_i^{(0)}$ are constants and denote them by k, ξ and $\bar{U}^{(0)}$ respectively. As in the single system,the present problem can be represented by a weak form including an inequality.

Let v, σ and α be N-dimensional vector functions which are differentiable in $t \geq 0$. Let K be a set of N-dimensional vector functions which are continuous and within σ_0-neighborhood of α, that is,

$$K = K_\alpha = \{ \tau \in C(T)^N ; \ \max_i |\tau_i - \alpha_i| \leq \sigma_0 \ \text{ for any } t \in T \} ; \ \sigma_0 = k\bar{U}^{(0)}.$$

THEOREM 2.4 The initial value problem of the multiple masspoint system is equivalent to the following problem:

Seek v, σ, α which are differentiable and satisfy

$$(2.4)\quad \begin{cases} (\dot{\sigma} - k\dot{U}, \tau - \sigma)_{E^N} \geq 0 & \text{for any } \tau \in K, \\ \dot{\alpha} = (1 - \frac{1}{\zeta})(\dot{\sigma} - k\dot{U}) \\ \rho_i \dot{v}_i + \sigma_i - \sigma_{i+1} = 0 & i = 1, \ldots, N, \end{cases}$$

and $\sigma \in K, \sigma(0)=0, \sigma_{N+1}=0, \alpha(0)=0, v(0)=\dot{u}(0), v_0=0$, where $\dot{U}_i = v_i - v_{i-1}$.

2.4 Energy inequality (1). To prove the existence of a solution for the continuous problem, Duvaut-Lions [/] uses a penalization technique, introducing an elasto-visco-plastic problem. In our problem, however, we can directly get the required estimates from this discrete problem. First, a basic estimate is obtained as follows. Put $\tau = \alpha$ in the inequality of (2.4). Substituting the equation for α, we get

$$0 \leq \eta(\dot{\sigma} - k\dot{U}, \int_0^t [\dot{\sigma} - k\dot{U}]dt) - (\dot{\sigma}, \sigma) + k(\dot{U}, \sigma).$$

$$(\eta = 1 - \frac{1}{\zeta})$$

Since $(\dot{U}, \sigma) = \sum (v_i, \sigma_i - \sigma_{i+1}) = -\sum \rho_i (v_i, \dot{v}_i) = -1/2 \sum \rho_i (v_i)_t^2$, we have,

$$(-\frac{1}{2\eta})\|\alpha\|^2 + \frac{1}{2}\|\sigma\|^2 + \frac{k}{2}\sum \rho_i (v_i)^2$$

$$\leq \frac{k}{2}\sum \rho_i v_i^2(0)$$

2.5 Energy inequality (2). We shall estimate some "derivatives". We can derive first the following estimate.

$$(2.6)\quad \sum [\frac{\rho_i}{2}(\dot{v}_i)^2 + \frac{k}{2}(1 - \zeta)\dot{U}_i^2](t) \leq \sum [\frac{\rho_i}{2}(\dot{v}_i)^2 + \frac{k}{2}\dot{U}_i^2](0)$$

Once this estimate is obtained, other derivatives are easily estimated. The result is as follows. Let $\bar{E}_0$ be the quantity of the right side of (2.6). Then

$$\sum \dot{\sigma}_i^{\,2} \leq 2k\bar{E}_0,$$

$$(2.7) \qquad \sum (\sigma_{i+1} - \sigma_i)^2 \leq 2\mathrm{Max}\,\rho_i \bar{E}_0,$$

$$\sum \dot{\alpha}_i^{\,2} \leq \frac{8(1-\xi)k}{\xi^2}\,\bar{E}_0.$$

3. ELASTIC-PLASTIC VIBRATION OF A ROD

3.1 A weak form of the equation of motion. We introduce a weak form of the original problem and show that this form is a natural extension of the equation of motion of the multiple masspoint system considered in the previous sections. Let $\Omega = (0,1)$ and define

$$K = K_\alpha = \{ \tau \in L^\infty(T;L_2(\Omega));\ \text{a.e. } T, |\tau - \alpha| \leq \sigma_0 \ \text{ a.e. } \Omega \}$$

for $\alpha \in L^\infty(T;L_2(\Omega))$. Then our problem is:

Find (v,σ,α) such that

$$v, \sigma \in L^\infty(T;W_2^1(\Omega)), \qquad \dot{v}, \dot{\sigma}, \alpha, \dot{\alpha} \in L^\infty(T;L_2(\Omega))$$

and a.e. T,

$$(3.1) \qquad \left\{ \begin{aligned} & (\dot{\sigma} - kv_x, \tau - \sigma)_{L_2(\Omega)} \geq 0 \qquad \text{for any } \tau \in K, \\ & \dot{\alpha} = (1 - \tfrac{1}{\xi})(\dot{\sigma} - kv_x) \\ & \dot{v} - \sigma_x = 0, \end{aligned} \right.$$

where $\sigma \in K$, $v(0,x)=a(x)$:given, $v(t,0)=0$, $\sigma(0,x)=0$, $\sigma(t,1)=0$, $\alpha(0,x)=0$

It is evident that if the original problem has a classical solution,then $v=\dot{u}$ and σ satisfy this equation, α being the parameter representing the center of the yield surface .

3.2 Finite element approximations. We use the finite element solutions and take limit to get the solution for (3.1).

We first devide the interval Ω into N elements of equal length h. Let i be the point with coordinate $i*h(i=0,1,..,N)$ and e_i the element $[(i-1)*h,i*h]$. We use three basis functions:

$\hat{\varphi}_i(x)$: piecewise linear basis,

$\bar{\varphi}_i(x)$: characteristic function of $[i*h-\frac{h}{2},i*h+\frac{h}{2}]$,

$\bar{\bar{\varphi}}_i(x)$: characteristic function of element e_i.

Then one of the simplest finite element approximation to the original problem is

$$(3.2)\quad \left\{\begin{array}{ll} (\ddot{\bar{u}}, \bar{\varphi}_i) + \sum_e (\sigma(\hat{u}), \hat{\varphi}_{i,x})_e = 0 & i=1,..,N \\ \dot{\sigma}(\hat{u}) = \begin{cases} k\dot{\hat{u}}_x & \text{in the elastic region} \\ (1-\zeta)k\dot{\hat{u}}_x & \text{in the plastic region.} \end{cases} \end{array}\right.$$

where

$$\bar{u} = \sum_{i=1}^{N} u_i(t)\,\bar{\varphi}_i(x)$$

$$\hat{u} = \sum_{i=1}^{N} u_i(t)\,\hat{\varphi}_i(x),$$

and $u_i(0)=0, \dot{u}_i(0)=a(ih)$. ($a(x)$ is assumed to be smooth and $a(0)=0$).

Since σ is constant on each element, we put

$$\sigma_i = \sigma(\hat{u})\big|_{e_i},$$

to get

$$\sum_e (\sigma(\hat{u}), \hat{\varphi}_{i,x})_e = \sigma_i - \sigma_{i+1} \qquad (\sigma_{N+1} = 0).$$

Then the finite element equation (3.2) is written as

$$(3.3)\quad \left\{\begin{array}{ll} \rho_i\ddot{u}_i + \sigma_i - \sigma_{i+1} = 0 & i=1,..,N \\ \dot{\sigma}_i = \begin{cases} \frac{k}{h}(\dot{u}_i - \dot{u}_{i-1}) & \text{for the elastic element} \\ (1-\zeta)\frac{k}{h}(\dot{u}_i - \dot{u}_{i-1}) & \text{for the plastic element,} \end{cases} \end{array}\right.$$

where ρ_i = h for $i \neq N$ and = h/2 for $i=N, u_0=0$, which is just the same equation considered in the previous sections.

3.3 Convergence of the finite element solutions.

Define

$$\bar{\bar{\sigma}} = \sum_{i=1}^{N} \sigma_i \bar{\bar{\varphi}}_i , \qquad \hat{\sigma}_* = \sum_{i=0}^{N} \sigma_{i+1} \hat{\varphi}_i .$$

According to Theorem 2.4, we can rewrite the system (3.3) as

$$(3.4) \quad \begin{cases} (\dot{\bar{\bar{\sigma}}} - k\hat{v}_x, \bar{\bar{\tau}} - \bar{\bar{\sigma}})_{L_2} \geq 0 & \text{for any } \bar{\bar{\tau}} \in \bar{\bar{K}}, \\ \dot{\bar{\bar{\alpha}}} = (1 - \frac{1}{\xi})(\dot{\bar{\bar{\sigma}}} - k\hat{v}_x) & \\ (\dot{\bar{v}}, \bar{\varphi}_i) + (\bar{\bar{\sigma}}, \hat{\varphi}_{i,x}) = 0 & i=1,..,N \end{cases}$$

where

$$\bar{\bar{K}} = \{ \bar{\bar{\tau}} \; ; \; \operatorname{Max}_{\Omega} | \bar{\bar{\tau}} - \bar{\bar{\alpha}} | \leq \sigma_0 \quad \text{for any } t \in T \}.$$

Thus we can take limit by the usual procedure and finally have

<u>THEOREM 3.1</u> There exists a unique solution (v, σ, α) for (3.1), which is the limit of the finite element solutions and, at the same time, regarded as the limit of the solutions for the multiple masspoint systems.

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