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DYNAMIC BUCKLING LOADS OF SHALLOW STRUCTURES AND COMPARISON OF BACKBONE CURVES RESULTED BY FIVE NUMERICAL METHODS

by

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1. INTRODUCTION

Shallow structures like arches and spherical shells subjected to static loadings have load-deflection curves of the type shown in Fig.1. Governing equations of these curves contain quadratic terms, together with cubic ones. These quadratic terms make the nonlinear characteristic unsymmetrical and cause snap-through or direct snapping phenomena in the static and dynamic buckling problems. Therefore, it is the first step for treating the stability of shallow structures to examine the characteristic,e.g., backbone curve and response curve, of the nonlinear equations of motion with the gradratic and cubic terms. We have many analytical methods of nonlinear equations of motion and are often puzzled to select the suitable method for the problem to be analyzed. This is one of the reasons that we compared, in section three of the present paper, five kinds of analytical methods by numerically constructing backbone curves.

It is well known that the dynamic buckling loads of shallow structures are considerably lower than the static values, though exceptional cases exist [1], and considerable attention has been paid to determine the reliable critical values. As for the analytical methods for the dynamic buckling loads, the following three kinds of methods have usually been in use: (1) Numerical Integration Method [2], (2) Infinitesimal Stability Analysis [1] and (3) Energy Criterion, including a Lyapunov type of approach [3]. The next section deals with the third method to determine the dynamic buckling loads of the shallow structures due to rectangular pulse.

2. DYNAMIC BUCKLING LOADS DUE TO RECTANGULAR PULSE

Energy criteria, based on the observation of the total potential energy surfaces, have been used to determine the lower bound of the dynamic buckling load under impulse and step loadings. Here let us apply the above energy criterion to the dynamic buckling problem of shallow structure under a rectangular pulse and make a relation between the critical load level and time duration of the rectangular loading.

Let us consider the nonlinear equations of motion:

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{K}\mathbf{d} + \mathbf{g}(\mathbf{d}) = \mathbf{q} \tag{1}$$

where $\mathbf{M}$, $\mathbf{K}$, $\mathbf{d}$, $\mathbf{g}(\mathbf{d})$ and $\mathbf{q}$ are mass matrix, stiffness matrix, displacement vector, nonlinear terms and load vector, respectively. If we introduce the function

$$V = \dot{\mathbf{d}}^T \mathbf{M} \dot{\mathbf{d}} + \mathbf{d}^T \mathbf{K} \mathbf{d} + 2\,U(\mathbf{d}), \quad \frac{\partial U}{\partial \mathbf{d}} = \mathbf{g}(\mathbf{d}) \tag{2}$$

which is proportional to the total energy, $\dot{V} \equiv \frac{dV}{dt}$ along the solution of Eq.(1) takes the form

$$\dot{V} = 2\,\dot{\mathbf{d}}^T \mathbf{q}. \tag{3}$$

From the schwartz inequality,

$$\begin{aligned} \dot{\mathbf{d}}^T \mathbf{q} = \dot{\mathbf{d}}^T \mathbf{M}^{-1} \mathbf{M} \mathbf{q} &\leqq \{\mathbf{q}^T \mathbf{M}^{-1} \mathbf{q}\}^{\frac{1}{2}} \{\dot{\mathbf{d}}^T \mathbf{M} \dot{\mathbf{d}}\}^{\frac{1}{2}} \\ &\leqq \mu(t)\, V^{\frac{1}{2}}(t) \end{aligned} \tag{4}$$

with $\mu(t) = \{\mathbf{q}^T \mathbf{M}^{-1} \mathbf{q}\}^{\frac{1}{2}}$. Inserting Eq.(3) into Eq.(4) leads to

$$\dot{V}(t) \leqq 2\mu(t)\, V^{\frac{1}{2}}(t) . \tag{5}$$

Integration of Eq.(5) yields

$$V^{\frac{1}{2}}(t) \leqq V^{\frac{1}{2}}(0) + \int_0^{t_d} \mu(t)\, dt . \tag{6}$$

If the total energy at $t = t_d$ does not exceed the minimum value of the relative maximum points or saddle points near the initial state on the strain energy curve of $q = 0$, the system is stable [4]. Therefore, when $\mathbf{d}^*$ represents the displacement vector which gives the minimum value of the relative maximum points or saddle points, the sufficient condition for stability can be written as

$$V^{\frac{1}{2}}(0) + \int_0^{t_d} \mu(t)\, dt \leqq [\, \mathbf{d}^{*T} \mathbf{K}\, \mathbf{d}^* + 2\, U(\mathbf{d}^*)]^{\frac{1}{2}} . \tag{7}$$

Since $\mathbf{d}^*$ corresponds to the first critical equilibrium point in the vicinity of the initial state, the components of $\mathbf{d}^*$ can be numerically obtained by using the static load-deflection curves.

In the case of the rectangular pulse with the time duration of t_d , Eq.(7) becomes

$$\mu \leqq [\, \mathbf{d}^{*T} \mathbf{K}\, \mathbf{d}^* + 2\, U(\mathbf{d}^*)]^{\frac{1}{2}} / t_d \tag{8}$$

which gives the relation between the critical load level and time duration, as the critical load level q_{cr} is determined from

$$\mu_{cr} = \{ \mathbf{q}^T \mathbf{M}^{-1} \mathbf{q} \}^{\frac{1}{2}} . \tag{9}$$

For the illustrative example, let us consider the shallow arch as shown in Fig.2[5]. Fig.3 and 4 show the relation between the dynamic critical load level f_d and the time duration parameter t_d/t_o (t_o :the natural period) for the shape parameter $H = 4$ and $H = 8$, respectively. The present results are good agreement with the numerical integration results by means of the Runge-Kutta-Gill method. Fig.5 and 6 represent the wave forms at the positions of ■ and ▲ in Fig.3 and 4, respectively. These numerical results first appeared in the reference of [5].

3. COMPARISON OF NUMERICAL ANALYSIS OF THE NONLINEAR EQUATION OF MOTION

The purpose of this section is to examine the characteristics of five analytical methods of the nonlinear equation of motion, comparing the numerical results for the backbone curves.

As an illustrative model for the nonlinear equation of motion, we adopt a shallow sinusoidal arch supported with hinges (Fig.2.). Restricting the deformation to the first characteristic mode with no damping, we get the equation of motion with the quadratic term:

$$\ddot{x} + \left(1+\frac{H^2}{2}\right)x - \frac{3H}{4}x^2 + \frac{1}{4}x^3 = 0 \qquad (10)$$

where H is the same shape parameter as used in section 2. Introducing the nondimensional parameters, τ and ξ , as

$$\tau = \sqrt{1+\frac{1}{2}H^2}\; t \quad , \qquad x = 2\sqrt{1+\frac{1}{2}H^2}\;\xi \qquad (11)$$

we obtain the nondimensionalized equation of motion:

$$\ddot{\xi} + \xi + \epsilon\xi^2 + \xi^3 = 0 \qquad (12)$$

in which $\epsilon = -3H \big/ 2\sqrt{1+\frac{1}{2}H^2}$.

In the numerical calculation, we adopt two cases of $H=3$ and $H=8$, which correspond to the shallow and deep arch, respectively, from the standpoint of the static buckling problem.

As five approximate methods, we adopt (1) Averaging method and its improved first order approximation, (2) Bogoliuboff-Mitropolsky's asymptotic method, (3) Perturbation method,

(4) Duffing's iteration method and (5) Harmonic balance method. For the perturbation method, we deal with four types of power series expansion, that is in the neighborhood of ω_0 and ω_0^2 , and in the vicinity of ω and ω^2 . The notations ω_0 and ω denote the linear eigen frequency and the unknown frequency in the nonlinear system, respectively.

In order to compare the numerical results from each method, it is necessary to conform the correct solution. However, as it is impossible to derive the rigorous solution of Eq.(12), the results by the harmonic balance method will be used for the comparative study. In the numerical calculation by using the harmonic balance method for Eq.(12), it is found that the Fourier series converges very rapidly and sufficient accuracy is obtained by taking only up to second term (see Figs.7 and 14). Here ten waves approximation is used and the resulted backbone curve is depicted in Fig. 7

Fig.8 through 14 show the results by the asymptotic method, perturbation methods of four types, Duffing's iteration method and harmonic balance method of two waves approximation, respectively. The result by the averaging method is not shown because of its unusual error. From these figures, the following comments will be given.

Every method Possesses a relatively good precision in the neighborhood of the linear oscillation, that is $0.95 \leqq \omega \leqq 1.0$. The equation of motion (Eq.10) with quadratic terms has both the softening and hardening properties. The present results show that the numerical methods which can reflect the above combined properties are only the Duffing's method and the harmonic balance method. And for the case with a large quadratic terms of $H=8$, the harmonic balance method only catches the nonlinear characteristics of the very large amplitude oscillation.

In a concluding remark, the numerical results explain that when we apply the perturbation method, asymptotic method, etc. to the nonlinear equation of motion with both the quadratic and cubic terms, it is necessary to introduce a technique which can express the shift between the hardening and softening properties.

The detailed procedures and discussions will appear to "Bulletin of Earthquake Resistant Structure Research Center , the Institute of Industrial Science, University of Tokyo, No.12, Dec., 1978".

References

[1] Lock,M.H. "Snapping of a Shallow Sinusoidal Arch under a Step Pressure Load," AIAA Journal, Vol.4, No.7, July, 1966.
[2] Ball,R.E. and Burt,J.A. "Dynamic Buckling of Shallow Spherical Shells," Journal of the Applied Mechanics, June, 1973.
[3] Plant,R.H. and Infante,E.F. "Bounds on Motions of some Lumped and Continuous Dynamic Systems," Journal of the Applied Mechanics, Dec., 1970.
[4] Hus,C.S. "On Dynamic Stability of Elastic Bodies With Prescribed Initial Conditions," International Journal of Engineering Science, Vol.4, 1966.
[5] Hangai,Y. and Matsui,N. "Dynamic Buckling under Step-Loading and Time Duration," Pre- Print of the Annual Meeting of the Architectural Institute of Japan, Oct., 1975 (in Japanese).
[6] Hayashi,C. "Nonlinear Oscillations in Physical Systems," McGraw-Hill, 1964.
[7] Meirovitch,L. "Methods of Analytical Dynamics," McGraw-Hill, 1970.
[8] Maesawa,S. "Nonlinear Ordinary Differential Equations," Diamond, 1969.
[9] Bogoliuboff,N.N. and Mitropolsky,Y.A. "Asymptotic Methods in the Theory of Non-linear Oscillations," Gordon & Breach, 1961.
[10] Evan-Iwanowski,R.M. "Resonance Oscillations in Mechanical Systems," Elsevier, 1976.
[11] Goto,H. and Hangai,Y. "Comparison of the Approximate Methods of Nonlinear Equations of Motion," Pre- Print of the Kanto Meeting of AIJ, July, 1978.

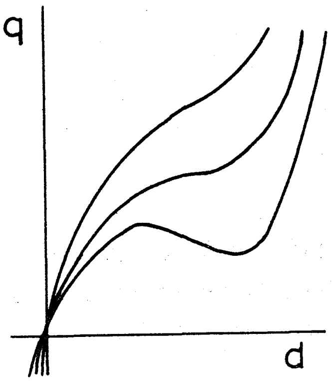


Fig.1: Typical Load-Deflection Curves

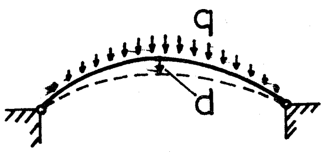


Fig.2: Shallow Arch

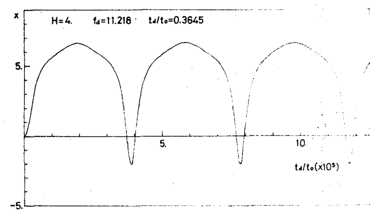
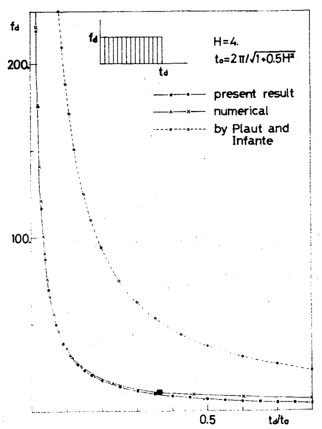


Fig.5: Waveform in H=4



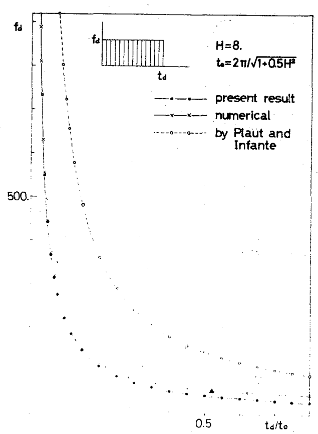


Fig.3: Critical Load for H=4 Fig.4: Critical Load for H=8

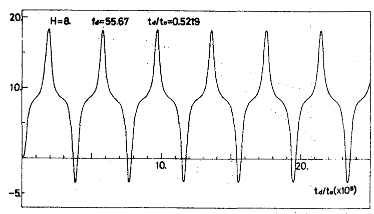


Fig.6: Waveform in H=8

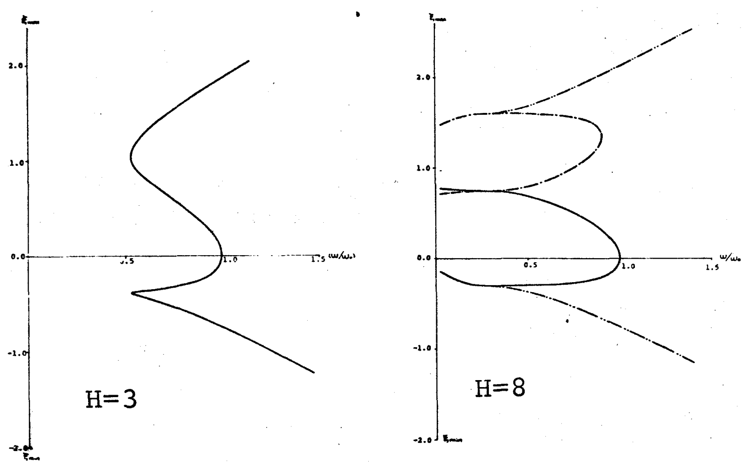


Fig.7: Backbone Curves by Harmonic Balance Method (Cosine 10 Waves App.)

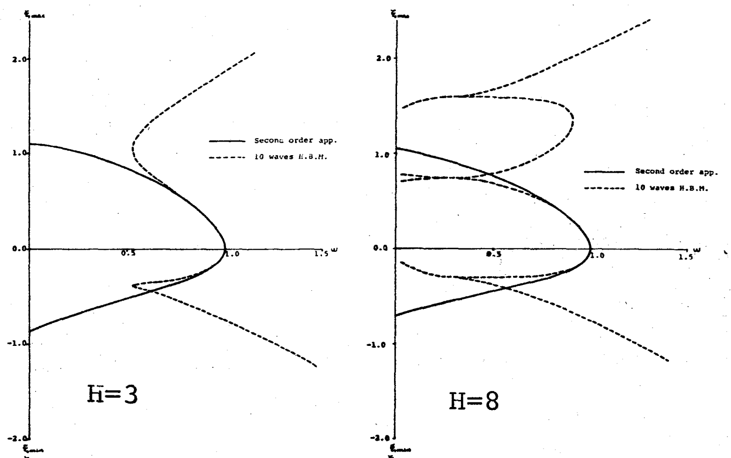


Fig.8: Backbone Curves
by Bogoliuboff-Mitropolsky's
Asymptotic Method

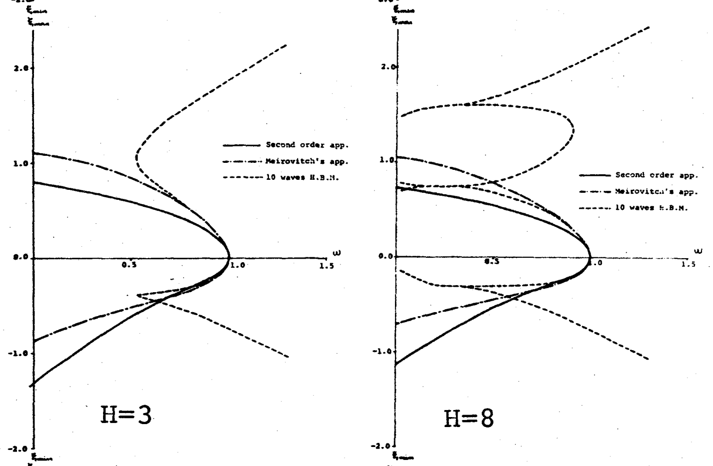


Fig.9: Backbone Curves
by Perturbation Method (No.1)

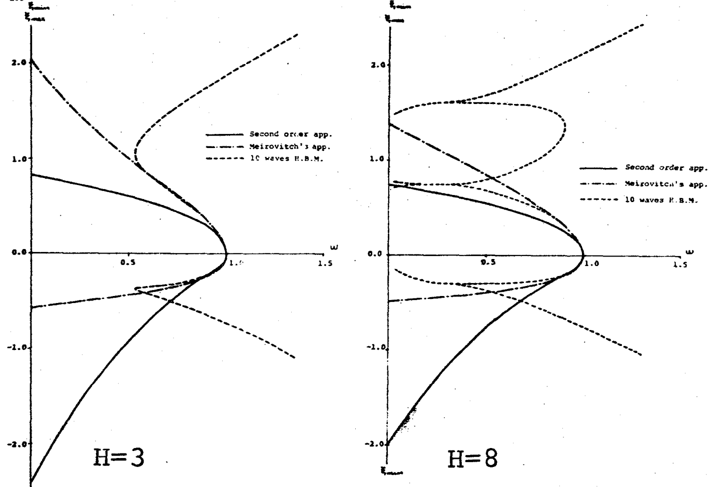


Fig.10: Backbone Curves
by Perturbation Method (No.2)

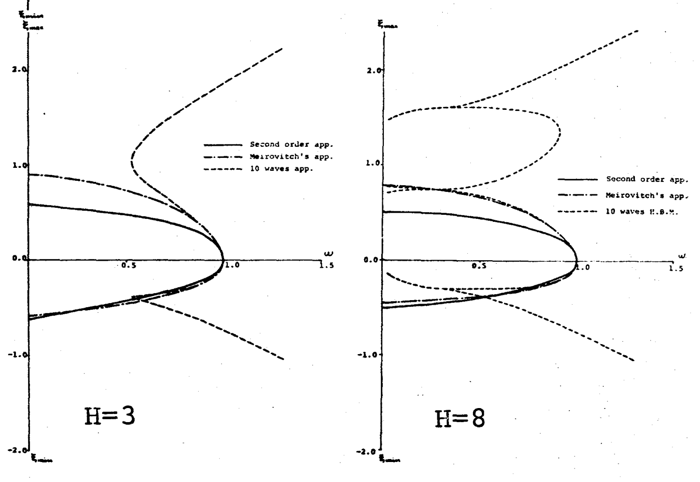


Fig.11: Backbone Curves
by Perturbation Method (No.3)

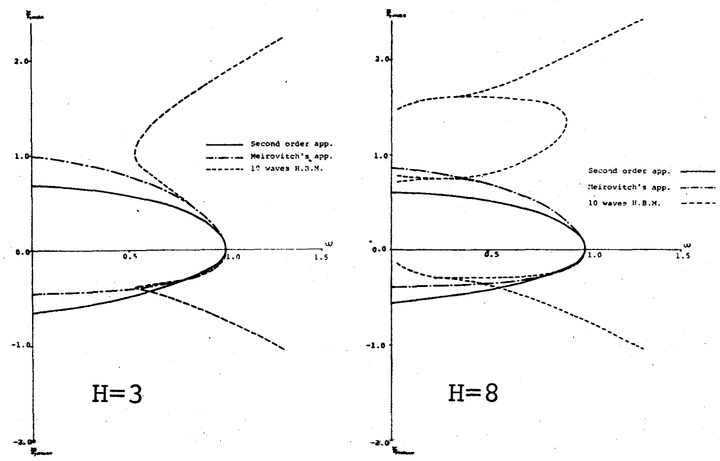


Fig.12: Backbone Curves
by Perturbation Method (No.4)

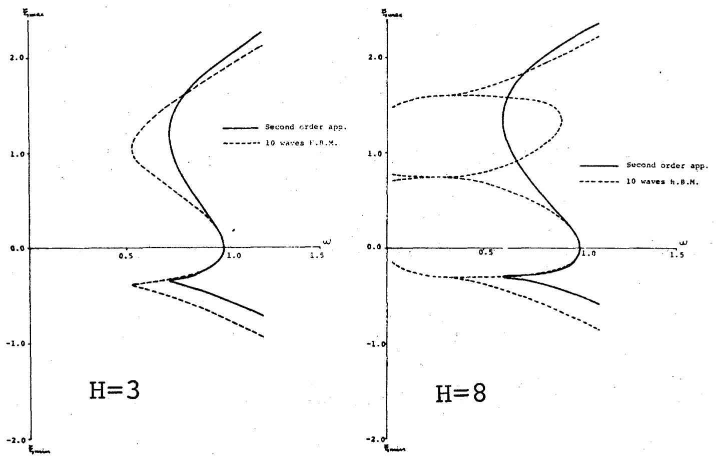


Fig.13: Backbone Curves
by Duffing's Iteration Method

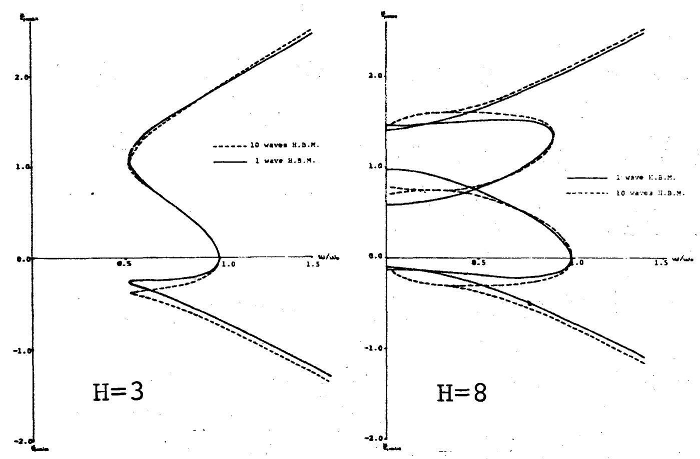


Fig.14: Backbone Curves
by Harmonic Balance Method
(Cosine 2 Waves App)