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On Pfaffian systems on $\mathbb{P}_2(\mathbb{C})$ with logarithmic singularities

R. Gérard*

Department of Mathematics
University of Tokyo
Hongo, Tokyo
and present address
Institut de Recherche Mathématique
Alsacien
7 rue René Descartes
67084 Strasbourg France

0. Introduction.

Let $\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_i$ be an algebraic subset of $\mathbb{P}_2(\mathbb{C})$ where for each i, $\mathcal{A}_i$ is irreducible and given by an irreducible polynomial equation in the homogeneous coordinates on $\mathbb{P}_2(\mathbb{C})$:

$$P_i(x_1, x_2, x_3) = 0.$$

We are considering Pfaffian systems of the form

(1) $$dz = \omega z$$

$$\omega = \sum_{i=1}^{n} A_i \frac{dP_i}{P_i}$$

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where the A_i's are constant square matrices of order m.

In this lecture we are going to speak about

1) the set $\mathcal{R}(1)$ of relations between the A_i's implied by the condition

$$d\omega = \omega \wedge \omega = 0$$

2) the relations in $\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$

3) the nature of the solutions of (1)

4) the Riemann-Hilbert problem.

In this lecture there are no deep results but only simple remarks.

1. <u>Some examples</u>.

1.1. $\mathcal{A} = \phi$, $\omega = 0$, $\pi_1(\mathbb{P}_2(\mathbb{C})) = 1$ and $\mathbb{C}^m$ is the vector space of solutions.

1.2. $\mathcal{A} = \bigcup_{i=1}^{3} \mathcal{A}_i$, $\mathcal{A}_i = \{x \in \mathbb{P}_2(\mathbb{C}) \mid x_i = 0\}$

$$\omega = \sum_{i=1}^{3} A_i \frac{dx_i}{x_i} . \qquad (1.2)$$

Then $\mathcal{R}(1)$:
$$\begin{cases} [A_i, A_j] = 0 & i \neq j \\ A_1 + A_2 + A_3 = 0 . \end{cases}$$

And

$$\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A}) \text{ is abelian.}$$

A fundamental matrix of solutions is given by:

$$x_1^{A_1}\ x_2^{A_2}\ x_3^{A_3}$$

and all solutions are elementary functions. The Riemann-Hilbert problem can easily be solved with a system of the form (1.2).

1.3. The Pfaffian system associated to the hypergeometric functions in two variables.

The hypergeometric function F_1 is given by the system of partial differential equations

$$x(1-x)(x-y)\frac{\partial^2 z}{\partial x^2} + [\gamma(x-y) - (\alpha+\beta+1)x^2 + (\alpha+\beta-\beta'+1)xy + \beta' y]\frac{\partial z}{\partial x}$$

$$-\beta y(1-y)\frac{\partial z}{\partial y} - \alpha\beta(x-y)z = 0$$

$$y(1-y)(y-x)\frac{\partial^2 z}{\partial y^2} + [\gamma(y-x) - (\alpha+\beta'+1)y^2 + (\alpha+\beta'-\beta+1)xy + \beta x]\frac{\partial z}{\partial y}$$

$$-\beta' x(1-x)\frac{\partial z}{\partial x} - \alpha\beta'(y-x)z = 0$$

$$(x-y)\frac{\partial^2 z}{\partial x\partial y} - \beta\frac{\partial z}{\partial x} + \beta\frac{\partial z}{\partial y} = 0.$$

But the map

$$z \longmapsto \begin{pmatrix} z \\ x\frac{\partial z}{\partial x} \\ y\frac{\partial z}{\partial y} \end{pmatrix}$$

transforms this "complicated" system into the following one of type (1) which is completely integrable.

$$dz = \omega z$$

$$\omega = \sum_{i=1}^{3} A_i \frac{dx_i}{x_i} + \sum_{i=1}^{3} B_i \frac{du_i}{u_i}$$

$$u_i = x_j - x_k \qquad j \neq k,\ j \neq i,\ k \neq i,$$

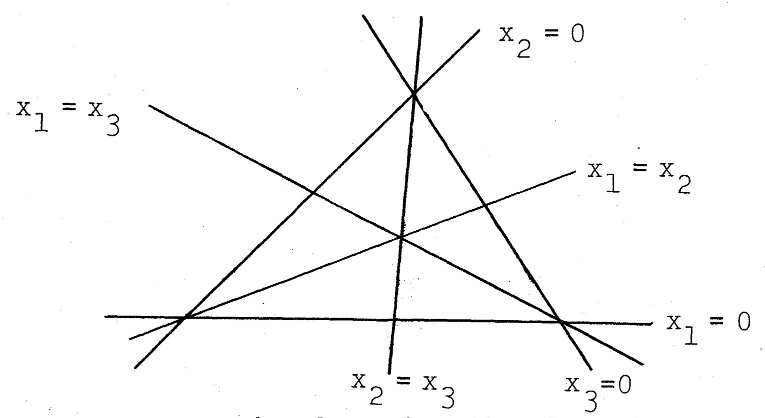
where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1-\gamma+\beta' & 0 \\ 0 & -\beta' & 0 \end{pmatrix} \qquad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha\beta' & -\beta' & \gamma-\alpha-\beta'-1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\beta \\ 0 & 0 & 1-\gamma+\beta \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ -\alpha\beta & \gamma-\alpha-\beta-1 & -\beta \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & -1 & -1 \\ \alpha\beta & \alpha+\beta & \beta \\ \alpha\beta' & \beta' & \alpha+\beta' \end{pmatrix} \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta' & \beta \\ 0 & \beta' & -\beta \end{pmatrix}$$

and we have $d\omega = \omega \wedge \omega = 0$. The singular set:



The same thing can be done for the functions F_2 and F_3 by using the map

$$z \longmapsto \begin{pmatrix} z \\ x\frac{\partial z}{\partial x} \\ y\frac{\partial z}{\partial y} \\ xy\frac{\partial^2 z}{\partial x \partial y} \end{pmatrix}$$

for F_2 the singular set is

$$\mathcal{A} = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_1 = x_3\} \cup \{x_2 = x_3\} \cup$$
$$\{x_1 + x_2 - x_3 = 0\} \cup \{x_3 = 0\} ,$$

for F_3 the singular set is

$$\mathcal{A} = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup \{x_1 = x_3\} \cup \{x_2 = x_3\} \cup$$
$$\{x_2 x_3 - x_1 x_3 - x_1 x_2 = 0\} .$$

We don't know if a transformation does exist for F_4. If it does exist it will be more complicated than for F_1, F_2, F_3.

1.4. Let $\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_i$ an arbitrary algebraic subset of $\mathbb{P}_2(\mathbb{C})$. Then let $\{A_i\}_{i=1,2,\ldots,n}$ a set of permutable matrices satisfying

$$\sum_{i=1}^{n} \deg(P_i) A_i = 0.$$

Then the system

$$dz = \omega z$$

with

$$\omega = \sum_{i=1}^{n} A_i \frac{dP_i}{P_i}$$

is completely integrable and has $\mathcal{A}$ as singular set. But we cannot call this system "generic".

Problem I. For any algebraic set $\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_i$ in $\mathbb{P}_2(\mathbb{C})$ does there exist a completely integrable Pfaffian system of type (1) with singularities on $\mathcal{A}$ and which is not example 1.4. ?

For some singular sets the answer is yes: examples F_1, F_2, F_3.

In example 1.4, we have a global fundamental matrix of solutions

$$\prod_{i=1}^{n} P_i^{A_i}$$

which is given by elementary functions. Then we can reformulate problem I in

Problem I'. Find the algebraic sets $\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_i$ in $\mathbb{P}_2(\mathbb{C})$ for which there exists a completely integrable Pfaffian system having singularities on $\mathcal{A}$ and which does not have a global fundamental matrix of solutions which is elementary.

2. The condition $d\omega = \omega \wedge \omega = 0$.

Let $\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_i$, $\mathcal{A}_i$: $P_i(x_1, x_2, x_3) = 0$ (irreducible).

2.1. The $\mathcal{A}_i$'s are normal crossing.

Consider $\omega = \sum_{i=1}^{n} A_i \frac{dP_i}{P_i}$ then it is easy to see that

$$d\omega = \omega \wedge \omega \Longleftrightarrow [A_i, A_j] = 0 \quad \text{for all } i, j \quad i \neq j.$$

And ω is well defined on $\mathbb{P}_2(\mathbb{C})$ if we add the condition

$$\sum_{i=1}^{n} (\deg P_i) A_i = 0.$$

And in this case problems I and I' are solved and the solutions are all elementary functions.

The Riemannn-Hilbert problem can be solved with a system of type (1). In fact $\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$ is abelian, then any linear representation

$$\chi : \pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A}) \longrightarrow GL(m, \mathbb{C})$$

is abelian.

The fundamental group has n-generators $g_1, g_2, \ldots, g_n$ related by

$$\prod_{i=1}^{n} g_i^{\deg P_i} = 1.$$

Set $\chi(g_i) = D_i$ and $\widetilde{A}_i = \frac{1}{2\pi i} \log D_i$. Then

$$[\widetilde{A}_i, \widetilde{A}_j] = 0$$

and

$$\sum_{i=1}^{n} (\deg P_i) \widetilde{A}_i = 2m\pi i \ I$$

and by choosing in a suitable way the determination of $\log D_i$, it is possible to find the set of A_i's satisfying

$$[A_i, A_j] = 0$$

and

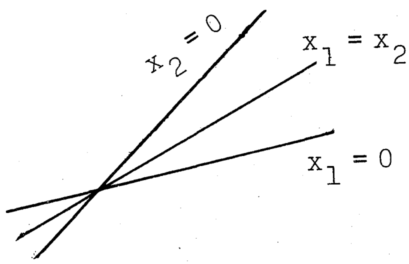
$$\sum_{i=1}^{n} (\deg P_i) A_i = 0.$$

2.2. The $\mathcal{A}_i$'s are not normal crossing.

2.2.1. An example.

$$\mathcal{A} = \bigcup_{i=1}^{3} \mathcal{A}_i, \quad \mathcal{A}_1 = \{x_1 = 0\}, \quad \mathcal{A}_2 = \{x_2 = 0\}$$

$$\mathcal{A}_3 = \{x_1 = x_2\}.$$



Set $\omega = A_1 \frac{dx_1}{x_1} + A_2 \frac{dx_2}{x_2} + A_3 \frac{d(x_1 - x_2)}{x_1 - x_2}$.

And $d\omega = \omega \wedge \omega$ gives

$$[A_1, A_1 + A_2 + A_3] = 0$$

$$[A_2, A_1 + A_2 + A_3] = 0$$

$$[A_3, A_1 + A_2 + A_3] = 0$$

which are trivial consequence of

$$A_1 + A_2 + A_3 = 0$$

which has to be satisfied to have a system on $\mathbb{P}_2(\mathbb{C})$.

This means that any set of matrices A_1, A_2, A_3 such that $A_1 + A_2 + A_3 = 0$ solves the problem I, the solutions are not elementary but nearly related the hypergeometric Gauss function of one variable.

2.2.2. Let us consider a Pfaffian form

$$\omega = \sum_{i=1}^{n} A_i \frac{dP_i}{P_i}$$

$\mathcal{A}_i$: $P_i = 0$, P_i irreducible. Then by a finite number of monoidal transformations (blowing up), we construct a manifold X with an analytic map

$$\sigma : X \longrightarrow \mathbb{P}_2(\mathbb{C}).$$

There exists in X, a divisor

$$\mathcal{A}^* = \bigcup_{j=1}^{m} \mathcal{A}_j^* \qquad (m > n)$$

where the $\mathcal{A}_j^*$ are normal crossing and for each $i \in [1,2,\ldots,n]$ there exists $j(i) \in [1,2,\ldots,m]$ such that

$$\sigma : \mathcal{A}^*_{j(i)} \xrightarrow{\simeq} \mathcal{A}_i .$$

But $\mathcal{A}^*$ contains some exceptional divisor coming from blowing up of the singularities of $\mathcal{A}$. Moreover σ is an isomorphism from

$$X - \mathcal{A}^* \quad \text{onto} \quad \mathbb{P}_2(\mathbb{C}) - \mathcal{A}.$$

This implies that

$$\pi_1(X - \mathcal{A}^*) \simeq \pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A}).$$

Denote by $\sigma^*(\omega)$ the inverse image of ω by σ, then $\sigma^*(\omega)$ has logarithmic pôles on $\mathcal{A}^*$. Then we have for each $i \in [1,2,\ldots,n]$

$$\text{Rés}_{\mathcal{A}_i}(\sigma^*\omega) = \text{Rés}_{\mathcal{A}_i}(\omega) = A_i$$

and for each exceptional divisor $\mathcal{B}$

$$\text{Rés}_{\mathcal{B}}(\sigma^*\omega) = \sum_{i=1}^{n} r_i \ \text{Rés}_{\mathcal{A}_i}(\omega)$$

where the r_i's are integers.

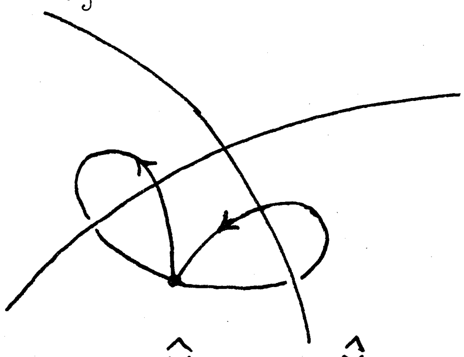
By local computations in X, we can easily prove the following

$$\omega \wedge \omega = 0 \Longleftrightarrow \sigma^*\omega \wedge \sigma^*\omega = 0 \Longleftrightarrow [\text{Rés}_{\mathcal{A}_i^*}\sigma^*(\omega), \text{Rés}_{\mathcal{A}_j^*}\sigma^*(\omega)] = 0$$

$$\text{for all } i, j \quad i \neq j \text{ with } \mathcal{A}_i^* \cap A_j^* \neq \phi .$$

This result can also be seen by using the following remark.

Locally in X near a point $M \in \mathcal{A}_i^* \cap \mathcal{A}_j^*$, choose a simple path γ_i surrounding $\mathcal{A}_i^*$ and a path γ_j surrounding $\mathcal{A}_j^*$ having the same origin and such that



$$\int_{\gamma_i} \frac{dP_i^*}{P_i^*} = 2\pi\sqrt{-1}$$

$$\int_{\gamma_j} \frac{dP_j^*}{P_j^*} = 2\pi\sqrt{-1} .$$

Denote by $\hat{\gamma}_i$ and $\hat{\gamma}_j$ the local homotopy classes of γ_i and γ_j. We have

$$\hat{\gamma}_i \hat{\gamma}_j = \hat{\gamma}_j \hat{\gamma}_i$$

(the local fundamental group is abelian). This means that there exists a two cell S_{ij} homotopic to zero such that

$$\partial S_{ij} = \gamma_i \gamma_j \gamma_i^{-1} \gamma_j^{-1}$$

and

Proposition 1.

$$\int_{S_{ij}} \sigma^*(\omega) \wedge \sigma^*(\omega) = + 4\pi^2 [\text{Rés}_{\mathcal{A}_j^*} \sigma^*(\omega), \text{Rés}_{\mathcal{A}_i^*} \sigma^*(\omega)]$$

and as a corollary

$$\omega \wedge \omega = 0 \Longrightarrow [\text{Rés}_{\mathcal{A}_j^*} \sigma^*(\omega), \text{Rés}_{\mathcal{A}_i} \sigma^*(\omega)] = 0$$

for all $i, j \quad i \neq j$.

Let us summarize the results:

The complete integrability condition is equivalent to the commutation of the résidues of $\sigma^*(\omega)$ for $\mathcal{A}_i^*$ and $\mathcal{A}_j^*$ when $\mathcal{A}_i^* \cap \mathcal{A}_j^* \neq 0$.

This means that for $\omega = \sum_{i=1}^{n} A_i \frac{dP_i}{P_i}$ we have

$$\omega \wedge \omega = 0 \Longleftrightarrow \begin{cases} \sum_{i=1}^{n} (\deg P_i) A_i = 0 \\ [\text{Rés}_{\mathcal{A}_i^*} \sigma^*(\omega), \text{Rés}_{\mathcal{A}_j^*} \sigma^*(\omega)] = 0 \\ \text{where} \end{cases}$$

$$\text{Rés}_{\mathcal{A}_j^*} \sigma^*(\omega) = \text{Rés}_{\mathcal{A}_j} \omega$$

where $\mathcal{A}_j$ is not an exceptional divisor and

$$\text{Rés}_{\mathcal{A}_j^*} \sigma^*(\omega) = \sum_{i=1}^{n} r_i^j \, \text{Rés}_{\mathcal{A}_i} \omega$$

where r_i^j are integers.

2.3. The group of the relation $\omega \wedge \omega = 0$.

Let us construct a group G in the following way.

To each irreducible component $\mathcal{A}_i$ of $\mathcal{A}$ is associated in an abstract way a generator g_i of G. Then to each relation among the A_i's we associate a relation between the g_i's in the following way.

To $\sum_{i=1}^{n} P_i A_i = 0$ $\quad P_i = \deg P_i$ les us associate

$$g_1^{P_1} g_2^{P_2} \cdots g_n^{P_n} = 1 \quad \text{and} \quad \tau(g_1^{P_1} g_2^{P_2} \cdots g_n^{P_n}) = 1$$

for all circular permutation τ of the factors.

To $[\text{Rés}_{\mathcal{A}_i^*}\sigma^*(\omega), \text{Rés}_{\mathcal{A}_j^*}\sigma^*(\omega)] = 0$

$$[g_1^{r_1^i} g_2^{r_2^i} \cdots g_n^{r_n^i}, g_1^{r_1^j} \cdots g_n^{r_n^j}] = 1$$

and

$$[\tau(g_1^{r_1^i} \cdots g_n^{r_n^i}, \tau(g_1^{r_1^j} \cdots g_n^{r_n^j})] = 1$$

for each circular permutation τ of the factors.

Examples.

1. See 1.2.

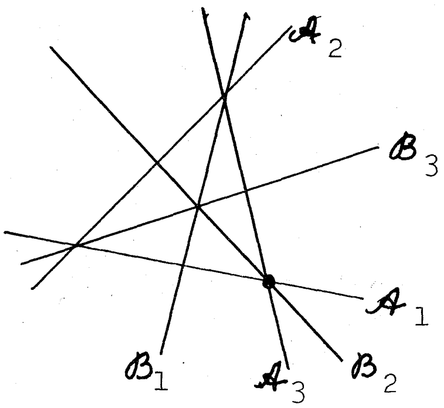
$$G = \{g_1, g_2, g_3; [g_i, g_j] = 1, g_1 g_2 g_3 = 1, \tau(g_1 g_2 g_3) = 1\}$$

and G is a free abelian with two generators.

2. See 2.1. Normal crossing case.

G is a free abelian with (n-1)-generators.

3. See 1.3. Hypergeometric function F_1.



The group G has 6 generators

$$a_1, a_2, a_3, b_1, b_2, b_3$$

corresponding to $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$.

The relations are up to cyclic permutations as indicated above:

$$[a_1, b_1] = [a_2, b_2] = [a_3, b_3] = 1$$

$$[a_1, b_2a_3] = [a_1, b_3a_2] = 1$$

$$[a_2, b_3a_1] = [a_2, b_1a_3] = 1$$

$$[a_3, b_1a_2] = [a_3, b_2a_1] = 1$$

$$[b_1, b_2b_3] = [b_1, a_2a_3] = 1$$

$$[b_2, b_1b_3] = [b_2, a_1a_3] = 1$$

$$[b_3, b_1b_2] = [b_3, a_1a_2] = 1$$

and

$$a_1a_2a_3b_1b_2b_3 = 1.$$

4. See 2.2.1.

The group G has three generators g_1, g_2, g_3 and up to cyclic permutation as indicated we have the relations

$$[g_i, g_1g_2g_3] = 1 \quad i = 1, 2, 3.$$

but also $g_1g_2g_3 = 1$. As a consequence G is a free group with two generators.

3. Some problems.

I. If problem I in section 1 is solvable for an algebraic set $\mathcal{A}$, what are the relations between

1) the monodromy of the Pfaffian system and the group G

2) $\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$ and G ?

II. In which cases is the Riemann-Hilbert problem solvable by a Pfaffian system of type (1) ? Or such that $\sigma^*(\omega)$ is of type (1) in each coordinate system in X ?

Remark: In many particular examples when $\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$ is known we have $G = \pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$.

To finish this summary let me thank Professor M. Oka of the University of Tokyo with whom I have had a very interesting discussion about fundamental groups of the complementary of an algebraic curve in $\mathbb{P}_2(\mathbb{C})$.