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Trace formula for a nonpositively curved manifold

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This is a preliminary report of my forthcoming paper which will be published elsewhere.

1. Introduction. The intention of this report is to give an application of a geometrical version of Selberg's trace formula to the heat equation asymptotics of a compact manifold of non-positive curvature.

The original form of trace formula given by Selberg himself works well on the class of weakly symmetric spaces. Applying it to the upper half plane, he could construct the so-called Selberg's zeta function and gave a functional equality and a proof of generalized Riemannian conjecture for the zeta function.

As ones know, the Selberg's trace formula can be considered as a non-abelian generalization of Poisson summation formula:

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{m\in\mathbb{Z}} \hat{f}(m),$$

where $\hat{f}(x) = \int_{-\infty}^{\infty} e^{-2\pi i xy} f(y)\,dy$. Putting especially $f(x) = e^{-4\pi^2 x^2 t}$, one obtains

$$\sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} = \frac{1}{\sqrt{4\pi t}} \sum_{m=-\infty}^{\infty} e^{-m^2/4t}.$$

This equality has a geometrical meaning, namely $\{4\pi^2 n^2 ;\ n\in\mathbb{Z}\}$ is just the spectrum of the Riemannian manifold $S^1 = \mathbb{R}/\mathbb{Z}$ (the circle), and $\{m\in\mathbb{Z}\}$ is the label of homotopy classes of closed paths in S^1.

In this point of view, many mathematicians have intended to generalize this formula to general manifolds. To explain some results in this direction, we make use of the following notations: Let (M, g) be a compact manifold with a Riemannian metric g, and let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \uparrow \infty$ be the spectrum of its Laplacian Δ acting on functions. The set of closed geodesics in M will be denoted by $\mathrm{Geo}(M)$ and by $\mathrm{Geo}^{\alpha}(M)$ the component of $\mathrm{Geo}(M)$. Y.Colin de Verdiere [5] and J.Chazarain [3] proved that for a generic (M, g), the Θ-series

$$\Theta(1/z) = \sum_i \exp(-\lambda_i/z) \quad , \quad \mathrm{Re}\, z > 0$$

can be written in the form

$$\sum_{\alpha} f_{\alpha}(z) \exp(-z\ell^2(\alpha)/4),$$

in which $\ell(\alpha)$ stands for the length of closed geodesics in $\mathrm{Geo}^{\alpha}(M)$, and $f_{\alpha}(z)$ is a function expanded as

$$f_{\alpha}(z) \underset{z \to \infty}{\sim} \exp\left(\sqrt{-1}\,\sigma_{\alpha} \frac{\pi}{2}\right) \left(\frac{z}{4\pi}\right)^{n_{\alpha}/2} \left(a_{\alpha 0} + a_{\alpha 1} z^{-1} + a_{\alpha 2} z^{-2} + \cdots\right),$$

where $n_{\alpha} = \dim \mathrm{Geo}^{\alpha}(M)$, σ_{α} is the Morse index, and the statement "$z \to \infty$" means that a fixed $\mathrm{Re}\, z = s$ is chosen and that "$\mathrm{Im}\, z \to +\infty$".

In general it is difficult to calculate the higher order coefficients $a_{\alpha i}$. For the case of strictly negative curvature, H.Donnelly [6] gave an algorithm for computing the $a_{\alpha i}$. (A.Morchanov [12] also treated the heat equation asymptotics for such a case. But his argument seems incomplete. For instance he claimed that the first coefficient obtained by his method agrees with that of McKean [10] for the case when M is a surface of negative constant curvature. But this is incorrect.)

If we take away the assumption of strict negativity and assume only nonpositivity for curvature, then the matter becomes somewhat complicated, and the arguments in [6] can not be directly appliable to this case.

In this paper, noting that a geometrical version of trace formula works well on a class of manifolds of nonpositive curvature satisfying a condition of non-degeneracy, we will establish the asymptotics of the Θ -series for such manifolds, and an algorithm for a calculus of the coefficients $a_{\alpha i}$. One should note that locally symmetric spaces belong to this class.

In the course of argument, we need off diagonal asymptotics for heat kernels on simply connected manifolds of non-positive curvature. We should note here that Y.Kannai [8] treated the off diagonal estimates for general case, but his results are not available for our purpose because we need the fact that the remainder terms can be estimated by a function with at most exponential growth at infinity. (The author is very much indebted to Prof. N.Ikeda for indicating the reference [8]).

2. Notations. First of all, I will establish some notations which will be needed later. For more information you should refer to W.Klingenberg "Lectures on closed geodesics" Springer-Verlag 1978, or J.Milnor "Morse theory" Princeton 1963.

Let M be a compact connected C^∞-Riemannian manifold of dimension n. The inner product of two vectors X_p, $Y_p \in T_pM$ will be denoted by $\langle X_p, Y_p \rangle$. The curvature tensor R is given by the relation

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle. The set of closed paths : $S^1 \longrightarrow M$ of class H^1 will be denoted by $\Omega(M)$, which is considered , in a natural way, a Hilbert manifold. For a smooth curve $c \in \Omega(M)$, the tangent space $T_c\Omega(M)$ is identified with the space of vector fields of class H^1 along c, $H^1(c^{-1}TM)$. The manifold $\Omega(M)$ is endowed with the complete metric given by

$$\langle X, Y \rangle_1 = \int_{S^1} \langle X, Y \rangle \, dt + \int_{S^1} \langle \nabla X, \nabla Y \rangle \, dt$$

We consider a functional E on $\Omega(M)$:

$$E(c)=\frac{1}{2}\int_{S^1}\|\dot{c}\|^2 dt,$$

which is called the energy or action integral. The differential of E is caluculated as

$$d_c E(X)=\int_{S^1}\langle \dot{c}, \nabla X\rangle dt,$$

which deduces that the critical point set of E is just the set of closed geodesics, $c: \nabla\dot{c}=0$. For simplicity we write Geo(M) for that. Let $c \in \mathrm{Geo}(M)$. Then the Hessian of E at c is given by

$$\mathrm{Hess}_c E(X,Y)=-\int_{S^1}\langle X, \nabla^2 Y+R(\dot{c},Y)\dot{c}\rangle dt.$$

Putting $J_c(X) = \nabla^2 X + R(\dot{c}, X)\dot{c}$, we have immediately

$$\text{Null space of } \mathrm{Hess}_c E = \mathrm{Ker}\, J_c.$$

We call J_c the Jacobi's differential operator.

Definition. M is said to be non-degenerate if the function E is non-degenerate in the sense of Bott, that is

i) Geo(M) is a submanifold in $\Omega(M)$,

ii) $T_c\mathrm{Geo}(M) = \mathrm{Ker}\, J_c$

It is known that M is non-degenerate for "generic" metrices.

We consider now the geodesic flow $\{\varphi_t\}: TM \longrightarrow TM$ which is given by

$$\varphi_t(p,v)=(\exp_p tv, \tfrac{d}{dt}\exp_p tv).$$

If $c \in \mathrm{Geo}(M)$, then $\varphi_t(\dot{c}(0))$ is a periodic orbit, and the differential $d_p\varphi_1 : T_{(p,v)}TM \longrightarrow T_{(p,v)}TM$ $(c(0) = p, \dot{c}(0) = v)$

is called the (linearized) Poincare mapping associated with the closed geodesic c. Using the connection on TM, we have a direct decomposition:

$$T_{(p,v)}TM = T^H \oplus T^V,$$

where T^H is the horizontal part, and T^V is the vertical part. Let

$$H_{(p,v)} : T_pM \longrightarrow T^H$$

be the isomorphism which is defined by

$$H_{(p,v)}(u) = \dot{\alpha}(0),$$

where $\alpha(t) = (\exp tv, U(t))$, $U(t)$ being the vector field along $\exp tu$ obtained by parallel translating v. Let

$$V_{(p,v)} : T_pM \longrightarrow T^V$$

be the canonical isomorphism. From now on, we identify $T_{(p,v)}TM$ with the direct sum $T_pM \oplus T_pM$ via the isomorphisms H, V.

<u>Lemma</u> 2.1. $P_{(p,v)}(Y(0), \nabla Y(0)) = (Y(1), \nabla Y(1))$, where $Y(t)$ is a Jacobi field along c.

From this lemma it follows that the eigenvalues of the Poincare mapping P is uniquely determined by the geodesic c. We will denote by $\det(P_c - I)$ the product of eigenvalues which are differ from 1.

3. Manifold of nonpositive curvature. This section will describe the behavior of closed geodesics in a manifold of nonpositive curvature and will give a usefull criterion of nondegeneracy for such manifolds.

Let M be a compact manifold of nonpositive curvature and with fundamental group Γ, which acts isometrically and proper discontinuously on the universal covering $\widetilde{M}$ as deck transformations. It is a standard fact that the exponential mapping $\exp : T_x\widetilde{M} \longrightarrow \widetilde{M}$ is a diffeomorphism, so that M is a $K(\pi, 1)$-manifold. The free homotopy classes of closed paths, $[S^1, M]$, can be then identified with the conjugacy classes in Γ in a natural way. We denote by $[\Gamma] = \{[\gamma]\}$ the set of conjugacy classes, and by $M_{[\gamma]}$ the set of closed geodesics c with $[c] = [\gamma]$, $[c]$ being the homotopy class of c.

For each $\gamma \in \Gamma$, we put $f_\gamma(x) = d(x, \gamma x)^2$, d being the distance function on $\widetilde{M}$. We denote by $\widetilde{M}_\gamma$ the critical point set of f_γ. We will review some basic properties of f_γ obtained by V.Ozols [13].

Lemma 3.1. A point x lies in $\widetilde{M}_\gamma$ if and only if γ preserves the minimizing geodesic from x to γx.

Proof. Let V^γ be the fundamental vector field on $\widetilde{M}$, defined by

$$\exp V^\gamma(x) = \gamma x.$$

Given a vector $v \in T_x\widetilde{M}$, we can form a surface in $\widetilde{M}$ by

$$\alpha(s, t) = \exp(sV^\gamma(\exp tv)).$$

Then $T = \alpha_*(\frac{\partial}{\partial s})$, $X = \alpha_*(\frac{\partial}{\partial t})$ are vector fields along the surface α satisfying $\nabla_X T = \nabla_T X$, and we have

$$\begin{aligned} X_{(0,t)} f &= \frac{\partial}{\partial t} d^2(\exp tv, \gamma \exp tv) \\ &= \int_0^1 \frac{\partial}{\partial t} \| T \|^2 \, ds \end{aligned}$$

$$= 2\int_0^1 \frac{d}{ds} \langle X, T\rangle ds = 2\langle X, T\rangle\Big|_0^1$$

By definition of X, $X_{(1,0)} = \gamma_* X_{(0,0)}$.

We now suppose $x \in \widetilde{M}_\gamma$, and let $\widetilde{c}_x$ be the minimizing geodesic from x to γx, so $\widetilde{c}_x(s) = \alpha(s, 0)$. We prove $\gamma\dot{\widetilde{c}}_x(0) = \dot{\widetilde{c}}_x(1)$. For this let $v \in T_x\widetilde{M}$ be any orthogonal vector of $\dot{\widetilde{c}}_x(0)$. Then $0 = \langle X, T\rangle\Big|_0^1 = \langle X, \dot{\widetilde{c}}_x(1)\rangle - \langle v, \dot{\widetilde{c}}_x(0)\rangle = \langle X, \dot{\widetilde{c}}_x(1)\rangle$, so that $\gamma_* v$ is orthogonal to $\dot{\widetilde{c}}_x(1)$, which implies $\gamma\dot{\widetilde{c}}_x(0) = \dot{\widetilde{c}}_x(1)$.

Conversely suppose that γ preserves a geodesic c. If $v \in T_x\widetilde{M}$ is tangent to c, then $v\cdot f_\gamma = \langle X, T\rangle\Big|_0^1 = \langle \gamma_* v, \gamma_*\dot{c}(0)\rangle - \langle v, \dot{c}(0)\rangle = 0$. If v is normal to c, then $v\cdot f_\gamma = \langle X, T\rangle\Big|_0^1 = \langle \gamma_* v, \dot{c}(1)\rangle - \langle v, \dot{c}(0)\rangle = 0$, which implies $x \in \widehat{M}_\gamma$.

Proposition 3.2. $\widetilde{M}_\gamma$ is non-empty, and a connected totally geodesic submanifold possibly with boundary on which f_γ attains its absolute minimum.

Proof. Non-emptyness follows from the compactness of M. Easy calculations show

$$\frac{\partial^2}{\partial t^2} f_\gamma(\exp tv) = 2\int_0^1 \langle \nabla_T X, \nabla_T X\rangle - \langle R(T,X)T, X\rangle ds.$$

By the assumption of curvature, we have

$$\frac{\partial^2}{\partial t^2} f_\gamma(\exp tv) \geqq 0,$$

in other words, f_γ is a convex function on $\widetilde{M}$, from which our assertions immediately follow.

The minimum of f_γ will be denoted by $\ell(\gamma)^2$.

We denote by Γ_γ the centralizer of γ in Γ. The set $\widetilde{M}_\gamma$ is invariant under Γ_γ-action and contractible.

Lemma 3.3. Let x be any interior point of $\widetilde{M}_\gamma$, and let $v \in T_x\widetilde{M}_\gamma$ be a transeverse vector to $\widetilde{c}_x$. Then the surface $\alpha(s, t)$ $(s \in R,\ -\varepsilon < t < \varepsilon)$ is totally geodesic, and has zero curvature.

Furthermore the vector fields T and X are parallel on α.

Proof. Since $\frac{\partial^2}{\partial t^2} f_\gamma(\alpha(s, t)) = 0$, we have

$$\int_0^1 \langle \nabla_T X, \nabla_T X \rangle - \langle R(T,X)T, X \rangle \, ds = 0,$$

so that $\nabla_T X = \nabla_X T = 0$, $\langle R(T,X)T, X\rangle = 0$. Hence it remains only to prove $\nabla_X X = 0$. But this follows from the

Lemma 3.4. Let $c(t)$ and $c'(t)$ be geodesics in $\widetilde{M}$. Then the function $t \longmapsto d^2(c(t), c'(t))$ is convex.

Proof. Easy and omitted.

We now define a mapping $\widetilde{\Phi}_\gamma : \widetilde{M}_\gamma \longrightarrow \Omega(M)$ as follows: For $x \in \widetilde{M}_\gamma$. let $\widetilde{c}_x : R \longrightarrow M$ be the unique geodesic with $\widetilde{c}_x(0) = x$ and $\widetilde{c}_x(1) = \gamma x$. Since $\gamma \widetilde{c}_x(t) = \widetilde{c}_x(t+1)$, the geodesic $\widetilde{c}_x$ can be considered as a lifting of a closed geodesic $c_x : S^1 \longrightarrow M$. Then we put $\widetilde{\Phi}_\gamma(x) = c_x$. It is clear that $\widetilde{\Phi}_\gamma(\widetilde{M}_\gamma) \subset M_{[\gamma]}$ and each c_x is of length $\ell(\gamma)$.

Lemma 3.5. The mapping $\widetilde{\Phi}_\gamma : \widetilde{M}_\gamma \setminus \partial \widetilde{M}_\gamma \longrightarrow \Omega(M)$ is smooth. Further the mapping $\widetilde{\Phi}_\gamma$ induces a homeomorphism $\Phi_\gamma : \widetilde{M}_\gamma / \Gamma_\gamma$ onto $M_{[\gamma]}$. In particular $M_{[\gamma]}$ is a manifold possibly with boundary, and $\pi(M_{[\gamma]})$ is isomorphic to Γ_γ.

This follows immediately from the definition of smooth structure on $\Omega(M)$.

The following is a main goal of this section.

Proposition 3.6. A manifold M of nonpositive curvature is non degenerate if and only if for any $\gamma \in \Gamma$, the function f_γ is non degenerate.

Proof. We adopt the notations in Lemma 3.1. By Lemma 3.5 $M_{[\gamma]}$ is without boundary if and only if so is $\widetilde{M}_\gamma$. The linear mapping

$$\begin{array}{ccc} \text{Null space of } \operatorname{Hess}_x f_\gamma & \longrightarrow & \operatorname{Ker} J_{c_x} \\ \Psi & & \Psi \\ v & \longmapsto & X \end{array}$$

is an isomorphism, so if $\dim T_x\widetilde{M}_\gamma = \dim$ Null space of $\operatorname{Hess}_x f_\gamma$, then $\dim M_{[\gamma]} = \dim \operatorname{Ker} J_{c_x}$, and vice versa.

<u>Lemma</u> 3.7. If M is nondegenerate, then the evaluation mapping $\begin{array}{ccc} i_{[\gamma]} : M_{[\gamma]} & \longrightarrow & M \\ \Psi & & \Psi \\ c & \longmapsto & c(0) \end{array}$ is a totally geodesic, isometric immersion.

Proof. The tangent space $T_c M_{[\gamma]}$ coincides with $\operatorname{Ker} J_c$, and the differential of $i_{[\gamma]}$ is just the restriction $X \mapsto X(0)$. Since each $X \in \operatorname{Ker} J_c$ is parallel, we have

$$\langle X, X \rangle_1 = \int_{S^1} \langle X, X \rangle \, ds = \langle X(0), X(0) \rangle ,$$

which implies $i_{[\gamma]}$ is isometric. Totally geodesicity is obvious.

<u>Corollary</u> 3.8. A manifold of strictly negative curvature is nondegenerate.

<u>Corollary</u> 3.9. A compact locally symmetric space of non-positive curvature is nondegenerate.

<u>Remark</u>. In our previous paper [15], we proved directly the above corollary, using the Morse theory. See also [17]

4. <u>Selberg's trace formula</u>. From now on we assume M is of nonpositive curvature and nondegenerate, so that $M_{[\gamma]}$ is a Riemannian manifold with the metric induced from that of M by immersion $i_{[\gamma]} : M_{[\gamma]} \longrightarrow M$. The canonical volume element on

$M_{[\gamma]}$ will be denoted $dv_{[\gamma]}$.

Let N_γ be the normal bundle of $\widetilde{M}_\gamma$ in $\widetilde{M}$. The centralizer Γ_γ acts on N_γ, and the exponential mapping $\exp : N_\gamma \longrightarrow \widetilde{M}$ is a Γ_γ-equivalent diffeomorphism.

It is convenient to introduce a volume element $d\mu_\gamma \cdot dv_\gamma$ on N_γ which is defined by the fiber product of dv_γ on $\widetilde{M}_\gamma$ and the ordinary Lebesgue measure $d\mu_{\gamma x}$ on the fibers $N_{\gamma x}$:

$$\int_{N_\gamma} d\mu_\gamma \cdot dv_\gamma = \int_{\widetilde{M}_\gamma} dv_\gamma(x) \int_{N_{\gamma x}} d\mu_{\gamma x}(v).$$

On the other hand, the pull back of $d\widetilde{v}$ by the exponential mapping yields another one on N_γ, so that we get a smooth function φ_γ on N_γ satisfying

$$(\exp)^*(d\widetilde{v}) = \varphi_\gamma \, d\mu_\gamma \cdot dv_\gamma .$$

We now turn to the normal bundle of the immersion $i_{[\gamma]}$: $M_{[\gamma]} \longrightarrow M$, to be denoted $N_{[\gamma]}$. The quotient N_γ/Γ_γ is identified with $N_{[\gamma]}$ via the natural projection $N_\gamma \rightarrow N_{[\gamma]}$ and the following diagram is commutative:

$$\begin{array}{ccc} N_\gamma & \longrightarrow & N_{[\gamma]} \\ \downarrow & & \downarrow \\ \widetilde{M}_\gamma & \longrightarrow & M_{[\gamma]} \end{array}$$

Further the exponential mapping yields a diffeomorphism $\exp$: $N_{[\gamma]} \longrightarrow \widetilde{M}/\Gamma_\gamma$. In the same way as above, we construct a volume element $d\mu_{[\gamma]} \cdot dv_{[\gamma]}$ on $N_{[\gamma]}$ and the function $\varphi_{[\gamma]}$ on $N_{[\gamma]}$ satisfying

$$(\exp)^*(d\widetilde{v}_\gamma) = \varphi_{[\gamma]} d\mu_{[\gamma]} \cdot dv_{[\gamma]},$$

where $d\widetilde{v}_\gamma$ is the volume element on $\widetilde{M}/\Gamma_\gamma$. It is obvious that φ_γ is the lifting of $\varphi_{[\gamma]}$ and $\varphi_{[\gamma]}(0) = 1$.

Lemma 4.1. $|\varphi_{[\gamma]}(v)| \leq C\exp(h\|v\|)$ for any normal vector $v \in N_{[\gamma]}$, and the constant C, h does not depend on v.

Proof. Take a point $p \in \widetilde{M}_\gamma$. Since $M_{[\gamma]}$ is compact, it is enough to show that

$$|\varphi_\gamma(v)| \leq C\exp(h\|v\|)$$

for any $v \in N_{\gamma p}$. Using the trivialization of the bundle N_γ given by

$$\begin{aligned} \mathcal{E} : \widetilde{M}_\gamma \times N_{\gamma p} &\cong N_\gamma \\ (q, v) &\mapsto \tau_{pq} v, \end{aligned}$$

where $\tau_{pq} : N_{\gamma p} \longrightarrow N_{\gamma q}$ denotes the parallel translation along the unique geodesic joining p to q, we have

$$\mathcal{E}^*(d\mu_\gamma \cdot dv_\gamma) = d\mu_{\gamma p} \times dv_\gamma ,$$

which implies

$$\varphi_\gamma(v) = |\det d_{(p,v)}(\exp \circ \mathcal{E})| .$$

But if $u_1 \in T_p\widetilde{M}_\gamma$ then

$$d_{(p,v)}(\exp \circ \mathcal{E})(u_1) = U_1(\|v\|)/\|v\| ,$$

and if $u_2 \in N_{\gamma p}$, then

$$d_{(p,v)}(\exp \circ \mathcal{E})(u_2) = U_2(\|v\|),$$

where U_1, U_2 are Jacobi fields along the normal geodesic $t \longmapsto \exp tv/\|v\|$ satisfying

$$U_1(0) = 0, \ \nabla U_1(0) = u_1 ,$$

$$U_2(0) = u_2, \qquad \nabla U_2(0) = 0.$$

From the theorem of comparison of Rauch and Berger ([4]), the norm $\|U_i(t)\|$ are bounded from above by functions with at most exponential growth at infinity, hence the lemma follows.

Let now $K(p, q)$ be a continuous function on $\widetilde{M} \times \widetilde{M}$ satisfying the relation

$$K(\mu p, \mu q) = K(p, q)$$

for all p, q in M and all μ in Γ. We make the following assumption: The sum

$$\sum_{\gamma \in \Gamma} |K(p, \gamma q)|$$

converges uniformly if p and q are in some compact region of $\widetilde{M}$. Since the function $(p, q) \longmapsto \sum_{\gamma \in \Gamma} K(p, \gamma q)$ is $\Gamma \times \Gamma$-invariant, it yields a continuous function on $M \times M$, to be denoted $k(x, y)$. Further the function $p \longmapsto K(p, \gamma p)$ is Γ_γ-invariant, hence it can be considered as a function of $\widetilde{M}/\Gamma_\gamma$. We write $K_{[\gamma]}(v) = K(\exp v, \gamma \exp v)$ via the identification $N_{[\gamma]} = \widetilde{M}/\Gamma_\gamma$.

<u>Proposition</u> 4.2. (A version of Selberg's trace formula)

$$\int_M k(x, x)\, dv(x) = \sum_{[\gamma] \in [\Gamma]} \int_{M_{[\gamma]}} dv_{[\gamma]}(x) \int_{N_{[\gamma]x}} K_{[\gamma]}(v)\, \varphi_{[\gamma]}(v)\, d\mu_{[\gamma]x}(v).$$

Proof. Let $\mathcal{D}$ be a fundamental domain for the action of Γ on $\widetilde{M}$, and let $\mathcal{D}_\gamma$ be that of the action of Γ_γ. We then get

$$\int_M k(x, x)\, dv(x) = \sum_{\gamma \in \Gamma} \int_{\mathcal{D}} K(p, \gamma p)\, d\widetilde{v}(p)$$
$$= \sum_{[\gamma] \in [\Gamma]} \sum_{\mu \in [\gamma]} \int_{\mathcal{D}} K(p, \mu p)\, d\widetilde{v}(p)$$

$$= \sum_{\omega \ni \in [\Gamma)} \sum_{\mu \in \Gamma/\Gamma_\gamma} \int_{\mu\mathcal{D}} K(p, \mu^{-1}\gamma\mu p)\ d\widetilde{v}(p)$$

$$= \sum_{\omega \ni \in [\Gamma)} \int_{\mathcal{D}_\gamma} K(p, \gamma\ p)\ d\widetilde{v}(p),$$

since $\bigcup_{\mu \in \Gamma/\Gamma_\gamma} \mu\mathcal{D}$ is a fundamental domain of the action of Γ_γ on $\widetilde{M}$. Regarding here $\widetilde{M}/\Gamma_\gamma$ as fiber space over $M_{[\gamma]}$, we have the assertion.

5. Heat kernels on M. We continue with the situation of § 2. First we review the basic properties of the heat kernel on a simply connected, nonpositively curved manifold $\widetilde{M}$.

Let $K(t;p, q)$ be the fundamental solution (heat kernel) of the heat equation on $\widetilde{M}$

$$\frac{\partial}{\partial t} K + \Delta K = 0$$

$$K(0;p, q) = \delta_q(\cdot).$$

In order to describe an asymptotical property of K, we let for brevity

$$\Theta(p, q) = \left| \det d_v \exp_p \right| \qquad (y = \exp_p v).$$

and define inductively

$$u_0(p, q) = \Theta^{-1/2}(p, q)$$

$$u_i(p, q) = \Theta^{-1/2}(p, q) \int_0^1 \Theta(p, \exp(\tau v))^{1/2} \Delta_2 u_{i-1}(p, \exp_p \tau v) \tau^{i-1} d\tau.$$

The comparison theorem and an inductive argument yield estimates

$$|u_i(p, q)| \leq C \exp h\, d(p, q)$$

for each integer $i \geq 0$, C and h being positive constants (the case $\dim M = 2$ is due to [5], and general case is due to P.H.Berard [1]).

Lemma 5.1. For any integer $N \geq 0$, there exists positive constants C, h such that

$$|(4\pi t)^{\frac{n}{2}} \exp(d(p,q)^2/4t) K(t;p,q) - \sum_{k=0}^{N} u_i(p,q)t^i|$$
$$\leqq C \exp(h d(p,q))\, t^{N+1}$$

for small positive t.

Proof. This is essentially due to [5], therefore the proof will be outlined.

Put

$$S_N(t;p,q) = (4\pi t)^{-n/2} \exp(-d(p,q)^2/4t) \sum_{i=0}^{N} u_i(p,q)t^i$$

$$T_N(t;p,q) = (\partial/\partial t + \Delta_2) S_N(t;p,q)$$

then

$$|S_N(t;p,q)| \leqq K t^{-n/2} e^{-d(p,q)^2/4t} e^{h d(p,q)}$$

$$|T_N(t;p,q)| \leqq L t^{N-n/2} e^{-d(p,q)^2/4t} e^{h d(p,q)}$$

and

$$K(t;p,q) = S_N(t;p,q) - \sum_{\lambda=1}^{\infty} (-1)^{\lambda} (T_N^{*\lambda} * S_N)(t;p,q),$$

where $*$ denotes the convolution. We need here the following estimate which is sharpened slightly than that of [5] : If two functions A(t;p, q) and B(t;p, q) are estimated by $K t^{\alpha} e^{-d(p,q)^2/4t} e^{h d(p,q)}$ and $L t^{\beta} e^{-d(p,q)^2/4t} e^{h d(p,q)}$ respectively and $\alpha, \beta \geqq -\frac{n}{2}$, then

$$|(A*B)(t;p,q)| \leqq KLC\, t^{\alpha+\beta+\frac{n}{2}+1} e^{-d(p,q)^2/4t} e^{h d(p,q)}$$

for small t, where C is a constant depending only upon n, h, and lower bound of sectional curvature of $\tilde{M}$. The proof can be done in the same way as [5], so we omit it. Applying this estimate to $A = S_N$ and $B = T_N$, we obtain

$$|T_N * S_N| \leqq KLC\, t^{N-\frac{n}{2}+1} e^{-d^2/4t} e^{hd},$$

and inductively

$$|T_N^{*\lambda} * S_N| \leqq KL^\lambda C^\lambda t^{\lambda(N+1)-\frac{n}{2}} e^{-d^2/4t} e^{hd}.$$

Hence

$$\left|\sum_{\lambda=1}^{\infty} (-1)^\lambda T_N^{*\lambda} * S_N\right| \leqq C' t^{-\frac{n}{2}+N+1} e^{-d^2/4t} e^{hd}$$

for small $t > 0$, as desired.

6. Asymptotic expansion. Using the estimates in §5, we easily observe that the sum

$$\sum_{\gamma \in \Gamma} K(t; p, \gamma q)$$

converges absolutely and uniformly on any compact subset in $\widehat{M} \times \widehat{M}$, and yields a fundamental solution $k(t;x, y)$ of the heat equation on M. Applying Proposition 4.2 to $K(t;p, q)$ we have

$$\Theta(t) = \sum e^{-\lambda_i t} = \int_M k(t;x,x)\, dv(x)$$

$$= \sum_{[\gamma] \in [\Gamma]} \int_{M_{[\gamma]}} dv_{[\gamma]}(x) \int_{N_{[\gamma]x}} K_{[\gamma]}(t; v)\, \varphi_{[\gamma]}(v)\, d\mu_{[\gamma]x}(v),$$

where $K_{[\gamma]}(t;\, v) = K(t;p,\gamma q)$ $\quad (p = \exp v)$. We put now

$$h_{[\gamma]}(t:x) = (4\pi t)^{\dim M_{[\gamma]}/2} \exp(\ell(\gamma)^2/4t) \int_{N_{[\gamma]x}} K_{[\gamma]}(t:v)\, \varphi_{[\gamma]}(v)\, d\mu_{[\gamma]x}(v)$$

so that $\Theta(t) = \sum_{[\gamma] \in [\Gamma]} (4\pi t)^{-\dim M_{[\gamma]}/2} \exp(-\ell(\gamma)^2/4t) \int_{M_{[\gamma]}} h_{[\gamma]}(t:x)\, dv_{[\gamma]}(x)$.

Then we have

Proposition 6.1. The function $h_{[\gamma]}$ can be expanded as

$$h_{[\gamma]}(t:x) \sim a^0_{[\gamma]}(x) + a^1_{[\gamma]}(x)\, t + \cdots \quad \text{as } t \downarrow 0$$

where the coefficients $a^{i}_{[\gamma]}(x)$ are smooth functions on $M_{[\gamma]}$.

The proof will be broken up into several steps. To begin, we make an estimate of the function $d(p, \gamma p)$. For this we write $d_\gamma(v) = d(\exp v, \gamma \exp v)$, $v \in N_\gamma$, and let $d_{[\gamma]}$ be the function on $N_{[\gamma]}$ induced from d_γ. It is obvious $d_{[\gamma]}(0) = \ell(\gamma)$.

<u>Lemma</u> 6.2. We can find constants $C_1 > 0$, C_2 such that

$$d_{[\gamma]}(v) \geqq C_1 \| v \| + C_2$$

for all $v \in N_{[\gamma]}$.

Proof. Let $v \in N_\gamma$ with $\| v \| = 1$, and put

$$\varphi(s) = d(\exp sv, \gamma \exp sv).$$

We recall here the second variation formula for arc length which asserts

$$\frac{d^2\varphi}{ds^2} = \varphi(s)^{-3} \int_0^1 \langle T, T \rangle \langle \nabla_X T, \nabla_X T \rangle - \langle \nabla_X T, T \rangle^2 dt - \varphi(s)^{-1} \int_0^1 \langle R(X, T)X, T \rangle dt$$

where we adopt the notation in § 3. The assumption for curvature and the Schwartz inequlity shows $\frac{d^2\varphi}{ds^2} \geqq 0$, so that φ is a convex function. Further

$$\frac{d^2\varphi}{ds^2}(0) = \frac{1}{2\varphi(0)} \int_0^1 \langle \nabla_T X, \nabla_T X \rangle - \langle R(X T)X, T \rangle dt > 0$$

from which our assertion follows.

We introduce the notations: For a positive integer N,

$$u_{i\gamma}(v) = u_i(\exp v, \gamma \exp v) \quad i = 0, 1, \ldots, N,$$

$$U_{N+1\gamma}(t;\ v) = (4\pi t)^{n/2} \exp(d^2/4t)\, t^{-N-1} (K - S_N)(t;\ \exp v, \gamma \exp v),$$

and $u_{i[\gamma]}(v)$, $U_{N+1[\gamma]}(v)$ are the induced functions on $N_{[\gamma]}$.
From Lemma 5.1, we obtain

$$\left|U_{N+1[\gamma]}(t;v)\right| \leqq C \exp(hd_{[\gamma]}(v)),$$

and by definition

$$h_{[\gamma]}(t;x) = (4\pi t)^{\dim M_{[\gamma]}/2 - n/2} e^{\ell(\gamma)^2/4t} \sum_{i=0}^{N} t^i \int_{N_{[\gamma]x}} e^{-d^2_{[\gamma]}(v)/4t} u_{i[\gamma]}(v) \varphi_{[\gamma]}(v)\, d\mu_{[\gamma]x}(v)$$

$$(*) \qquad + (4\pi t)^{\dim M_{[\gamma]}/2 - n/2} e^{\ell(\gamma)^2/4t} t^{N+1} \int_{N_{[\gamma]x}} e^{-d^2_{[\gamma]}(v)/4t} U_{N+1[\gamma]}(t;v) \varphi_{[\gamma]}(v)\, d\mu_{[\gamma]x}(v).$$

Hence, to prove Proposition 6.1, we must establish asymptotic properties of the integrals

$$I_i(t) = \int_{N_{[\gamma]x}} \exp(-d^2_{[\gamma]}(v)/4t) u_{i[\gamma]}(v)\, \varphi_{[\gamma]}(v)\, d\mu_{[\gamma]x}(v),$$

$$I_{N+1}(t) = \int_{N_{[\gamma]x}} \exp(-d^2_{[\gamma]}(v)/4t) U_{N+1[\gamma]}(t;v)\, \varphi_{[\gamma]}(v)\, d\mu_{[\gamma]x}(v).$$

More generally consider the integral of the form

$$I(t) = \int_{\mathbb{R}^s} \exp(-f(\xi)/t)\, g(\xi)\, d\xi,$$

where $f(\xi)$ and $g(\xi)$ are smooth functions on $\mathbb{R}^s$ satisfying

$$f(\xi) \geqq 0$$
$$f(\xi) \geqq c_1 \|\xi\|^2 + c_2 \qquad C_1 > 0$$
$$|g(\xi)| \leqq c_3 \exp c_4 f(\xi)^{1/2}.$$

Furthermore we suppose that the origin is only one critical point of $f(\xi)$, and Hessian of f at the origin is positive definite. Then we have a real form of stationary phase method:

<u>Lemma</u> 6.3. $I(t) \underset{t\downarrow 0}{\sim} (2\pi t)^{s/2} \exp(-f(0)/t)(c_0 + c_1 t + \ldots.)$.

Proof. Some observations show that without loss of generality we may assume $g(\xi) \in C_0(\mathbb{R}^s)$. Therefore we let suppose that the support of $g(\xi)$ is contained in a neighborhood U in which there exists a Morse type coordinate system $(\eta_1, \dots, \eta_s)$:

$$f(\eta_1, \dots, \eta_s) = \eta_1^2 + \cdots + \eta_n^2 + f(0).$$

Putting here $g_1(\eta_1, \dots, \eta_s) = g(\eta_1, \dots, \eta_s)\left|\det\left(\frac{\partial \xi^j}{\partial \eta_i}\right)\right|$ we get

$$I(t) = e^{-f(0)/t} \int_{\mathbb{R}^s} e^{-(\eta_1^2 + \cdots + \eta_s^2)/t} g_1(\eta_1, \dots, \eta_s)\, d\eta_1 \cdots d\eta_s$$

$$= t^{s/2} e^{-f(0)/t} \int_{\mathbb{R}^s} e^{-\|\eta\|^2} g_1(\sqrt{t}\,\eta)\, d\eta.$$

Making the Taylor expansion of g_1 and using the fact that

$$\int_{\mathbb{R}^s} e^{-\|\eta\|^2} \eta^\alpha\, d\eta = 0 \quad \text{if } |\alpha| = \alpha_1 + \cdots + \alpha_s \text{ is odd},$$

we are done.

We now revert to the proof of Prop. 6.1. We easily observe that the function $d^2_{[\gamma]}$ defined on the fiber $N_{[\gamma]x}$ has only one critical point 0, at which the Hessian is positive definite. Hence we are in the situation where we can apply the above lemma to the integrals $I_i(t)$, and get

$$I_i(t) = \exp(-\ell(\gamma)^2/4t)(4\pi t)^{n/2 - \dim M_{[\gamma]}/2} \times$$

$$\times (a_{i0} + a_{i1} t + \cdots + a_{iN-i} t^{N-i} + O(t^{N+1-i})).$$

Further

$$|I_{N+1}(t)| \leq C \int_{N_{[\gamma]x}} \exp(-d_{[\gamma]}^2(v)/4t)\, e^{h d_{[\gamma]}(v)} \varphi_{[\gamma]}(v)\, d\mu_{[\gamma]x}(v)$$

$$\leq e^{-\ell(\gamma)^2/4t}\, O(t^{\frac{n}{2} - \dim M_{[\gamma]}/2}).$$

Substituting these estimates into $(*)$ and adding up the coefficients, we have

$$h_{[\gamma]}(t; x) = \sum_{i=0}^{N} a^i_{[\gamma]}(x)\, t^i + O(t^{N+1}), \quad a^i_{[\gamma]}(x) = \sum_{j+k=i} a_{jk}.$$

Smooth dependence of the coefficients $a^i_{(n)}(x)$ is obvious from the construction.

We now give an explicit calculation for $a^i_{(n)}(x)$. For this let $f(\xi)$ and $g(\xi)$ be functions given in Lemma 6.3. Substituting the formal expansion of f, g by homogeneous polynomials

$$f(\xi) = f(0) + \frac{1}{2}\,\mathrm{Hess}\, f(0)(\xi,\xi) + \sum_{k=3}^{\infty} f_k(\xi),$$

$$g(\xi) = \sum_{k=0}^{\infty} g_k(\xi)$$

into $I(t)$, and making the change of variables $\xi = \sqrt{t}\,\eta$, one has

$$I(t) = t^{s/2} e^{-f(0)/t} \int_{\mathbb{R}^s} \exp\left(-\frac{1}{2}\mathrm{Hess}\, f(0)(\eta,\eta)\right) \exp\left(-\sum_{k=1}^{\infty} t^{\frac{k}{2}} f_{k+2}(\eta)\right) \times$$
$$\times \left(\sum_{j=0}^{\infty} g_j(\eta)\, t^{j/2}\right) d\eta$$

$$= t^{s/2} e^{-f(0)/t} \int_{\mathbb{R}^s} \exp\left(-\frac{1}{2}\mathrm{Hess}\, f(0)(\eta,\eta)\right) \sum_{k=0}^{\infty} t^{k/2} Q_k(\eta)\, d\eta$$

where $Q_k(\eta)$ is a polynomial of η whose coefficients are polynomials of derivatives $D^{\alpha} f(0)$, $D^{\beta} g(0)$ ($|\alpha| \geq 3$, $|\beta| \geq 0$). Let A be a symmetric matrix whose (i,j)-component is $\partial^2 f(0)/\partial\eta_i\,\partial\eta_j$. Then the change of variable $\eta = A^{-\frac{1}{2}}x$ yields

$$I(t) = t^{s/2} e^{-f(0)/t} \int_{\mathbb{R}^s} \exp\left(-\frac{1}{2}|x|^2\right) \sum_{k=0}^{\infty} t^{k/2} Q_k(A^{-\frac{1}{2}}x)\, |\det A|^{-1/2}\, dx$$

so

$$I(t) \sim (2t\pi)^{\frac{s}{2}} e^{-f(0)/t} |\det A|^{-1/2} \sum_{k=0}^{\infty} a_{2k}\, t^k$$

$$a_{2k} = \int_{\mathbb{R}^s} \exp\left(-\frac{1}{2}|x|^2\right) Q_k(A^{-1/2}x)\, dx$$

where we have used the fact

$$\int_{\mathbb{R}^s} \exp\left(-\frac{1}{2}|x|^2\right) Q_k(A^{-1/2}x)\, dx = 0 \quad \text{if } k \text{ is odd.}$$

The coefficient a_{2k} is given by a universal polynomial of $A^{-1/2}$, $D^{\alpha} f(0)$ and $D^{\beta} g(0)$. For instance

$$a_0 = g(0)$$

$$\begin{aligned} a_1 = \frac{g(0)}{12} \sum & \frac{\partial^3 f(0)}{\partial \xi_i \partial \xi_j \partial \xi_k} \rho_{i\alpha} \rho_{j\beta} \rho_{k\gamma} \frac{\partial^3 f(0)}{\partial \xi_{i'} \partial \xi_{j'} \partial \xi_{k'}} \rho_{i'\alpha'} \rho_{j'\beta'} \rho_{k'\gamma'} \times \\ & \times g_{\alpha\beta\gamma\alpha'\beta'\gamma'} \\ & - \frac{1}{6} \sum \frac{\partial^3 f(0)}{\partial \xi_i \partial \xi_j \partial \xi_k} \rho_{i\alpha} \rho_{j\beta} \rho_{k\gamma} \rho_{\ell\delta} \frac{\partial g(0)}{\partial \xi_\ell} g_{\alpha\beta\gamma\delta} \\ & + \frac{1}{2} \sum \frac{\partial^2 g(0)}{\partial \xi_i \partial \xi_j} \rho_{i\alpha} \rho_{j\beta} g_{\alpha\beta}\,, \end{aligned}$$

in which in general we put

$$g_{\alpha_1 \cdots \alpha_t} = \int_{\mathbb{R}^r} x_{\alpha_1} \cdots x_{\alpha_t} e^{-|x|^2/2}\, dx\,, \qquad (\rho_{\alpha\beta}) = A^{-1/2}.$$

The formula gets rapidly out of hand.

Finally we would like to show that the first coefficient $a^0_{[\gamma]}$ coincides with $\det(P_\gamma - I)^{-1/2}$, where P_γ is the linearized Poincare mapping associated with c_x.

For this we note the Poincare mapping induces

$$P_{[\gamma]} : N_{[\gamma]x} \oplus N_{[\gamma]x} \longrightarrow N_{[\gamma]x} \oplus N_{[\gamma]x}$$

and $\det(P_\gamma - I) = \det(P_{[\gamma]} - I)$. We then put

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} (\langle v_i, U_j(1)\rangle)_{ij} & (\langle v_i, V_j(1)\rangle)_{ij} \\ (\langle v_i, \nabla U_j(1)\rangle)_{ij} & (\langle v_i, \nabla V_j(1)\rangle)_{ij} \end{pmatrix}$$

where $U_i(t)$, $V_i(t)$ are Jacobi fields along $c_x : [0,1] \longrightarrow M$, with

$$\begin{cases} U_i(0) = v_i \\ \nabla U_i(0) = 0 \end{cases} \qquad \begin{cases} V_i(0) = 0 \\ \nabla V_i(0) = v_i \end{cases}$$

$v_1, \ldots, v_s$ ($s = \dim N_{[\gamma]x}$) being an orthogonal basis of $N_{[\gamma]x}$.
Since

$$\det(P_{[\gamma]} - I) = \det\begin{pmatrix} A-I & B \\ C & D-I \end{pmatrix}$$

$$= \det(C + (D-I)B^{-1}(I-A)) \cdot \det B,$$

it is enough to see that

$$|\det B| = \Theta(p, \gamma p)$$

$$|\det(C + (D-I)B^{-1}(I-A))| = |\det \mathrm{Hess}\, d^2_{[\gamma]}(0)| \cdot 2^{-s}$$

For this let $v_{s+1}, \ldots, v_n \in T_x M_{[\gamma]}$ be an orthogonal basis, and $V_i(t)$ ($i=s+1,\ldots,n$) be Jacobi fields with

$$V_i(0) = 0, \qquad \nabla V_i(0) = v_i$$

Since $\langle v_i, V_j(1)\rangle = 0$ for $s+1 \leq i \leq n$, $1 \leqq j \leqq s$, we have

$$\Theta(p, \gamma p) = |\det\langle v_i, V_j(1)\rangle_{1=i,j=n}|$$

$$= |\det\langle v_i, V_j(1)\rangle_{1=i,j=s}|\,|\det\langle v_i, V_j(1)\rangle_{s+1=i,j=n}|$$

$$= |\det B|\,\Theta_{\widetilde{M}_\gamma}(p, \gamma p)$$

But easy observations show $\Theta_{\widetilde{M}_\gamma}(p, \gamma p) = 1$, which implies the first equality.

On the other hand

$$\mathrm{Hess}\, d^2_{[\gamma]}(0)(v_i, v_j) = 2\int_0^1 \langle \nabla X_i, \nabla X_j\rangle - \langle R(T, X_i)T, X_j\rangle\, dt$$

$$= 2\langle X_i, \nabla X_j\rangle\Big|_0^1,$$

where $X_i(t)$ is a Jacobi field such that $X_i(0) = X_i(1) = v_i$.

Hence we can find a matrix $E = (e_{ij})$ such that

$$X_j(t) = U_j(t) + \sum e_{kj} V_k(t).$$

By definition of X_j, we have

$$v_j = X_j(1) = U_j(1) + \sum e_{kj} V_k(1)$$

$$X_j(1) = \nabla U_j(1) + \sum e_{kj} \ V_k(1)$$

$$X_j(0) = \sum e_{kj} v_k$$

so that

$$\langle X_i, \nabla X_j \rangle \Big|_0^1 = \langle v_i, \nabla U_j(1) \rangle + \sum e_{kj} \langle v_i, \nabla V_k(1) \rangle - e_{ij}$$

$$= C + DE - E$$

But $\langle v_i, v_j \rangle = \langle v_i, U_j(1) \rangle + BE$, hence $E = B^{-1}(I - A)$.
This implies

$$\langle X_i, \nabla X_j \rangle \Big|_0^1 = C + (D - I)B^{-1}(I - A),$$

as desired.

Sumarizing these results we have

<u>Theorem.</u> Let M be a compact nondegenerate manifold of nonpositive curvature. Then for each $[\gamma] \in [\Gamma]$ there exists a function $h_{[\gamma]}(t; x)$ on $\mathbb{R}_+ \times M_{[\gamma]}$ such that

i) $\displaystyle Ⓗ(t) = \sum_{[\gamma] \in [\Gamma]} (4\pi t)^{-\dim M_{[\gamma]}/2} \exp\left(-\ell(\gamma)^2/4t\right) \int_{M_{[\gamma]}} h_{[\gamma]}(t; x)\, dv_{[\gamma]}(x),$

ii) each $h_{[\gamma]}(t; x)$ can be expanded as

$$h_{[\gamma]}(t; x) \underset{t \downarrow 0}{\sim} a^0_{[\gamma]}(x) + a^1_{[\gamma]}(x)\, t + \cdots\cdots ,$$

where $a^i_{[\gamma]}(x)$ are smooth functions on $M_{[\gamma]}$, which can be represented as universal polynomials of normal derivatives of $d^2_{[\gamma]}$, $\varphi_{[\gamma]}$, $u_{i[\gamma]}$, and $(\text{Hess } d^2_{[\gamma]})^{-1/2}$. The first term is given by

$$a^0_{[\gamma]}(x) = \left|\det(P_\gamma - I)\right|^{-1/2}.$$

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