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On parabolicity of a Riemann surface

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1. On the occasion of the Colloquium held at Joensuu in August, 1978, Professor Grunsky asked the author personally the following question.

Let a_1, a_2, ... be a sequence of positive numbers decreasing to zero, and denote the points $-1 + ia_k$ and $1 + ia_k$ in the w-plane by α_k and β_k respectively. Let s_k be the segment $\alpha_k\beta_k$. Let $\alpha = -1$ and $\beta = 1$, and s_o be the segment $\alpha\beta$. Consider an extended plane P slit along s_o, s_1, s_2, ..., and an extended plane P_k slit only along s_k for k = 1, 2, ... Connect P_k crosswise with P through s_k. Let Q_1, Q_2, ... be extended planes slit along s_o, and identify the upper shore of Q_1 with the lower shore of P, the upper shore of Q_2 with the lower shore of Q_1, and so on. Denote by R the resulting simply connected Riemann surface $P \cup P_1 \cup P_2 \cup \cdots \cup Q_1 \cup Q_2 \cup \cdots$. The question is as to whether R is of parabolic type.

2. Later the author asked Professor Grunsky for the motivation of the problem. He replied: "The problem occurred to me in connecting with a problem which I treated in a paper presented at the conference in Jyväskylä in 1970: "Analytische Fortsetzung über offene Randkomponenten einer barandeten Riemannschen Fläche" (Lecture Notes in Math. No. 419, Springer,

1974, pp. 143-155). In 1.4. (p.148), case B), I needed an ad hoc hypothesis, which also occurs in the summary of the results, pp. 154-155, δ). If my remembrance of my former endeavours is correct, I can dispense with this assumption on the basis of the theorem you just have proved. I do not know when, or whether at all, I shall come back to this field which I do not appreciate so much any more; on the other hand it would, most likely, not be too difficult and timeconsuming to give the finishing touch to this work, and so I am very glad to have the solution of our problem. But just now, I am too busy with other things (univalent functions)."

The author is thankful to him for all.

3. Theorem. R is of parabolic type.

Proof. It will be sufficient to show that the family Γ of curves starting from a closed disk Δ in P and tending to the ideal boundary of R has infinite extremal length. We may assume that Δ lies above s_1. Denote by P^+ (P^- resp.) the upper (lower resp.) half of P.

Divide Γ into four families. The first family Γ_1 consists of curves c of Γ such that some terminal part of c is contained in $P^- \cup Q_1 \cup Q_2 \cup \cdots$. The second (third resp.) family Γ_2 (Γ_3 resp.) consists of curves of Γ each of which contains a sequence of points of P^+ converging to α (β resp.). The fourth family Γ_4 consists of curves of $\Gamma - \Gamma_2 - \Gamma_3$ each of which contains a sequence of points of P^+ converging to a point of $s_o - \{\alpha\} - \{\beta\}$. Then $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

Since $Q_1 \cup Q_2 \cup \cdots$ forms a "half" of a logarithmic surface, the extremal length $\lambda(\Gamma_1) = \infty$. To prove $\lambda(\Gamma_2) = \infty$, map the part R' of R lying above the left half plane $\mathrm{Re}\, w < 0$ conformally onto the left half plane $\mathrm{Re}\, z < 0$, and denote by $w = f(z)$ the composition of the inverse mapping onto R' and the projection into the w-plane. Take any $c \in \Gamma_2$ and let $\{w_n\}$ be a sequence of points of $c \cap P^+$ which converges to α and whose image sequence converges to z_o on the imaginary axis. For any $c' \in \Gamma_2$ we can find a sequence $\{\gamma_n\}$ of arcs in P^+ such that γ_n connects w_n and c' for each n and its length tends to 0 as $n \to \infty$. By applying Koebe's theorem to $f(z)$ in $\mathrm{Re}\, z < 0$ we see that $f^{-1}(\gamma_n)$ tends to z_o as $n \to \infty$. Thus the image of $c' \cap R'$ contains a sequence of points tending to z_o. By symmetry R is mapped conformally outside a point or a segment on the imaginary axis. The image of c' and hence the image of every curve of Γ_2 contains a sequence of points tending to z_o. It follows that $\lambda(\Gamma_2) = \infty$. Similarly $\lambda(\Gamma_3) = \infty$.

Finally let Λ_n $(n \geqq 2)$ be the subfamily of Γ_4 such that the cluster set of the part in P of each curve of Λ_n is contained in $(-1 + 1/n, 1 - 1/n) \subset s_o$. Evidently $\Gamma_4 = \cup_n \Lambda_n$. To prove $\lambda(\Lambda_n) = \infty$, denote by $\alpha_k^{(n)}$ and $\beta_k^{(n)}$ the points $-1 + 1/n + ia_k$ and $1 - 1/n + ia_k$ respectively, and map P_k conformally onto a rectangle D_k of height one so that the end segments $\alpha_k\alpha_k^{(n)}$ and $\beta_k^{(n)}\beta_k$ correspond to the sides of length one; observe that D_1, D_2, ... have the same shape. Given a curve of Λ_n, its image in D_k connects opposite sides if k is large. Define a density ρ_k in P_k by means of the constant density $1/k$ in D_k, and let ρ be the density on R equal to ρ_k in P_k for $k = 1, 2, \ldots$ and to 0 elsewhere. Then $\int_c \rho ds = \infty$ for every $c \in \Lambda_n$ and $\iint \rho^2 dxdy < \infty$. Hence $\lambda(\Lambda_n) = \infty$ for every n so that $\lambda(\Gamma_4) = \infty$. Thus $\lambda(\Gamma) = \infty$.