数理解析研究所講究録
第364巻 1979年 5-15

EXISTENCE OF QUASICONFORMAL MAPPINGS BETWEEN RIEMANN SURFACES

by

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The purpose of this paper is to give a criterion for the existence of a direct or indirect quasiconformal mapping between two given Riemann surfaces in terms of their Royden compactifications as follows:

THEOREM. *There exists a quasiconformal mapping of a Riemann surface* R *onto another* S *if and only if their Royden compactifications* R^* *and* S^* *are homeomorphic.*

Before proceeding to the proof of the above theorem we recall the definition of the Royden compactification and that of the quasiconformal mapping for the convenience of the reader and we also state some of their properties, known and new, which will be used in our proof. We denote by $M(R)$ the class of bounded continuous Tonelli functions f on a Riemann surface R with the finite Dirichlet integrals $D_R(f) = \int_R |\mathrm{grad}\ f(z)|^2\, dxdy$ which forms a Banach algebra equiped with the norm $||f|| = \sup_R |f| + (D_R(f))^{1/2}$. The algebra $M(R)$ is referred to as the *Royden algebra* associated with R (cf. e.g. Sario-Nakai [3; p.148]). The *Royden compactification* R^* of a Riemann surface R is a topological space with the following four conditions (cf. e.g. [3; p.154]): a) R^* is a compact Hausdorff space; b) R^* contains R as an open and dense subspace; c) Every function in

$\mathcal{M}(R)$ can be continuously extended to R^* so that $\mathcal{M}(R) \subset C(R^*)$, the space of continuous functions on R^*; d) $\mathcal{M}(R)$ separates points of R^*. The existence and uniqueness of R^* is seen as follows (cf. e.g. [3; p.155]): We denote by $X = X(R)$ the *character space* over $\mathcal{M}(R)$, i.e. X is the set of multiplicative linear functionals χ on $\mathcal{M}(R)$ with $\langle 1, \chi \rangle = 1$. Since X is a closed subset of the unit sphere of the dual space $\mathcal{M}(R)^*$ of $\mathcal{M}(R)$ with the weak star topology $\sigma(\mathcal{M}(R)^*, \mathcal{M}(R))$, X is a compact Hausdorff space. For a point p in R we can define a χ_p in X by $\langle f, \chi_p \rangle = f(p)$. It can be easily seen that $p \mapsto \chi_p$ defines a topological injection from R to X and therefore we can view R as a topological subspace of X. It is also easily checked that X satisfies the four conditions a)-d). Let R^* be a Royden compactification of R. For each p in R^* we can consider a χ_p in X defined by $\langle f, \chi_p \rangle = f(p)$. Then it is seen that $p \mapsto \chi_p$ defines a homeomorphism between R^* and X. Thus we have seen the existence and the uniqueness of the Royden compactification R^* of any Riemann surface R and that R^* is represented as the character space $X(R)$ over $\mathcal{M}(R)$. Needless to say R is closed if and only if $R^* = R$. For a subset Z of R we denote by $\overline{Z}$ the closure of Z in R^*, not in R.

Consider a sequence $(A_n)_{n=1}^{\infty}$ of annuli A_n contained in a simply connected subregion U_n in $R - \overline{R}_{n(k)}$, where $(R_n)_{n=1}^{\infty}$ is a canonical exhaustion of an open Riemann surface R, such that $\overline{U}_n \cap \overline{U}_m = \emptyset$ $(n \neq m)$ and $k(n) \to +\infty$ $(n \to +\infty)$. For convenience we call such a sequence $(A_n)_{n=1}^{\infty}$ as a *distinguished sequence* of annuli on R. The closed set $R - A_n$ consists of two components one of which X_n is compact and the other Y_n is noncompact. We call $X = \bigcup_{n=1}^{\infty} X_n$ the *inside* of the distinguished sequence $(A_n)_{n=1}^{\infty}$ and $Y = \bigcap_{n=1}^{\infty} Y_n$ the *outside* of $(A_n)_{n=1}^{\infty}$. Clearly

$X \cap Y = \emptyset$ in R but $\overline{X} \cap \overline{Y}$ may or may not be empty in R^*. The following observation plays one of decisive roles in our proof:

LEMMA 1. *The inside* X *and the outside* Y *of a distinguished sequence* $(A_n)_{n=1}^{\infty}$ *of annuli on an open Riemann surface* R *have disjoint closures in* R^*, *i.e.* $\overline{X} \cap \overline{Y} = \emptyset$, *if and only if*

$$(1) \qquad \sum_{n=1}^{\infty} 1/\text{mod } A_n < +\infty.$$

Here $\text{mod } A_n$ is the *modulus* $\log \mu_n$ of the annulus A_n, i.e. A_n is conformally equivalent to the circular ring $1 < |z| < \mu_n$, so that if w_n is the harmonic measure of the inner boundary $(\partial A_n) \cap X_n$ of A_n with respect to A_n, then $\text{mod } A_n = 2\pi/D_R(w_n)$, where w_n is extended to R by setting $w_n = 0$ on Y_n and 1 on X_n. We first assume the validity of (1). Let $w = \sum_{n=1}^{\infty} w_n$. Then it is a continuous Tonelli function on R with $0 \leqq w \leqq 1$ and

$$D_R(w) = \sum_{n=1}^{\infty} D_R(w_n) = 2\pi \sum_{n=1}^{\infty} 1/\text{mod } A_n < +\infty.$$

Therefore w belongs to $M(R)$ and a fortiori w is continuous on R^*, which implies that $\overline{X} \cap \overline{Y} = \emptyset$. Conversely suppose that $\overline{X} \cap \overline{Y} = \emptyset$. Then there exists a function h in $C(R^*)$ with $h|\overline{X} > 1$ and $h|\overline{Y} < 0$. Since $M(R)$ is dense in $C(R^*)$, we can find a g in $M(R)$ such that $g|\overline{X} > 1$ and $g|\overline{Y} < 0$. The function f defined by $f(p) = \max(\min(g(p),1),0)$ pointwise on R again belongs to $M(R)$ by the lattice property of $M(R)$ (cf. e.g. Constantinescu-Cornea [1; p.69], or [3; p.147]). By the Dirichlet principle, $D_R(w_n) \leqq D_{A_n}(f)$. Therefore

$$2\pi \sum_{n=1}^{\infty} 1/\text{mod } A_n = \sum_{n=1}^{\infty} D_R(w_n) \leqq \sum_{n=1}^{\infty} D_{A_n}(f) = D_R(f) < +\infty,$$

i.e. the validity of (1) is derived from $\overline{X} \cap \overline{Y} = \emptyset$ in R^*. The proof of Lemma 1 is herewith complete.

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The fact that the Royden compactification R^* of any open Riemann surface R is not metrizable is usually understood as a drawback of this compactification to develope Analysis on R^*. This situation, however, can be conveniently made use of in our proof. Namely, we have

LEMMA 2 (cf. e.g. [1; p.103], or [3; p.156]). *A point* p *in* R^* *belongs to the Royden boundary* $R^* - R$ *of* R *if and only if* p *is not* G_δ.

Since this can be derived easily from Lemma 1, we give here a proof of this well known fact, although it is essentially the same as known ones. Since any point in R is G_δ, we only have to show that any point p in $R^* - R$ is not G_δ. Contrariwise suppose that p is G_δ. Then there exists a sequence $(V_n)_{n=1}^{\infty}$ of open neighborhoods of p in R^* such that $\overline{V}_{n+1} \subset V_n$ $(n = 1, 2, \ldots)$ and $\bigcap_{n=1}^{\infty} \overline{V}_n = \{p\}$. Take an arbitrary sequence $(U_n)_{n=1}^{\infty}$ of simply connected regions U_n in $(V_n - \overline{V}_{n+1}) \cap R$. Next take an annulus A_n in each U_n with $\mathrm{mod}\, A_n = n^2$. Then $(A_n)_{n=1}^{\infty}$ is a distinguished sequence of annuli and both of X_n and Y_n converge to p and therefore $\overline{X} \cap \overline{Y}$ contains p, i.e. $\overline{X} \cap \overline{Y} \neq \emptyset$, which contradicts the finiteness of the sum of $1/\mathrm{mod}\, A_n$ for $n = 1, 2, \ldots$. This completes the proof of Lemma 2.

The *maximal dilatation* $K(T)$ of a homeomorphism T of a Riemann surface R onto another S is $\inf \{c \mid c^{-1} \mathrm{mod}\, Q \leq \mathrm{mod}\, TQ \leq c\, \mathrm{mod}\, Q,$ $Q \in \{Q\}\}$. Here $\{Q\}$ is the family of quadrilaterals Q in R, each Q consisting of a Jordan region Q' and four distinguished points $z_1, \ldots, z_4$ on $\partial Q'$. Map Q' conformally onto a rectangle Q'' such that $z_1, \ldots, z_4$ correspond to the four vertices of Q''. Let a and b be the lengths of the sides of Q'' which correspond to $z_1 z_2$ and $z_2 z_3$. The ratio a/b is determined uniquely by Q and is denoted by mod Q. A homeomorphism T with finite K(T) is referred to as a *quasiconformal mapping* of R onto S. This is called the *geometric definition* of quasiconformality (cf. e.g. Lehto-Virtanen [2; p.17]). Note that we also include the orientation reversing homeomorphisms into our class of quasiconformal mappings in addition to the usual orientation preserving ones. We denote by $A \approx B$ if two topological spaces A and B are homeomorphic. Similarly we denote by $R \mathrel{\approx\!\!\!\approx} S$ if two Riemann surfaces R and S are quasiconformally equivalent, i.e. if there exists a quasiconformal mapping of R onto S. We need the following so called *analytic definition* of quasiconformality:

LEMMA 3 (cf. e.g. [2; p.176]). *A homeomorphism* T *of a Riemann surface* R *onto another* S *is quasiconformal if the local expression* T(z) *of* T *is a Tonelli function and there exists a constant* K *such that* $\max_\alpha |\partial_\alpha T(z)| \leqq K \min_\alpha |\partial_\alpha T(z)|$ *almost everywhere on* R, *where* ∂_α *is the derivative at* z *in the direction* $e^{i\alpha}$.

Suppose there exists a homeomorphism T of a Riemann surface R onto another S. Take an arbitrary point p in R, a parametric disk U about p, a local parameter z on U with $z(p) = 0$ and $z(U) = \{|z| < 1\}$, and a local parameter ζ on TU with $\zeta(Tp) = 0$ and $\zeta(TU) = \{|\zeta| < 1\}$. In terms of the local expression $\zeta = T(z)$ of T, we consider

$$\begin{cases} M(r) = \max_{|z| = r} |T(z)|, \\ m(r) = \min_{|z| = r} |T(z)| \end{cases}$$

for each r in the interval (0, 1), and define the *circular dilatation* $\delta(p)$ of T at p by the following upper limit:

$$\delta(p) = \limsup_{r \to 0} M(r)/m(r).$$

It is easy to see that $1 \leqq \delta(p) \leqq +\infty$ and $\delta(p)$ is determined only by p not depending on the special choice of parametric disks and local parameters. The following characterization of quasiconformality will play a decisive role in our proof:

LEMMA 4 (cf. e.g. [2; p.187]). *A homeomorphism* T *of a Riemann surface* R *onto another* S *is quasiconformal if and only if the circular dilatation* $\delta(p)$ *of* T *is bounded on* R.

We need one more preparation of technical nature. Let T be a homeomorphism of a Riemann surface R onto another S so that $R \approx S$. Suppose moreover that R and S are interiors of compact bordered surfaces $\bar{R}$ and $\bar{S}$ whose borders ∂R and ∂S consist of finite numbers of disjoint quasiconformal curves (cf. [2; p.101]). We allow the case ∂R and ∂S are empty so that R and S are closed. Such surfaces R and S will be referred to as being *finite*. Suppose moreover that T can be extended to a homeomorphism $\bar{T}$ of $\bar{R}$ onto $\bar{S}$ such that the circular dilatation $\bar{\delta}(p)$ of $\bar{T}$ is bounded in a neighborhood of ∂R. Then we say that R and S are *canonically homeomorphic*. If R and S are closed, then no additional requirment other than $R \approx S$ is imposed upon to conclude that R and S are canonically homeomorphic.

LEMMA 5. *If* T *is a canonical homeomorphism of a finite Riemann surface* R *onto another* S, *then there exists a quasiconformal mapping* T_1 *of* R *onto* S *such that* $T = T_1$ *in a neighborhood of* ∂R. *In particular, if* R *and* S *are closed and* $R \approx S$, *then* $R \approxeq S$.

If ∂R and ∂S are not empty, then the result is well known at least for the planar R and S (cf. e.g. [2; p.102]) and the extension to the present setting is not difficult. If R and S are closed, then remove a disk U from R and V from S and consider the conformal mapping T of U onto V. Clearly T can be extended to a homeomorphism of R onto S such that T is conformal in a neighborhood of $\overline{U}$. Then $R - \overline{U}$ and $S - \overline{V}$ are canonically homeomorphic, and $R \approxeq S$ can be deduced. This lemma is by any mean simple and elementary for the *two* dimensional case. However we do not know whether this is also valid for higher dimensional cases. If this is certainly the case, then we can also extend our theorem to the case of Riemannian manifolds R and S of general dimensions.

We now proceed to the *proof of Theorem*. We will actually prove a bit more. Consider the following three conditions for Riemann surfaces R and S: [A] $R \approxeq S$; [B] $M(R) \simeq M(S)$ (*algebraically isomorphic*); [C] $R^* \approx S^*$. We will show that [A] → [B], [B] → [C], and [C] → [A]. Thus, in particular, we see that *the Royden algebra is determined by its maximal ideal space (i.e. its character space)*. Let T be a quasiconformal mapping of R onto S. By Lemma 3 we can see that $f \circ T$ belongs to $M(R)$ for every f in $M(S)$ and $f \mapsto f \circ T$ gives rise to an algebraic isomorphism of $M(S)$ onto $M(R)$ (cf. e.g. [3; pp.212-213]), i.e. [A] → [B]. Next suppose that there exists an algebraic isomorphism t of $M(S)$ onto $M(R)$. For each χ in $X(R)$ define $t^*\chi$ in $X(S)$ defined by $\langle f, t^*\chi\rangle = \langle tf, \chi\rangle$ for every f in $M(S)$. Then $t^*: X(R) \to X(S)$ gives rise to a homeomorphism T^* of R^* onto S^* (cf. e.g. [3; pp.213-214]), i.e. we have shown [B] → [C]. In passing we remark that if $tf = f \circ T$, then T^* is an extension of T. Therefore we see that *any quasiconformal mapping* T *of a Riemann surface* R *onto another* S *can be uniquely extended to a homeomorphism* T^* *of* R^* *onto* S^*. This, of course, has been long known since Royden introduced the conpactification now bearing his name.

The essential part of our proof is to show the implication [C] → [A]. Let T^* be a homeomorphism of R^* onto S^*. By Lemma 2, the restriction T of T^* on R is a homeomorphism of R onto S. If R and S are closed, then, by Lemma 5, $R \approx S$. Therefore, hereafter in the proof, we assume that R and S are open. We also fix a canonical exhaustion $(R_n)_{n=1}^{\infty}$ of R. Consider the circular dilatation $\delta(p)$ of T on R and

$$\delta_n = \sup_{p \in R-\overline{R}_n} \delta(p).$$

Since δ_n is nonincreasing as n increases, either $\delta_n = +\infty$ for every n or $\lim_{n \to \infty} \delta_n < +\infty$. We maintain that the latter is the case.

Suppose contrariwise that the former alternative occurs. Then we can find a point p_n in $R - \overline{R}_n$ with

$$\delta(p_n) > \exp(n^2)$$

for each n. We can moreover assume that $p_n \neq p_m$ $(n \neq m)$. Fix a parametric disk U_n about p_n contained in $R - \overline{R}_n$ with its closure and a local parameter z on U_n with $z(p_n) = 0$ and $z(U_n) = \{|z| < 1\}$ for each n. We can also choose U_n so as to satisfy $\overline{U}_n \cap \overline{U}_m = \emptyset$ $(n \neq m)$. Fix an n and let ζ be a local parameter on TU_n with $\zeta(Tp_n) = 0$ and $\zeta(TU_n) = \{|\zeta| < 1\}$. Using the local expression $\zeta = T(z)$ of T on U_n, we can choose an r_n in the interval $(0, 1)$ so small that

$$M(r_n)/m(r_n) > \exp(n^2).$$

Let B_n be the annulus in TU_n on S bounded by the inner circle ℓ_ζ: $|\zeta| = m(r_n)$ and the outer circle L_ζ: $|\zeta| = M(r_n)$. Since, then, $\text{mod}\, B_n = \log(M(r_n)/m(r_n))$, we conclude that

$$\text{(2)} \qquad \text{mod}\, B_n > n^2 \quad (n = 1, 2, \ldots).$$

The closed region F_1 bounded by the counter image ℓ_z of ℓ_ζ under T has at least one point z_1 in common with the circle $|z| = r_n$ and the complement F_2 in U_n: $|z| < 1$ of the Jordan region bounded by the counter image L_z of L_ζ under T has at least one point z_2 in common with the circle $|z| = r_n$. Clearly $z_1 \neq z_2$. We denote by A_n the annulus in U_n bounded by ℓ_z and L_z. By embedding U_n: $|z| < 1$ in the complex plane $|z| < +\infty$ and enlarging F_2 by adding $|z| \geqq 1$, the annulus A_n can be viewed as an annulus in the plane $|z| < +\infty$ separating two points z_1 and 0 from two points z_2 and ∞. By the Teichmüller module theorem (cf. e.g. [2; p.58]), we obtain the following estimate of the modulus of the annulus A_n:

$$\text{mod } A_n \leqq 2\, G(|z_1|^{1/2}(|z_1| + |z_2|)^{-1/2})$$

where $G(t)$ is the modulus of the Grötzsch extremal region that is the unit disk $\{|z| < 1\}$ less the closed segment $[0, t]$ on the real line $(0 < t < 1)$. Since $|z_1| = |z_2| = r_n$, the right hand side of the above displayed inequality is $2\, G(2^{-1/2}) = \pi$ (cf. e.g. [2; p.63]). Hence we have the following estimates:

$$\text{(3)} \qquad \text{mod } A_n \leqq \pi \quad (n = 1, 2, \ldots).$$

Observe that $(A_n)_{n=1}^\infty$ is a distinguished sequence of annuli A_n on R and $(TA_n)_{n=1}^\infty$ is also a distinguished sequence of annuli $TA_n = B_n$ on S. Let X and Y be the inside and the outside of $(A_n)_{n=1}^\infty$. Then TX and TY are the inside and the outside of $(TA_n)_{n=1}^\infty$. The inequalities (3) imply that

$$\sum_{n=1}^\infty 1/\text{mod } A_n = +\infty,$$

which, by Lemma 1, yields $\overline{X} \cap \overline{Y} \neq \emptyset$ in R^*. Similarly by the inequalities (2) we observe that

$$\sum_{n=1}^{\infty} 1/\text{mod } TA_n = \sum_{n=1}^{\infty} 1/\text{mod } B_n \leqq \sum_{n=1}^{\infty} 1/n^2 < +\infty,$$

and again by Lemma 1, we conclude that $\overline{TX} \cap \overline{TY} = \emptyset$ in S^*. On the other hand,

$$\overline{TX} \cap \overline{TY} = T^*\overline{X} \cap T^*\overline{Y} = T^*(\overline{X} \cap \overline{Y}) \neq \emptyset$$

since $\overline{X} \cap \overline{Y} \neq \emptyset$. This is clearly a contradiction. We have thus shown that the sequence $(\delta_n)_{n=1}^{\infty}$ has a finite limit.

Fix an arbitrary K in the interval $(\lim_{n\to\infty} \delta_n, +\infty)$. Then we can choose an $n > 1$ such that $\delta_{n-1} = \sup_{p \in R-\overline{R}_{n-1}} \delta(p) < K$. This means, by Lemma 4, that T is a quasiconformal mapping of each component of $R - \overline{R}_{n-1}$ onto the corresponding one of $S - T\overline{R}_{n-1}$ with a maximal dilatation less than K. Thus R_n and TR_n are finite Riemann surfaces canonically homeomorphic in the sense of Lemma 5. Hence T can be deformed to a quasiconformal mapping T_1 of R_n onto TR_n such that $T = T_1$ in a neighborhood of ∂R_n. Let $T_2 = T$ on $R - R_n$ and $T_2 = T_1$ on R_n. It is readily seen that T_2 is a quasiconformal mapping of R onto S, and we have shown the implication [C] → [A].

The proof of Theorem is herewith complete.

Although $R \approxeq S$ and $R^* \approx S^*$ are equivalent and any homeomorphism T^* of R^* onto S^* is an extension of a certain homeomorphism T of R onto S, T need not be a quasiconformal mapping. Such T's are characterized as *Royden mappings* (cf. [3; pp.216-221]) which are generalizations of quasiconformal mappings. As a consequence of the proof of Theorem, we see that a homeomorphism T of R onto S is a Royden mapping if and only if T is a quasiconformal mapping outside a compact subset of R. In other words, a Royden mapping is a homeomorphism which is quasiconformal in a neighborhood of the ideal boundary. We restate this in the following

COROLLARY. *A homeomorphism* T *of a Riemann surface* R *onto another* S *can be extended to a homeomorphism between their Royden compactifications if and only if* T *is quasiconformal outside a compact subset of* R.

In fact, if T can be extended to a homeomorphism T^* of R^* onto S^*, then, as we have shown in the proof of the implication [C] → [A], T is a quasiconformal mapping of $R - \overline{R}_{n-1}$ onto $S - \overline{TR}_{n-1}$ for some $n > 1$. Conversely, if T is quasiconformal outside a compact subset of R, then T is quasiconformal from $R - \overline{R}_{n-1}$ onto $S - \overline{TR}_{n-1}$ for some $n > 1$. As we have shown in the proof of [C] → [A], there exists a quasiconformal mapping T_2 of R onto S such that $T_2 = T$ on $R - R_n$. By the proofs of [A] → [B] and [B] → [C], T_2 can be extended to a homeomorphism T_2^* of R^* onto S^*. Let $T^* = T_2^*$ on $R^* - R_n$ and $T^* = T$ on R_n. It is seen off hand that T^* is a homeomorphism from R^* onto S^* and $T^*|R = T$. This completes the proof of Corollary.

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