数理解析研究所講究録
第 364 巻 1979 年 45-51

QUASICONFORMAL CIRCLES AND DISTORTION THEOREMS

by

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1. Introduction. In 1971 the important papers by O. Lehto [9] and R. Kühnau appeared independently at almost same time. In these papers for instance the following interesting theorem was proved by the different methods.

THEOREM. *Let* f, $f(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\cdots$ *be a schlicht analytic function in* $|z|>1$ *with a quasiconformal extension to the plane such that* $|\mu(z)| \le k$ *where* $\mu(z)$ *is the complex dilatation. Then*

$$\sum_{n=1}^{\infty} n\,|a_n|^2 \le k^2$$

with equality if and only if

$$f(z) = \begin{cases} z + a_0 + a_1/z & \text{for } |z|>1 \\ z + a_0 + a_1\bar{z} & \text{for } |z| \le 1. \end{cases}$$

It is trivial that this theorem is a generalization of "Flächensatz" in the function theory. After these papers many ones of this type appeared, for instance , D.K.Blevins [2], [3] Z. Göktürk [4] and J.McLeavey [11]. We also prove here some results of this type analogous to Blevins one.

Let the chordal distance between the points w_1 and w_2 in the extended complex w-plane $\bar{C}$ be denoted by $q(w_1, w_2)$. Then we define the chordal cross-ratio of the quadruple w_1, w_2, w_3, w_4 in $\bar{C}$ by

$$X(w_1, w_2, w_3, w_4) = \frac{q(w_1, w_2)\,q(w_3, w_4)}{q(w_1, w_3)\,q(w_2, w_4)}.$$

A Jordan curve Γ_k in $\bar{C}$ is called a k-circle $(0 < k \leq 1)$, if for all ordered quadruples of points on Γ_k

(1) $$X(w_1, w_2, w_3, w_4) + X(w_2, w_3, w_4, w_1) \leq \frac{1}{k}.$$

As was shown by D.K.Blevins [2], a Jordan curve $\gamma_k(d) = \{ |\arg(w-d)| = \arcsin k \}$ is a k-circle which plays an important role in this paper. In the following we will be concerned with schlicht analytic maps of the annulus into the domains bounded by a circle $|w| = r$ and a k-circle. Then we reduce a distortion theorem with respect to functions schlicht and analytic in a unit disk by tending r to 0.

We will use the following notations. $R(r, \Gamma_k)$ and $R(r, d_0, d)$ are ring domains bounded by the circle $|w| = r$ and Γ_k, $\gamma_k(d) \cup [d_0, d]$ respectively. $D(\Gamma_k)$ and $D(d_0, d)$ are the simply connected domains bounded by a k-circle Γ_k and $\gamma_k(d) \cup [d_0, d]$ respectively.

2. Distortion theorems. In this section we will derive distortion theorems with respect to functions schlicht and analytic in an annulus and a unit disk.

THEOREM 1. *Let* $w = f(z)$ *be a schlicht analytic function in an annulus* $r' < |z| < 1$ *such that under the mapping the circle* $|w| = r$ *corresponds to the circle* $|z| = r'$ *and the image ring domain* D_f *lies in* $R(r, \Gamma_k)$. *Then under the assumption that* Γ_k *contains the infinity and the fixed positive point* d *there holds the inequality*

$$|f(z)| \geq F(|z|)$$

for all z *in the annulus* $r' < |z| < 1$ *where* $w = F(z)$ *is a schlicht* *analytic function which maps the circle* $|z| = r'$ *onto the circle* $|w| = r$ *and* $r' < |z| < 1$ *into* $R(r, \gamma_k(d))$ *with* $F(1) = d_0$ *and* $D_F = R(r, d_0, d)$. *The equality holds only if* $f(z) = F(e^{i\theta}z)$ (θ*; real*).

Proof. Let d_1 be the distance between the origin and the outer boundary of D_f. We perform the circular symmetrization with respect to the negative real axis. Let D_f^* be the symmetrization of D_f. Then by the well known theorem on the circular symmetrization we have

(2) $\qquad \text{Mod } D_f \leq \text{Mod } D_f^*$.

The equality holds if and only if D_f^* is obtained from D_f by a simple rotation around the origin. On the other hand as was shown by D.K.Blevins [2] using (1), the symmetrization Γ_k^* of Γ_k lies in $\{ |\arg (w - d)| \geq \arcsin k\}$. Therefore we have $D_f^* \subset R(r, d_1, d)$, which implies

$$\text{Mod } D_f^* \leq \text{Mod } R(r, d_1, d).$$

So from (2),

$$\text{Mod } D_f \leq \text{Mod } R(r, d_1, d).$$

Since modulus is invariant under the conformal mapping, we have

$$\text{Mod } D_f = \text{Mod } R(r, d_0, d) = \log 1/r'.$$

Mod $R(r, d_0, d)$ is a continuous and monotonuously increasing function of d_0, and therefore

(3) $\qquad d_0 \leq d_1$.

Without loss of generality we can assume that

(4) $\qquad \min_{|z|=\rho} |f(z)| = |f(\rho)| \qquad (r' < \rho < 1)$.

Let $z = z(\zeta)$ ($\rho = z(1)$) be a schlicht analytic function that maps an annulus $r'' < |\zeta| < 1$ onto $r' < |z| < 1$ slit along the segment $[\rho, 1]$.

The composite function $w = f(z(\zeta))$ maps the annulus $r'' < |\zeta| < 1$ into the ring domain $R(r, \Gamma_k)$. From (3) and (4) we have

(5) $|f(\rho)| \geq d_0'$

where d_0' is uniquely determined by the equation

$$\log 1/r'' = \text{Mod } R(r, d_0', d).$$

By the uniqueness of the mapping function $w = F(z(\zeta))$ $(F(z(1)) = d_0')$ there holds

(6) $F(\rho) = d_0'$.

From (4), (5) and (6) we have

$$|f(\rho)| \geq F(\rho)$$

and $|f(z)| \geq F(|z|)$

which is the desired inequality. The equality holds only if

$$f(z) = F(e^{i\theta} z) \quad (\theta; \text{real}).$$

Now we prove a distortion theorem with respect to functions schlicht in $|z| < 1$, by tending r to zero in the above result.

THEOREM 2. *Let* f,

$$f(z) = z + a_2 z^2 + \ldots\ldots$$

be a schlicht analytic function in $|z| < 1$ *and* D_f *the image domain of* $|z| < 1$. *Under the condition that* D_f *is contained in a domain bounded by a* k-*circle* Γ_k *which contains the infinity and the fixed positive point* d, *there holds the inequality*

$$|f(z)| \geq F(|z|)$$

where $F(z) = z + A_2 z^2 + \ldots\ldots$ *is a schlicht analytic mapping which maps* $|z| < 1$ *into* $D(\gamma_k(d))$ *with* $D_F = D(d_0, d)$ *and* $F(1) = d_0$.

Proof. For an arbitrary small number r (>0), denote by $D_f(r)$ a ring domain obtained by deleting a disk $|w| \leq r$ from D_f. Let Mod $D_f(r) = \log 1/r'$ $(r' = r'(r))$ and denote by $w = f_r(z)$ a function mapping the annulus $r' < |z| < 1$ conformally onto $D_f(r)$. In the same manner we construct a schlicht analytic function $w = F_r(z)$ which maps the annulus $r' < |z| < 1$ onto a ring domain $R(r, d_0(r), d)$ with $F_r(1) = d_0(r)$ for same pair (r', r) with one in the case of mapping $w = f_r(z)$.

Applying theorem 1 to the functions $f_r(z)$ and $F_r(z)$, we have

$$|f_r(z)| \geq F_r(|z|).$$

Tending r to zero, we have

$$|f(z)| \geq F(|z|).$$

We have sketched the proof. To prove exactly, we must use other results for instance

$$\lim_{r \to 0} \frac{r'}{r} = 1.$$

As another application of theorem 1 we prove the following

THEOREM 3. *Let* $l(f, \theta)$ *be the Lebesgue linear measure of the set* $\{w;\ \arg w = \theta\} \cap D_f$. *Under the condition of theorem 2 there hold the inequalities*

$$l(f, 0) \geq d_0, \quad l(f, \pi) \geq \frac{dd_0}{d - d_0}.$$

These inequalities are best possible.

Proof. The first inequality is trivial from theorem 2, so we omit the proof for it. For the proof of the second inequality we consider the linear transformation

$$\zeta = \zeta(w) = \frac{dw}{w - d}$$

and $\qquad w = f(-z).$

The composite function $\zeta = \zeta(f(-z))$ is schlicht and analytic in $|z| < 1$ and normalized as

$$\zeta(f(-z)) = z + \cdots\cdots$$

The linear transformation

$$\zeta = \frac{dw}{w - d}$$

maps the k-circle onto a k-circle, and the infinity and the positive constant d are mapped onto d and the infinity respectively. Therefore we can apply the first inequality for the composite function $\zeta = \zeta(f(-z))$.

Let the ray $\{w;\ \arg w = \pi\}$ intersect the boundary ∂D_f of the image domain D_f at $-q_1, -q_2, \cdots\cdots$ We assume that the sequence is arranged as follows;

$$(0 <)\ q = q_1 < q_2 < \cdots\cdots$$

Then it is trivial that

$$\zeta(-q) = \frac{dq}{q + d} \geq d_0$$

that is,

$$q \geq \frac{dd_0}{d - d_0}.$$

On the other hand,

$$l(f, \pi) \geq q,$$

which implies the desired inequality

$$l(f, \pi) \geq \frac{dd_0}{d - d_0}.$$

Remark. When we tend d to the infinity, above results reduce to the well known theorems in the function theory.

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