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開放系のゆらぎ

京大理.物理　　富田和久

§1. 揺ぎと統計力学的記述

統計的記述の必然性 —— 2種の揺ぎ

§2. 熱雑音と非平衡開放系

中心極限定理 — 非可逆循環 — Hopf分岐と異常ゆらぎ

§3. 混沌相と非熱雑音

点安定性 — 軌道安定性 — 構造安定性

§3-1. 混沌相の例

非自律系 (Brussel模型) — 自律系 (B-Z 反応)

§3-2 エルゴード性と不変測度

Logistic chaos — histogram と周期点の分布 —

測度論的エントロピー

§3-3 混沌相から大局的記述 (megaloscopy !) へ

§3-4 混沌相の積極的意義

以下の論文は, 本講演にもとづいて執筆し, 後日西独における Symposium on Synergetics (Elmau, May, 1979) において報告されたものである。

Chaos and Its Description

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1. Introduction

There may be various different physical origins for the phenomenon called "fluctuation". Here, however, we confine ourselves to those fluctuations for which the best possible description is a stochastic one. Furthermore, we are more concerned here with the new aspects which emerge only in open system.

Physical measurements are essentially associated with a finite space-time extension. This means that a mathematical determinism does not actually ensure a physically deterministic prediction. In fact, the latter is made possible only by the existence of an orbital stability. Therefore, even if a dynamical system is clearly defined in mathematical sense, a deterministic description becomes impossible in the realm of physical observation, when there appears an orbital instability. In this case a stochastic description is a necessity rather than a convenience.

Among those fluctuations which may best be described in a stochastic language, there are two different kinds of *noise* which have been treated in the literature, namely (a) the thermal noise and (b) the non-thermal noise.

2. Thermal Noise in a Non-equilibrium State

Thermal noise is a kind of noise which appears as a residual fluctuation when we project a conservative system, having many microscopic degrees of freedom, onto a macroscopic dissipative system, consisting of a few thermodynamic degrees of freedom. This kind of noise is a result of a convolution of very many microscopic degrees of freedom (say N in number), and the resulting statistical measure is well represented by a Gaussian distribution (Central Limit Theorem). However, the variance $\sigma \equiv \langle \xi(0)\xi(0) \rangle$ is of order 1/N as compared with the mean value, and the distribution is thus very sharp.

At thermal equilibrium the stochastic information is completely specified when the variance σ is given. This is due to the existence of the detailed balance condition of ONSAGER [1]. In an off-equilibrium condition, however, a second quantity should be added in order to specify the stochastic information completely. This quantity is called "irreversible circulation" [2-6] and is defined by

$$\alpha \equiv \frac{1}{2} < [\xi(0), \dot{\xi}(0)] > .$$

This represents the rotational flow of the distribution, and indicates the degree to which Onsager's condition is violated, therefore it may be taken as a measure of the distance from thermal equilibrium. Expressed in different words, the antisymmetric part of the correlation function $<\xi(0)\xi(t)>$ is needed in addition to the part that is symmetric with respect to time reversal. The existence of the antisymmetric part has been confirmed in several cases, e.g. in nerve membrance [7] reactor noise [6] etc.. However, in the case of chemical reaction it is technically not easy to observe this part. Biased Brillouin scattering [8] which is predicted in a fluid under heat conduction seems to be another example.

One of the differences between a conservative system and a macroscopic dissipative system is that in the latter there may appear a limit cycle, i.e. a periodic motion which is asymptotically stable or attracting. Assoicated with the Hopf bifurcation, or a hard mode instability through which the limit cycle emerges, there appears an anomalous increase in the irreversible circulation α, provided the bifurcation is normal in character [2, 3]. This resembles an ordinary phase transition, or soft-mode instability, and the circulation in the fluctuation grows into the orbital revolution through the hard mode instability. The anomalous increase should make observation easier; however, the range of control parameter in which the fluctuation becomes large is fairly small. If it can be observed α may be taken as a fore-runner of the macroscopic orbital revolution which is to follow, and may have a use as such [6].

3. Chaotic Phase and Non-thermal Noise

A second difference between a conservative system and a macroscopic dissipative system is that in the latter there may appear a chaotic motion which is attracting [9]. Let us consider a pair of limit cycles which are interacting with each other in order to get some feeling for the generation of chaos. Three different cases are expected [*].

(i) The case in which the ratio of the two unperturbed frequencies is sufficiently close to that of small integers. In this case a synchronization is expected and a single limit cycle results.

(ii) The case in which the ratio of the two unperturbed frequencies is far from that of small integers. In this case a quasi-periodic motion results involving two different frequencies.

(iii) In the case in which the ratio of frequencies and amplitudes are both appropriate, there may appear repeated subharmonic bifurcations in a finite range of the parameter. When all possible periodic orbits becomes unstable, there seems to result a chaotic motion. The motion in a short time span resembles periodic motion with a certain period; however, orbital instability inhibits its continuation, thus leading to chaotic behaviour. Although chaotic in behaviour, the motion is structurally stable, and may be called a phase (chaotic phase). RUELLE and TAKENS [10] considers that this kind of chaotic phase is the proto-type of hydrodynamic turbulence.

(iv) A fourth case should be admitted in which chaos appears without having the cascades of subharmonic bifurcations.

In what follows several examples of chaotic phase treated by us are described and an attempted stochastic description is presented.

4. Examples of Chaotic Phase

(1) Forced Brussels Model (A non-autonomous example) [11-14]

This well known model exhibiting a limit cycle has been investigated under the influence of a periodic external excitation. Solving the corresponding equations, i.e.

$$\frac{dX}{dt} = X^2Y - BX - X + A + a\cos\omega t \,, \tag{1}$$

$$\frac{dY}{dt} = -X^2Y + BX \,, \tag{2}$$

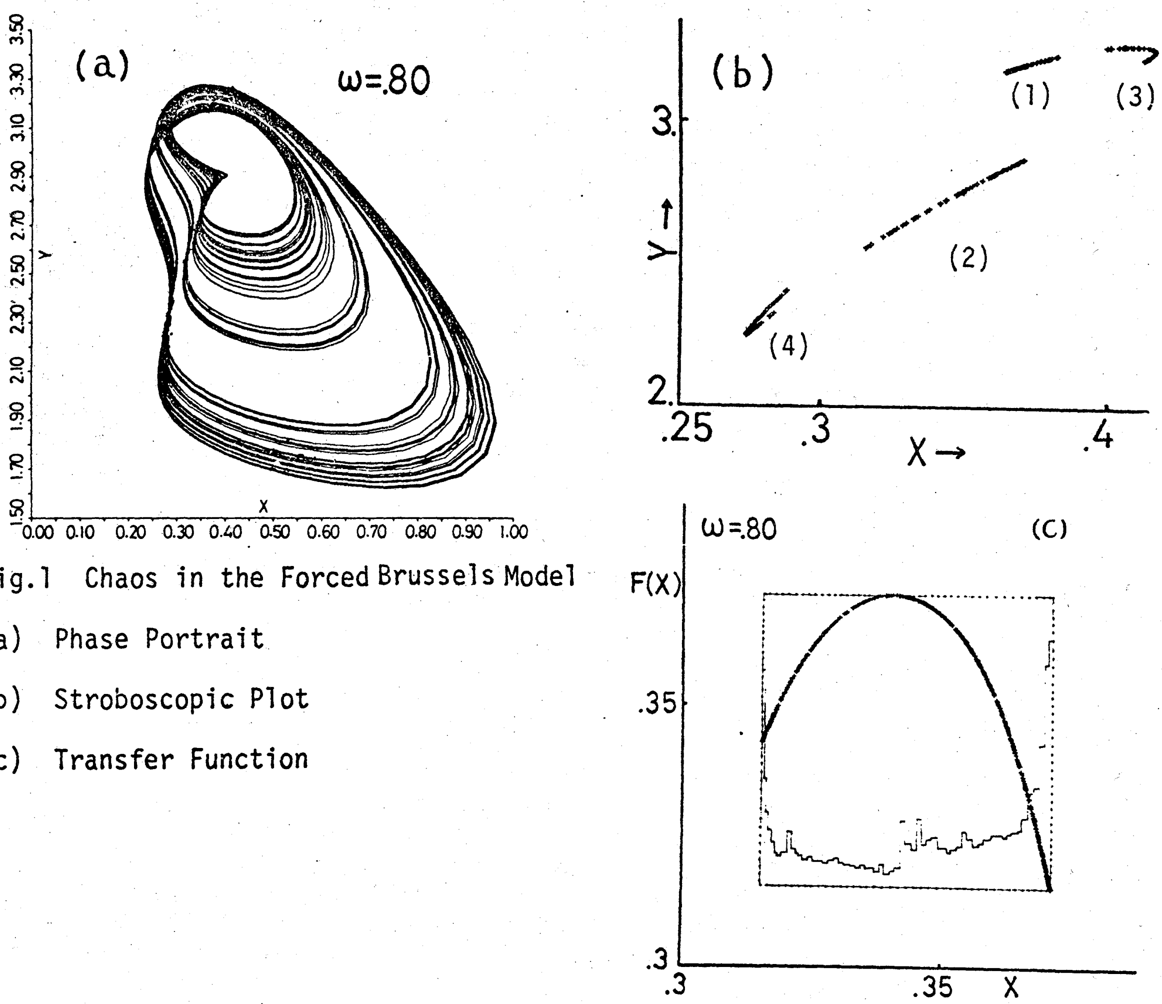

Fig.1 Chaos in the Forced Brussels Model

(a) Phase Portrait

(b) Stroboscopic Plot

(c) Transfer Function

depending on the values of parameters a and ω, we found all three types of result, mentioned in the previous section. In particular when the amplitude a is increased in the subharmonic resonance band $\omega \sim 2\omega_n$ (ω_n is the natural frequency), there appears a converging cascade of bifurcations and beyond the limit there appears an apparently chaotic phase of which an example is given in Fig. 1. Here (a) is the phase portrait, (b) is the Poincaré plot, which is simply a stroboscopic plot in this case. If this plot consisted of four isolated points it would mean a strict four point periodicity. The fact that the four islands have a remarkable continuous spread of their own indicates that the motion is chaotic. In addition, the spread is fairly one dimensional for each island. Using this fact one may use still another plot (c) in which the consecutive visits to one particular island (here the island (2)) are plotted in the form $x_{n+1}=F(x_n)$. It is remarkable that the transfer function F(x) has a simple form, which is nearly quadratic in the present case. In this form it is fairly clear that nature of the motion is similar to the logistic chaos. We have also computed the histogrammatic invariant measure for this case, and the Liapounov characteristic exponent was found to be +0.451. (cf. §5)

(ii) Belousov-Zhabotinsky Reaction under Stirred Flow Conditions (an autonomous example)

Although there are several experimental reports on chaos in this particular reaction [15-18] the theoretical possibility has been in dispute [19, 20]. We have investigated our simplified model [21] under stirred flow conditions [22]. In terms of scaled quantities the model is governed by the equations

$$\frac{d\xi}{d\tau} = (1-\phi)\xi + \eta - \xi\eta - \xi\xi \ , \tag{3}$$

$$\frac{d\eta}{d\tau} = -(1+\phi)\eta + \zeta - \xi\eta + m \ , \ (m = m_0 + \phi n_0) \tag{4}$$

$$\rho \frac{d\zeta}{d\tau} = -(1+p\phi)\zeta + \xi - \xi\xi \ . \tag{5}$$

Here ξ, η, ζ and m_0 correspond to $[HB_rO_2]$, $[B_r^-]$, $[C_e^{4+}]$ and $[CH_2(COOH)_2]$, respectively, and ϕ stands for the flow rate. The control parameters are m_0 and ϕ in this case. In a relatively narrow region in the (ϕ, m_0) plane the system has three steady states and in this region we have found a practically chaotic behaviour by analogue computation. An example is shown Fig. 2. In (a) the phase portrait is shown projcted onto (ξ, ζ) plane. The corresponding correlation spectrum is shown in (b). Although there exist recognizable periods, the noisy character of the motion is fairly obvious.

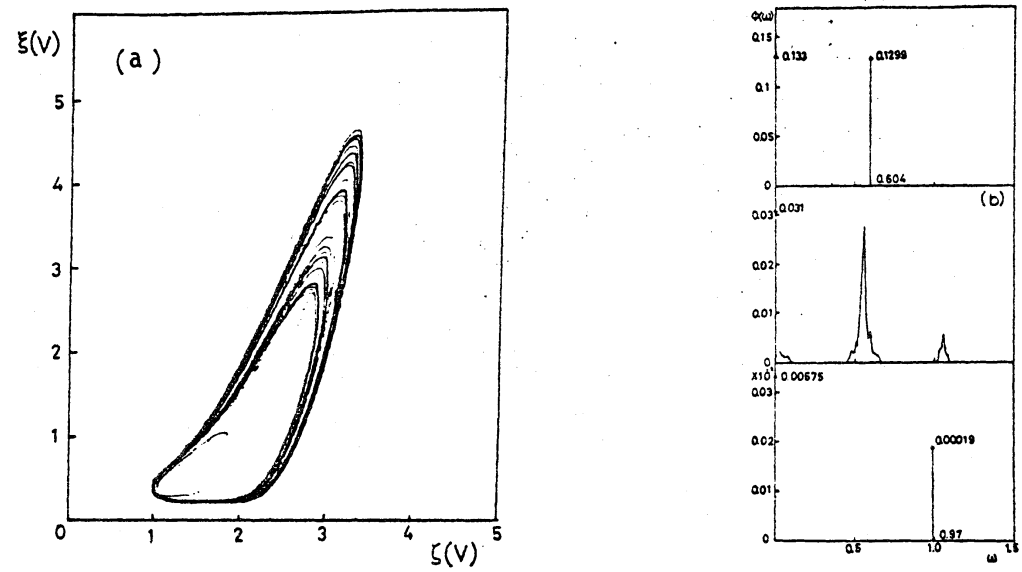

Fig. 2 Chaos in the Belousov-Zhabotinsky Reaction (Kyoto Model) (a) Phase Portrait projected onto (ξ, ζ) plane (b) Correlation spectrum of ξ. The middle figure corresponds to (a).

5. The Invariant Measure associated with Chaos [23]

As is indicated in the preceding example, there are cases in which the stroboscopic map looks fairly one dimensional, although the space behind the chaos has many dimensions (cf. Fig. 1b). Based on this experience we adopted the one dimensional logistic model

$$x_{n+1} = A\, x_n\ (1-x_n) \equiv F(x_n) \qquad (0 < x_i < 1) \tag{6}$$

to look into the invariant measure associated with chaos.

5.1 Histogram (Time Sequence Construction)

It is not very difficult to obtain an invariant measure associated with a chaotic motion by constructing a histogram (with divisions of finite size) according to

$$\rho_{x_0}(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N-1} \delta(x-F^{(i)}(x_0)) \tag{7}$$

when $\rho_{x_0}(x)$ is independent of the initial point x_0, then the motion is said to be ergodic and the invariant measure, thus defined, is unique. As an example the case A=3.9 is shown in Fig. 3, in which the histogramatic measure is shown by the broken line.

5.2 The Covering associated with the Fixed Points (Phase Average)

Instead of tracing the time sequence, we now try to find an equivalent to the histogramatic measure starting from the fixed points in the phase space. We have investigated a covering by cells which are associated with each fixed point, and found that the histogramatic measure is reproduced only when appropriate weights are assigned to the respective cells.

Let $x_j^{(N)}$ (j=1, 2, ..., N) be N-periodic points satisfying

$x_j^{(N)}=F^{(N)}(x_j^{(N)})$, and define a density associated with these periodic points, i.e.

$$\rho^{(N)}(x) = \sum_{j=1}^{M_N} p_j^{(N)} \delta(x-x_j^{(N)}) \, , \qquad (8)$$

where M_N is the total number of N-periodic points and the $p_j^{(N)}$ are to be determined so as to satisfy the physical requirements. The following two cases have been investigated.

(A) The case in which $p_j^{(N)}=p^{(N)}=1/M_N$.

In this case the rusulting measure does not tend to the histogramatic measure even for large enough N.

(B) The case in which $p_j^{(N)}= c_N/|F^{(N)'}(x_j^{(N)})|$.

The idea behind this particular assignment of weight to each cell is rougly the consideration of the residence time in a particular cell. The resulting measure does in fact converge to the histogramatic measure when N becomes large, provided the N-periodic points are unstable, as is shown in Fig. 3 by the solid line. When the N-periodic points are stable, the above assignments lead to M_N isolated fixed points under $F^{(N)}$.

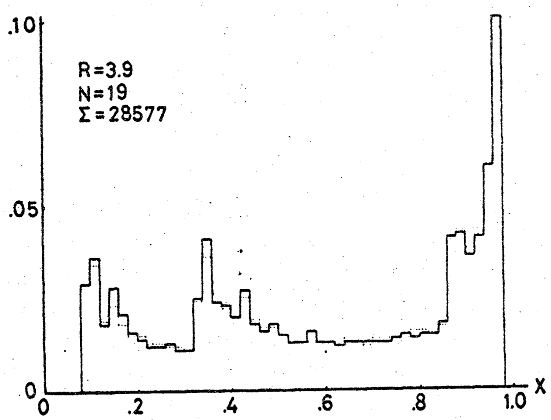

Fig.3 Invariant Measure for the Logistic Chaos (A=3.9; Broken Line: Histogram; Solid Line: Weighted fixed point distribution)

5.3 Measure Theoretic Entropy (Kolmogorov Entropy)

The measure theoretic entropy $h_\mu(F)$, which is associated with the invariant measure μ with respect to the transformation F, is defined in the following way by using a partition $A=\{A_i\}$ of the phase space, i.e.

$$h_\mu(F) \equiv \sup_{A} \lim_{N\to\infty} \frac{1}{N} h_\mu(F, A^{(N)}) \, . \qquad (9)$$

Here

$$h_\mu(F, A^{(N)}) \equiv - \sum_i \mu(A_i^{(N)}) \ln \mu(A_i^{(N)}) \, , \qquad (10)$$

and

$$A^{(N)} \equiv A \vee F^{(-1)}(A) \vee F^{(-2)}(A) \vee \cdots \vee F^{(-(N-1))}(A) \, . \qquad (11)$$

In the case of the logistic model, let us start with a partition which bisects the whole interval at the maximum of the transfer function, i.e. at x=1/2. In this case the partition $A^{(N)}$ is easily shown to be identical with the partition which consists of cells defined by the consecutive minimum and maximum in the transfer function $F^{(N)}$. Consequently, each cell contains at most one fixed point of $F^{(N)}$, which is the desired relationship with the distribution of fixed points.

Remembering that

$$\mu(A_i^{(N)}) = \int_{A_i^{(N)}} \rho(x)\,dx \ , \qquad (12)$$

and adopting the expression (A) or (B) for $\rho(x)$, the measure given to the cells which do not involve fixed points will vanish and the corresponding K-entropy may be evaluated as follows.

In case (A) we are lead to the expression

$$h_\mu(F) = \lim_{N\to\infty} \frac{1}{N} \ln M_N \ , \qquad (13)$$

which is equivalent to the topological entropy [24], provided the ratio between the number of $A^{(N)}$ cells and M_N is not divergent (This seems to be the case for the logistic model).

In case (B) one is lead to the expression

$$h_\mu(F) = \int \rho(x) \ \ln |F'(x)| \ dx \ , \qquad (14)$$

where it is assumed that $c_N \ll e^{\alpha N}$. Therefore, the measure theoretic entropy coincides with the Liapounov characteristic number. This is a relation already confirmed for the histogramatic measure, and is taken to be a support that the weighted distribution (B) of fixed points is a legitimate measure.

5.4 Variation Principle

Another support for the weighted fixed point distribution (B) is obtained from the variation principle. BOWEN and RUELLE [23] have proved that for Axiom A dynamical system

$$\Phi(\mu) = h_\mu(F) - \int \mu(dx) \ \ln |F'(x)| \le 0 \ , \qquad (15)$$

and the ergodic measure μ may be determined by maximizing $\Phi(\mu)$.

Let us suppose that this variation principle applies to the present case. Noting that $\mu(dx)=\rho(x)dx$, and varying $p_i^{(N)}$ in the expression of $\rho(x)$, one is lead by the variation principle to the result

$$p_i^{(N)} = c_N \ / \ |F^{(N)\prime}(x_i^{(N)})| \ , \qquad (16)$$

where C_N is to be determined by the normalization. This is clearly identical with the weight assumed in the distribution (B), and is taken to be a support of its validity.

6. Discussion

The validity which has been demonstrated for the weighted fixed point distribution (B) in a number of ways has its own physical background. As is experienced in the simulations, the motion of the representation points on a short time scale often resembles motion with a particular period; however, in the long run the representative point cannot remain on that particular periodic orbit, because every periodic orbit is unstable. This is the mechanism through which chaos emerges. According to this picture it is not difficult to understand that the ergodic measure is related to the periodic point distribution, and the particular form of the weight (16) is interpreted as a kind of mean residence time of the representative point in the neighbourhood of the particular periodic point.

As we extracted the macroscopic or thermodynamic degrees of freedom from the microscopic ones by way of projection, it is conceivable to extract the "*megaloscopic*" digrees of freedom from the *macroscopic* or dissipative chaos. We are then left with a non-thermal noise of macroscopic origin. It resembles the thermal noise in so far as it corresponds to the sensitive dependence on the initial condition; however, in several significant respects it differs from the thermal noise. Namely,

(i) Non-thermal noise does not require many degrees of freedom as its background, therefore there is no reason for expecting the central limit theorem to apply. Consequently, the resulting distribution is not necessarily Gaussian, and may be different for different problems.

(ii) In contrast with thermal noise, for non-thermal noise the measure theoretic entropy does not in general coincide with the topological entropy.

(iii) Under the circumstances in which both the thermal and the non-thermal noises exist, the non-thermal noise is expected to predominate, simply because it is macroscopic. This means that the thermal noise is masked by the non-thermal in such a situation.

In the interpretation of an actually observed noise in a macroscopic system one should be aware of the above points.

Of course there is not the scale difference usually existent when the conventional extraction of macroscopic motion is made and to this extent the meaning of "megaloscopic motion" cannot be so clear cut. However, this may also lead to a positive use of megaloscopy. In contrast to the fairly sharp response on the *macroscopic* level (small noise), the response on the *megaloscopic* level must have a considerably larger flexibility (large noise). From this point of view it is tempting to anticipate the existence of a macroscopic chaos on the part of the observer as a background to pattern recognition. Suppose there are a number of orbits which are markedly different on the macroscopic level, and yet belong to the range of a common macroscopic chaos. One may associate these different orbits with a single pattern, in so far as a common megaloscopic motion may be extracted from them. The existence of macroscopic chaos in this context will lead to a definitely larger flexibility in the global identification of patterns. This seems much more natural than the digital mechanism popular nowadays as the basis of pattern recognition.

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