数理解析研究所講究録
第 367 巻 1979 年 229-231

Comment on the most probable paths of diffusion processes.

Detlef Dürr
Universität Münster
Institut für Theoretische Physik I
44 Münster, Germany

The physicists naive point of view is this: Given a Focker-Planck-Equation defining a diffusion process X_t with drift $f(x)$ and diffusion $D(x)$, both functions as nice as necessary. Then the density $p(s,x,t,y)$ of the transition probability of X_t is desired to be represented as a path-integral of the form:

$$(1)\quad p(s,x,t,y) = \int_{\substack{\text{paths}\\ x\to y}} d[z]\exp\left(-\int_s^t OM(\dot z, z, u)du\right)$$

Next maximizing the integrand, i.e. minimizing

$$(2)\quad \int_s^t OM(\dot z,z,u)du$$

by a variation principle yields a function $x_m(t)$. Introducing $x_m(t)$ into the exponent of (1) should give some approximation for (1)

$$(3)\quad p(s,x,t,y) \propto \exp\left(-\int_s^t OM(\dot x_m, x_m, u)du\right)$$

Onsager and Machlup [1] initiated this concept, applying it to the exceptional Ornstein-Uhlenbeck-process. The integrand $OM(\dot z,z,u)$ is called Onsager-Machlup-function (OMF) and $x_m(t)$ the most probable path of X_t.

In physical as well as biological or chemical systems (see Kitaharas report) nonlinear stochastic differential equations are of interest and the Onsager-Machlup theory has been extended to the general case [eg. 2,3,4]. In [5] a mathematical version of the Onsager-Machlup theory for one-dimensional systems has been presented, stimulated by a paper of Stratonovich [6]. Ito [7] made a nice generalisation to more dimensional processes and you can find the mathematical tools in Takahashis elegant approach.

Let us introduce:

(4) $C_{x_o}(I) = \{x(u) \text{ continuous},\ x(s)=x_o, u\in[s,t]\}$ and with

$z\in C^2 \cap C_{x_o}(I)$ the tube

(5) $K(z,\varepsilon) = \{x\in C_{x_o}(I) \mid \|x-z\| < \varepsilon,\ \varepsilon>0\}$, where

(6) $\|x\| = \sup_{u\in I}|x|$.

Let μ_x denote the measure on the Borelalgebra of $C_{x_o}(I)$ corresponding to X_t, then the probability of finding a path $x(u)$ of X_t in the tube (5) is given by

(7) $\mu_x(K(z, \varepsilon))$.

Now we are able to give the

Definition 1: Let a function $z_m(u)$exist, maximizing (7) but independent of ε. If $z_m(t)$ may be obtained by a variation of $\int_s^t OM(\dot{z},z,u)du$ in the sense of a Lagrange formalism, then $OM(\dot{z},z,u)$ is called OMF and $z_m(t)$ the most probable path of X_t.

Remark: This definition includes the request for universality of the OMF at least for a class of diffusion processes.

Takahashis report gives rise to another remark. In (7) different tubes (for different z) of the same thickness ε are to be compared. Then the machinery works pretty well for processes with constant diffusion. In case of process depending diffusion the process X_t may be transformed by the Itô-formula to a process $Y_t=h(X_t)$ with constant diffusion. But with

(8) $h^{-1}(K(z, \varepsilon)) = K(h^{-1}(z), \varepsilon_{h^{-1}}(z))$ we have

(9) $\mu_x(K(h^{-1}(z), \varepsilon_{h^{-1}}(z))) = \mu_y(K(z, \varepsilon))$.

This shows: Comparing tubes of the same thickness for Y_t to determine the most probable path of Y_t means to compare for X_t tubes of different thickness. So, as long as in (6) $|\cdot|$is defined by the Euclidean metric (as it is always the case for one-dimensional processes) the most probable path of Y_t is not the transformed one of X_t.

In Takahashis procedure Y_t is defined in normal coordinates in the Riemannian metric given by the diffusion $\mathbf{D}$ of the generator of X_t which is used to define$|\cdot|$in (6).

For this special choice of metric the most probable path of Y_t determines via the transformation h the most probable path of X_t in a geometrical sense.

This should result in a refinement of the Definition 1 specifying in which sense, i.e. in which metric the most probable path is of interest.

At the moment it seems that from the way of modeling diffusion processes for physical, biological and chemical systems the Euclidean metric is more natural. So to find the most probable path for this case is still desirable.

I wish to thank Prof.H. Hasegawa for his invitation to the meeting and my colleague Dr. A. Bach for four years of good cooperation. I'm indebted to Prof. S. Albeverio for his patience in discussions and his encouragement.

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