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Sperical Setions of a Homogeneous Vector Bundle

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In this note, we show some results on the integral representation of eigensections of invariant differential operators on a homogeneous vector bundle over a riemannian symmetric space. First we determine the structure of the algebra $\mathbb{D}$ of invariant differential operators on a homogeneous vector bundle, define the notions of eigensections and spherical sections corresponding to a given finite-dimensional representation of the algebra $\mathbb{D}$ and obtaine the dimension formula and an integral representation of spherical sections. The notion of spherical sections is a generalization of that of zonal spherical functions. The dimension formula of the space of spherical sections will be crucial for the problem of Poisson integral representation of eigensections, which I would like to solve during this summer.

Precisely speaking, let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$ of finite center, K a maximal compact subgroup with Lie algebra $\mathfrak{k}$, $G = KAN$ an Iwasawa decomposition with split torus A and $g = K(g)e^{H(g)}n(g)$ $(g \in G,\ K(g) \in K,\ e^{H(g)} \in A,\ n(g) \in N)$ the decomposition of g in G corresponding to $G = KAN$, where $H(g)$ is an element in the

Lie algebra $\mathfrak{a}$ of A. Let E denote the homogeneous vector bundle over G/K associated to a given irreducible unitary representation τ of K and $\mathbb{D}_\tau$ denote the algebra of invariant differential operators on E. Let $\mathfrak{G}$ (resp. $\mathfrak{K}$) denote the universal enveloping algebra of $\mathfrak{g}_c$ (resp. $\mathfrak{k}_c$) and $*$ denote the anti-automorphism of $\mathfrak{G}$ defined by $X^* = -X$ $(X \in \mathfrak{g})$. Let $\mathfrak{G}^K$ be the centralizer of K in $\mathfrak{G}$, $\mathfrak{I}_\tau$ the kernel of $d\tau$ in $\mathfrak{K}$ and $\mathfrak{I}_\tau^*$ the set of $z^*(z \in \mathfrak{I}_\tau)$. Then $\mathbb{D}_\tau$ is canonically isomorphic to the algebra $\mathfrak{G}^K/\mathfrak{G}^K \cap \mathfrak{G}\mathfrak{I}_\tau^*$.

Let ρ denote the half sum of the roots corresponding to N. For $\lambda \in \mathfrak{a}_c^*$, put

$$P_{\tau,\lambda}(g) = e^{-(\lambda+\rho)H(g^{-1})}\tau(K(g^{-1})) \qquad (g \in G) .$$

Then there exists an algebra homomorphism, say $\chi_{\tau,\lambda}$, of $\mathbb{D}_\tau$ into $\mathrm{End}_M(V)$ such that

$$\Delta P_{\tau,\lambda}(g) = P_{\tau,\lambda}(g) \circ \chi_{\tau,\lambda}(\Delta) \qquad (\Delta \in \mathbb{D}_\tau) ,$$

where V denotes the representation space of τ and $\mathrm{End}_M(V)$ denotes the set of endomorphisms on V which commute with $\tau(m)(m \in M = Z_K(A)$ = the centralizer of A in K).

For an irreducible representation (σ, V_σ) of M, put $H_\sigma = \mathrm{Hom}_M(V_\sigma, V)$ and $H_{\tau,\sigma} = \mathrm{Hom}_M(V, V_\sigma)$. Then $\chi_{\tau,\lambda}$ defines a representation $\chi_{\tau,\sigma,\lambda}$ of $\mathbb{D}_\tau$ on H_σ by

$$\chi_{\tau,\sigma,\lambda}(\Delta)a = \chi_{\tau,\lambda}(\Delta) \circ a \qquad (a \in H_\sigma) .$$

Let $\mathcal{A}(E)$ denote the space of analytic sections of E. Given a representation (χ,H) of $\mathbb{D}_\tau$, we call $u \in \mathcal{A}(E)$ an eigensection of type χ, if there exists a finite number of $\mathbb{D}_\tau$-invariant subspaces H_i in $\mathcal{A}(E)$ such that as a representation space of $\mathbb{D}_\tau$, each H_i is isomorphic to a quotient representation of (χ,H). Let $\mathcal{A}(E,\chi)$ denote the space of eigensections of type χ.

Let $F_{\sigma,\lambda}$ denote the vector bundle over G/MAN associated to the representation $\sigma \otimes e^{-\lambda+\rho} \otimes 1$ of MAN on V_σ and let $\mathcal{B}(F_{\sigma,\lambda})$ denote the space of $F_{\sigma,\lambda}$-valued hyperfunctions on G/MAN. For $\varphi \otimes a \in \mathcal{B}(F_{\sigma,\lambda}) \otimes H_\sigma$, put

$$P_{\tau,\sigma,\lambda}(\varphi \otimes a) = \int_K P_{\tau,\lambda}(k^{-1}g)a\varphi(k)dk .$$

Then it is easy to see that $P_{\tau,\sigma,\lambda}$ gives a G-linear mapping of $\mathcal{B}(F_{\sigma,\lambda}) \otimes H_\sigma$ into $\mathcal{A}(E,\chi_{\tau,\sigma,\lambda})$.

Let $\mathcal{A}(E,\chi_{\tau,\sigma,\lambda})^\tau$ denote the subspace consisting of the sections in $\mathcal{A}(E,\chi)$ which transform according to τ under the left regular representation of K on $\mathcal{A}(E)$. We identify $V \otimes H_{\tau,\sigma}$ with the subspace of sections in $\mathcal{B}(F_{\sigma,\lambda})$ which transform according to τ under the left regular representation of K on $\mathcal{B}(F_{\sigma,\lambda})$ by

$$V \otimes H_{\tau,\sigma} \ni v \otimes b \longmapsto \varphi_{v,b}(k) = b\tau(k^{-1})v \in \mathcal{B}(F_{\sigma,\lambda})$$

$$(v \in V,\ b \in H_{\tau,\sigma},\ k \in K),$$

where $\varphi_{v,b}$ is regarded as an element in $\mathcal{B}(F_{\sigma,\lambda})$ by

$$\varphi_{v,b}(kan) = e^{(\lambda-\rho)H(a)}\varphi_{v,b}(k) .$$

Then the mapping

$$v \otimes b \otimes a \longmapsto P_{\tau,\sigma,\lambda}(\varphi_{v,b} \otimes a)$$
$$= \int_K P_{\tau,\lambda}(k^{-1}g)ab\tau(k^{-1})v\,dk$$

gives a linear mapping of $V \otimes H_{\tau,\sigma} \otimes H_\sigma$ onto $\mathcal{A}(E,\chi_{\tau,\sigma,\lambda})^\tau$. As a corollary, using the "subrepresentation theorem" of Casselman, we have a linear isomorphism

$$\mathcal{A}(E,\chi)^\tau \cong V \otimes (\mathbb{D}_\tau/\mathrm{Ker}\ \chi)^*$$

for any irreducible representation χ of $\mathbb{D}_\tau$.

Now it is very interesting to study when $P_{\tau,\sigma,\lambda}$ will give a G-isomorphism of $\mathcal{B}(F_{\sigma,\lambda}) \otimes H_\sigma$ onto $\mathcal{A}(E,\chi_{\tau,\sigma,\lambda})$. We have the following conjecture:

Under a certain regularity condition on λ, we can construct a boundary value mapping $\beta_{\tau,\sigma,\lambda}$ of $\mathcal{A}(E,\chi_{\tau,\sigma,\lambda})$ into $\mathcal{B}(F_{\sigma,\lambda}) \otimes H_\sigma$ such that

$$\beta_{\tau,\sigma,\lambda} \circ P_{\tau,\sigma,\lambda}(\varphi \otimes a) = \varphi \otimes c(\tau,\lambda)a, \quad \varphi \in \mathcal{B}(F_{\sigma,\lambda}),\ a \in H_\sigma \ ,$$

where $c(\tau,\lambda)$ is the generalized c-function.

It seems not so hard to prove it now because almost all necessary lemmas have been proved and we have only to see what differential operators in $\mathbb{D}_\tau$ will be necessary and suitable in order to determine the vector bundle to which the boundary values should belong.

References

(1) S. Helgason, Invariant differential operators on homogeneous manifolds, Bull. Amer. Math. Soc. 83 (1977), 751-774.

(2) J. Lepowsky, Algebraic results on representations of semisimple Lie groups, Trans. Amer. Math. Soc. 176 (1973), 1-44.

(3) K. Minemura, Invariant differential operators and spherical sections of a homogeneous vector bundle, preprint.

(4) K. Okamoto, Harmonic analysis on homogeneous vector bundles, Lecture Notes in Math., Vol. 266, pp. 255-271, Springer-Verlag, 1972.

(5) N. R. Wallach, Representations of semisimple Lie groups and Lie algebras, Canad. Math. Congress, 1978.