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On a decomposability of homogeneous linear system representations of a locally compact group

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§1. Linear system representations

A pair $H = \langle H_1, H_2 \rangle$ of complex linear spaces H_1, H_2 is called a linear system if a duality $\langle \xi, \eta \rangle$ is defined between H_1 and H_2. Namely, $\langle \xi, \eta \rangle$ is a complex bilinear form on $H_1 \times H_2$ with the property $\langle \xi, H_2 \rangle = 0$ only if $\xi = 0$ and $\langle H_1, \eta \rangle = 0$ only if $\eta = 0$. In this paper we consider H_1, H_2 as locally convex Hausdorff topological vector spaces with $\sigma(H)$-topology, that is, the topology generated by all functionals $\xi \longrightarrow \langle \xi, \eta \rangle$ on H_1, and by all functionals $\eta \longrightarrow \langle \xi, \eta \rangle$ on H_2 respectively.

Let X be a topological group or a topological algebra over the complex number field $\mathbf{C}$. A linear system representation (LSR, for short) of X means a pair $T = \langle T_1, T_2 \rangle$ of a representation T_1 of X on H_1 and an antirepresentation T_2 of X on H_2 such that $\langle T_1(x)\xi, \eta \rangle = \langle \xi, T_2(x)\eta \rangle$ for all $x \in X$, $\xi \in H_1$, and $\eta \in H_2$, and that the $\mathbf{C}$-valued functions $x \longrightarrow \langle T_1(x)\xi, \eta \rangle$ on X are continuous for all $\xi \in H_1$, $\eta \in H_2$.

Two LSR's $T = \langle T_1, T_2 \rangle$ on $H = \langle H_1, H_2 \rangle$ and $T' = \langle T_1', T_2' \rangle$

on $H' = \langle H_1', H_2' \rangle$ are called equivalent if there exists a pair $\Phi = \langle \Phi_1, \Phi_2 \rangle$ of linear isomorphisms Φ_1 of H_1 onto H_1' and Φ_2 of H_2 onto H_2' such that $\langle \Phi_1(\xi), \Phi_2(\eta) \rangle = \langle \xi, \eta \rangle$ for all $\xi \in H_1$, $\eta \in H_2$ and that $\Phi_1 T_1(x) \Phi_1^{-1} = T_1'(x)$, $\Phi_2 T_2(x) \Phi_2^{-1} = T_2'(x)$ for all $x \in X$.

A LSR $T = \langle T_1, T_2 \rangle$ of X on $H = \langle H_1, H_2 \rangle$ is called irreducible if every T_1-invariant non-trivial subspace of H_1 is $\sigma(H)$-dense in H_1, or equivalently, if every T_2-invariant non-trivial subspace of H_2 is $\sigma(H)$-dense in H_2.

Let G be a locally compact unimodular group, and L(G) the algebra of all continuous functions on G with compact supports, with multiplication defined by convolution. For every compact subset C of G, denote by $L_C(G)$ the normed space of all continuous functions on G whose supports are contained in C with supremum norm. Then L(G) is, as the inductive limit of $\{L_C(G)$; C is a compact subset of G$\}$, a topological algebra. A LSR $T = \langle T_1, T_2 \rangle$ of G on $H = \langle H_1, H_2 \rangle$ is called integrable with respect to L(G) if, for every function $f \in L(G)$, there exist linear operators $T_1(f)$ on H_1 and $T_2(f)$ on H_2 such that

$$\int_G \langle T_1(x)\xi, \eta \rangle f(x)dx = \langle T_1(f)\xi, \eta \rangle = \langle \xi, T_2(f)\eta \rangle$$

for all $\xi \in H_1$, $\eta \in H_2$, where dx denotes a Haar measure on G. For a compact subgroup K of G, it is called integrable with respect to L(K) if the restriction of T on K is integrable with respect to L(K).

§2. Decomposability of LSR's

Let $\mathcal{T}$ be a measure space with a σ-finite measure μ. Suppose there is given, for almost every $\tau \in \mathcal{T}$, a linear system $F^{\tau} = \langle F_1^{\tau}, F_2^{\tau} \rangle$. Two functions ζ, ζ', defined for almost all $\tau \in \mathcal{T}$ with its values $\zeta(\tau)$, $\zeta'(\tau)$ in F_i^{τ} $(i=1 \text{ or } 2)$, are identified if $\zeta(\tau) = \zeta'(\tau)$ for almost all $\tau \in \mathcal{T}$. Let F_1 be a vector space of functions (or, strictly speaking, equivalence classes of functions with respect to this identification) ξ on $\mathcal{T}$ with its values $\xi(\tau)$ in F_1^{τ}, and F_2, similarly, a vector space of functions η on $\mathcal{T}$ with its values $\eta(\tau)$ in F_2^{τ}. When we consider each element $\xi \in F_1$ as an equivalence class, we shall denote by $\dot{\xi}$ a representative function in ξ. Similarly we shall denote by $\dot{\eta}$ a representative function in $\eta \in F_2$. For a such pair F_1, F_2, we give the following three definitions.

DEFINITION 1. A pair F_1, F_2 will be called summable if $\tau \longrightarrow \langle \xi(\tau), \eta(\tau) \rangle$ is a $\mathbb{C}$-valued summable function on $\mathcal{T}$ for every $\xi \in F_1$, $\eta \in F_2$.

DEFINITION 2. A pair F_1, F_2 will be called regular if, for every function $\phi \in L^{\infty}(\mathcal{T}, \mu)$, $\xi \in F_1$ implies $\phi\xi \in F_1$, and $\eta \in F_2$ implies $\phi\eta \in F_2$, where $\phi\xi(\tau) = \phi(\tau)\xi(\tau)$, $\phi\eta(\tau) = \phi(\tau)\eta(\tau)$.

DEFINITION 3. A pair F_1, F_2 will be called saturating if, for arbitrary complete systems of representative functions $\{\dot{\xi}\,;\, \xi$

$\in F_1\}$ and $\{\dot{\eta} ; \eta \in F_2\}$, the set $\{\dot{\xi}(\tau) ; \xi \in F_1\}$ is $\sigma(F^\tau)$-dense in F_1^τ and $\{\dot{\eta}(\tau) ; \eta \in F_2\}$ is $\sigma(F^\tau)$-dense in F_2^τ for almost all $\tau \in \mathcal{T}$.

LEMMA 1. Let F_1, F_2 be a regular and saturating pair, then there exist $\xi_0 \in F_1$ and $\eta_0 \in F_2$ such that $\xi_0(\tau) \neq 0$ and $\eta_0(\tau) \neq 0$ for almost all $\tau \in \mathcal{T}$.

Let F_1, F_2 be a regular saturating summable pair. Then the bilinear form

$$\langle \xi , \eta \rangle = \int_{\mathcal{T}} \langle \xi(\tau) , \eta(\tau) \rangle \, d\mu(\tau)$$

gives a duality between F_1 and F_2. We shall call the linear system $F = \langle F_1 , F_2 \rangle$ with this duality a direct integral of F^τ, and denote it by

$$F = \langle F_1 , F_2 \rangle = \int_{\mathcal{T}} \langle F_1^\tau , F_2^\tau \rangle \, d\mu(\tau).$$

DEFINITION 4. Let X be a topological group or a topological algebra. A LSR $U = \langle U_1 , U_2 \rangle$ of X on a linear system $E = \langle E_1 , E_2 \rangle$ is called decomposable if the following three conditions are satisfied.

(1) The linear system $E = \langle E_1 , E_2 \rangle$ is isomorphic to a direct integral $F = \langle F_1 , F_2 \rangle = \int_{\mathcal{T}} \langle F_1^\tau , F_2^\tau \rangle \, d\mu(\tau)$.

(2) For almost all $\tau \in \mathcal{T}$, irreducible LSR's $V^\tau = \langle V_1^\tau , V_2^\tau \rangle$ are defined on $F^\tau = \langle F_1^\tau , F_2^\tau \rangle$.

(3) Denote by $V_1(x)\xi$, $V_2(x)\eta$ the functions defined by $[V_1(x)\xi](\tau) = V_1^\tau(x)\xi(\tau)$, $[V_2(x)\eta](\tau) = V_2^\tau(x)\eta(\tau)$. Then $\xi \in F_1$, $\eta \in F_2$ implies $V_1(x)\xi \in F_1$, $V_2(x)\eta \in F_2$ for all $x \in X$, and there exists

an isomorphism $\Phi = \langle \Phi_1, \Phi_2 \rangle$ of E onto F such that

$$V_1(x) = \Phi_1 U_1(x) \Phi_1^{-1}, \quad V_2(x) = \Phi_2 U_2(x) \Phi_2^{-1}$$

for all $x \in X$.

The LSR $U = \langle U_1, U_2 \rangle$ is called finite-dimensionally decomposable if, in addition, $F^\tau = \langle F_1^\tau, F_2^\tau \rangle$ are finite-dimensional for almost all $\tau \in \mathcal{T}$.

§3. Spherical LSR's of $L^\circ(\delta)$ and canonical LSR's of G

Let G be a locally compact unimodular group, K a compact subgroup of G, and δ an equivalence class of irreducible representations of K. The normalized trace of δ will be denoted by χ_δ, and the normalized Haar measure on K will be denoted by du.

For a LSR $T = \langle T_1, T_2 \rangle$ of G on $H = \langle H_1, H_2 \rangle$ which is integrable with respect to L(G) and L(K), we define continuous projections $P_1(\delta)$, $P_2(\delta)$ on H_1, H_2 respectively by

$$\int_K \langle T_1(u)\xi, \eta \rangle \overline{\chi_\delta}(u)du = \langle P_1(\delta)\xi, \eta \rangle = \langle \xi, P_2(\delta)\eta \rangle .$$

Put $H_1(\delta) = P_1(\delta)H_1$, $H_2(\delta) = P_2(\delta)H_2$, then $H(\delta) = \langle H_1(\delta), H_2(\delta) \rangle$ is a linear system with the duality $\langle\ ,\ \rangle$ restricted from H. For every function $f \in L(\delta) = \overline{\chi_\delta} * L(G) * \overline{\chi_\delta}$, the space $H_1(\delta)$ is invariant under $T_1(f)$, and $H_2(\delta)$ is invariant under $T_2(f)$. Hence we obtain a LSR $\tilde{T} = \langle \tilde{T}_1, \tilde{T}_2 \rangle$ of $L(\delta)$ on $H(\delta) = \langle H_1(\delta), H_2(\delta) \rangle$ where $\tilde{T}_1(f) = T_1(f)|H_1(\delta)$ and $\tilde{T}_2(f) = T_2(f)|H_2(\delta)$ for each $f \in L(\delta)$. If T is irreducible, then $\tilde{T}$ is also irreducible.

Now we fix a unitary matricial representation $u \longrightarrow D(u)$ of K which belongs to δ. We shall denote by d its degree and by

$d_{ij}(u)$ the (i,j)-coefficient of $D(u)$. Let $P_1^i(\delta)$, $P_2^i(\delta)$ be the continuous projections on H_1, H_2 respectively defined by

$$d\int_K \langle T_1(u)\xi , \eta\rangle \overline{d_{ii}}(u)du = \langle P_1^i(\delta)\xi , \eta\rangle = \langle \xi , P_2^i(\delta)\eta\rangle .$$

Put $H_1^i(\delta) = P_1^i(\delta)H_1$, $H_2^i(\delta) = P_2^i(\delta)H_2$, then the pairs $H^i(\delta) = \langle H_1^i(\delta) , H_2^i(\delta)\rangle$ are linear systems with the dualities restricted from H. Since $H_1^i(\delta)$ and $H_2^i(\delta)$ are invariant under $T_1(f)$ and $T_2(f)$ respectively for all functions $f \in L^\circ(\delta) = \{f^\circ ; f \in L(\delta)\}$, where $f^\circ(x) = \int_K f(uxu^{-1})du$, we obtain d LSR's of the algebra $L^\circ(\delta)$ on $H^i(\delta) = \langle H_1^i(\delta) , H_2^i(\delta)\rangle$ for $i = 1,\cdots,d$. These LSR's are mutually equivalent. A LSR $U = \langle U_1 , U_2\rangle$ of $L^\circ(\delta)$ will be called a spherical LSR corresponding to $T = \langle T_1 , T_2\rangle$ if it is equivalent to these LSR's of $L^\circ(\delta)$.

For a linear system $E = \langle E_1 , E_2\rangle$, we shall denote by E_1^d the vector space of all column vectors $\boldsymbol{\xi} = \begin{pmatrix}\xi_1\\ \vdots \\ \xi_d\end{pmatrix}$ with $\xi_i \in E_1$, and by E_2^d the vector space of all column vectors $\boldsymbol{\eta}$ whose components are in E_2. Then $E^d = \langle E_1^d , E_2^d\rangle$ is a linear system with the duality

$$\langle \boldsymbol{\xi} , \boldsymbol{\eta}\rangle = \sum_{i=1}^{d} \langle \xi_i , \eta_i\rangle .$$

LEMMA 2. Let $U = \langle U_1 , U_2\rangle$ be a LSR of $L^\circ(\delta)$ on $E = \langle E_1 , E_2\rangle$ which satisfies one of the following conditions,

(a) U is a spherical LSR corresponding to a LSR of G,

(b) U is irreducible and finite-dimensional.

Then there exists a unique LSR $\tilde{U} = \langle \tilde{U}_1 , \tilde{U}_2\rangle$ of $L(\delta)$ on $E^d = \langle E_1^d ,$

$E_2^d\rangle$ such that

$$\tilde{U}_1(\varepsilon_k * f)\xi = D(k)\begin{pmatrix} U_1(f)\xi_1 \\ \vdots \\ U_1(f)\xi_d \end{pmatrix}, \quad \tilde{U}_2(\varepsilon_k * f)\eta = {}^tD(k)\begin{pmatrix} U_2(f)\eta_1 \\ \vdots \\ U_2(f)\eta_d \end{pmatrix}$$

for all $k \in K$ and $f \in L^\circ(\delta)$, where $\varepsilon_k * f(x) = f(k^{-1}x)$ and ${}^tD(k)$ is the transposed matrix of $D(k)$, and right hand sides are formal products of matricies.

Let $U = \langle U_1, U_2\rangle$ be a finite-dimensional irreducible LSR of $L^\circ(\delta)$ on a linear system $E = \langle E_1, E_2\rangle$, and $\tilde{U} = \langle \tilde{U}_1, \tilde{U}_2\rangle$ the LSR of $L(\delta)$ on $E^d = \langle E_1^d, E_2^d\rangle$ which is given in Lemma 2. Then it is not difficult to show that $\tilde{U} = \langle \tilde{U}_1, \tilde{U}_2\rangle$ is irreducible. Choose non zero vectors $\xi_0 \in E_1^d$ and $\eta_0 \in E_2^d$ arbitrarily, and put

$$\mathfrak{M}_1 = \{f \in L(G)\ ;\ \tilde{U}_1(\overline{\chi_\delta} * g * f * \overline{\chi_\delta})\xi_0 = 0 \quad \text{for all } g \in L(G)\},$$

$$\mathfrak{M}_2 = \{g \in L(G)\ ;\ \tilde{U}_2(\overline{\chi_\delta} * g * f * \overline{\chi_\delta})\eta_0 = 0 \quad \text{for all } f \in L(G)\}.$$

Then $\mathfrak{M}_1$ is a closed maximal left ideal and $\mathfrak{M}_2$ is a closed maximal right ideal in $L(G)$. Now put

$$H_1 = L(G)/\mathfrak{M}_1, \qquad H_2 = \mathfrak{M}_2\backslash L(G).$$

Denoting by $[f]_1$ the coset in H_1 which contains f and by $[g]_2$ the coset in H_2 which contains g, the pair $H = \langle H_1, H_2\rangle$ is a linear system with the duality

$$\langle [f]_1, [g]_2\rangle = \langle \tilde{U}_1(\overline{\chi_\delta} * g * f * \overline{\chi_\delta})\xi_0, \eta_0\rangle.$$

Then the LSR $T = \langle T_1, T_2\rangle$ of G on $H = \langle H_1, H_2\rangle$, defined by

$$T_1(x)[f]_1 = [\varepsilon_x * f]_1, \qquad T_2(x)[g]_2 = [g * \varepsilon_x]_2,$$

is irreducible, and is called a canonical LSR of G corresponding to U. Of course it depends on the choice of ξ_0 and η_0, but it is unique up to equivalence.

§4. Decomposability of a homogeneous LSR of G

Let G, K, and δ be the same as in §3. Let $T = \langle T_1, T_2 \rangle$ be a LSR of G on a linear system $H = \langle H_1, H_2 \rangle$. Under the condition of integrability with respect to L(K), it is called G-homogeneous with respect to δ if every T_1-invariant subspace of H_1 containing $H_1(\delta)$ is $\sigma(H)$-dense in H_1, and if every T_2-invariant subspace of H_2 containing $H_2(\delta)$ is $\sigma(H)$-dense in H_2.

Suppose there exist $\sigma(H)$-dense T_1- or T_2-invariant subspaces H_1', H_2' of H_1, H_2 respectively, then $H' = \langle H_1', H_2' \rangle$ is a linear system with the duality restricted from H. We shall call the LSR $T' = \langle T_1', T_2' \rangle$, where $T_1' = T_1 | H_1'$ and $T_2' = T_2 | H_2'$, a dense contraction of T on H'.

THEOREM. Assume that G is second countable. Let $T = \langle T_1, T_2 \rangle$ be a LSR of G on $H = \langle H_1, H_2 \rangle$, which is integrable with respect to L(G), L(K), and is G-homogeneous with respect to δ. Suppose the corresponding spherical LSR of $L^\circ(\delta)$ is finite-dimensionally decomposable, then there exists a decomposable dense contraction T' of T on $H' = \langle H_1', H_2' \rangle$ which is integrable with respect to L(G) and L(K) and satisfies $H_1'(\delta) = H_1(\delta)$, $H_2'(\delta) = H_2(\delta)$.

Let's sketch the outline of the proof. Let $U = \langle U_1, U_2 \rangle$ be the corresponding spherical LSR of $L^\circ(\delta)$ on a linear system $E = \langle E_1, E_2 \rangle$. For simplicity we consider as follows.

(1) $E = \langle E_1, E_2 \rangle = \int_{\mathcal{T}} \langle E_1^\tau, E_2^\tau \rangle d\mu(\tau)$.

(2) For almost all $\tau \in \mathcal{T}$, finite-dimensional irreducible LSR's $U^\tau = \langle U_1^\tau, U_2^\tau \rangle$ of $L^\circ(\delta)$ are defined on $E^\tau = \langle E_1^\tau, E_2^\tau \rangle$.

(3) For every $\xi \in E_1$, we have $[U_1(f)\xi](\tau) = U_1^\tau(f)\xi(\tau)$, and for every $\eta \in E_2$, we have $[U_2(f)\eta](\tau) = U_2^\tau(f)\eta(\tau)$.

Consider the algebras $\mathcal{A}(G) = L^\infty(\mathcal{T}, \mu) \otimes_{\mathbb{C}} L(G)$ and $\mathcal{A}(\delta) = L^\infty(\mathcal{T}, \mu) \otimes_{\mathbb{C}} L(\delta)$. Let $\tilde{U} = \langle \tilde{U}_1, \tilde{U}_2 \rangle$ be the LSR of $L(\delta)$ which is given in Lemma 2 for U. For every element $\alpha = \sum_i \phi_i \otimes f_i \in \mathcal{A}(\delta)$, we define

$$\pi_1(\alpha)\boldsymbol{\xi} = \sum_i \tilde{U}_1(f_i)\phi_i\boldsymbol{\xi}, \qquad \pi_2(\alpha)\boldsymbol{\eta} = \sum_i \tilde{U}_2(f_i)\phi_i\boldsymbol{\eta}$$

($\boldsymbol{\xi} \in E_1^d$, $\boldsymbol{\eta} \in E_2^d$). By Lemma 1, there exist $\xi_0 \in E_1$ and $\eta_0 \in E_2$ such that $\xi_0(\tau) \neq 0$ and $\eta_0(\tau) \neq 0$ for almost all $\tau \in \mathcal{T}$. We put

$$\boldsymbol{\xi}_0 = \begin{pmatrix} \xi_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in E_1^d, \qquad \boldsymbol{\eta}_0 = \begin{pmatrix} \eta_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in E_2^d.$$

Then, using the second countability of G, we can prove the following

LEMMA 3. The subspace $\{\pi_1(\alpha)\boldsymbol{\xi}_0 ; \alpha \in \mathcal{A}(\delta)\}$ is $\sigma(E^d)$-dense in E_1^d, and $\{\pi_2(\alpha)\boldsymbol{\eta}_0 ; \alpha \in \mathcal{A}(\delta)\}$ is $\sigma(E^d)$-dense in E_2^d.

Let B(,) be a bilinear form on $\mathcal{A}(G) \times \mathcal{A}(G)$ defined by

$$B(\alpha, \beta) = \sum_{i,j} \langle \tilde{U}_1(\overline{\chi_\delta} * g_j * f_i * \overline{\chi_\delta})\phi_i\boldsymbol{\xi}_0, \psi_j\boldsymbol{\eta}_0 \rangle$$

for $\alpha = \sum_i \phi_i \otimes f_i$ and $\beta = \sum_j \psi_j \otimes g_j$. Now we put

$$\mathfrak{M}_1 = \{\alpha \in \mathcal{A}(G) ; B(\alpha, \beta) = 0 \text{ for all } \beta \in \mathcal{A}(G)\},$$

$$\mathfrak{M}_2 = \{\beta \in \mathcal{A}(G) ; B(\alpha, \beta) = 0 \text{ for all } \alpha \in \mathcal{A}(G)\}.$$

Then the pair $\mathfrak{A} = \langle \mathfrak{A}_1 , \mathfrak{A}_2 \rangle$, $\mathfrak{A}_1 = \mathcal{A}(G)/\mathfrak{M}_1$, $\mathfrak{A}_2 = \mathcal{A}(G)/\mathfrak{M}_2$, is a linear system with the duality

$$\langle [\alpha]_1 , [\beta]_2 \rangle = B(\alpha , \beta).$$

Now we construct a LSR $S = \langle S_1 , S_2 \rangle$ of G on $\mathfrak{A} = \langle \mathfrak{A}_1 , \mathfrak{A}_2 \rangle$ by

$$S_1(x)[\alpha]_1 = [\varepsilon_x * \alpha]_1, \qquad S_2(x)[\beta]_2 = [\beta * \varepsilon_x]_2$$

for every $x \in G$.

Since the LSR $\tilde{U} = \langle \tilde{U}_1 , \tilde{U}_2 \rangle$ of $L(\delta)$ on $E^d = \langle E_1^d , E_2^d \rangle$ is equivalent to $\tilde{T} = \langle \tilde{T}_1 , \tilde{T}_2 \rangle$ on $H(\delta) = \langle H_1(\delta) , H_2(\delta) \rangle$, there exists an isomorphism $\Psi = \langle \Psi_1 , \Psi_2 \rangle$ of E^d onto $H(\delta)$ such that $\tilde{T}_1(f) = \Psi_1 \tilde{U}_1(f) \Psi_1^{-1}$, $\tilde{T}_2(f) = \Psi_2 \tilde{U}_2(f) \Psi_2^{-1}$ for all $f \in L(\delta)$. For every element $\alpha = \sum_i \phi_i \otimes f_i \in \mathcal{A}(G)$, we put

$$\Phi_1([\alpha]_1) = \sum_i T_1(f_i)\Psi_1(\phi_i \xi_0), \quad \Phi_2([\alpha]_2) = \sum_i T_2(f_i)\Psi_2(\phi_i \eta_0).$$

Then $\Phi = \langle \Phi_1 , \Phi_2 \rangle$ is a homomorphism of $\mathfrak{A} = \langle \mathfrak{A}_1 , \mathfrak{A}_2 \rangle$ into $H = \langle H_1 , H_2 \rangle$, and, by Lemma 3, the images $H_1' = \Phi_1(\mathfrak{A}_1)$, $H_2' = \Phi_2(\mathfrak{A}_2)$ are $\sigma(H)$-dense T_1- or T_2-invariant subspaces of H_1, H_2 respectively. The dense contraction $T' = \langle T_1' , T_2' \rangle$ of T on $H' = \langle H_1' , H_2' \rangle$ is integrable with respect to $L(G)$, $L(K)$, and satisfies $H_1'(\delta) = H_1(\delta)$, $H_2'(\delta) = H_2(\delta)$. Moreover it is equivalent to $S = \langle S_1 , S_2 \rangle$.

On the other hand, using vectors $\xi_0(\tau) \in (E_1^\tau)^d$, $\eta_0(\tau) \in (E_2^\tau)^d$, we can construct the canonical LSR $T^\tau = \langle T_1^\tau , T_2^\tau \rangle$ of G on a linear system $H^\tau = \langle H_1^\tau , H_2^\tau \rangle$ corresponding to U with

$$\langle [f]_1^\tau , [g]_2^\tau \rangle = \langle \tilde{U}_1^\tau(\overline{\chi_\delta} * g * f * \overline{\chi_\delta})\xi_0(\tau) , \eta_0(\tau) \rangle,$$

$$T_1^\tau(x)[f]_1^\tau = [\varepsilon_x * f]_1^\tau, \quad T_2^\tau(x)[g]_2^\tau = [g * \varepsilon_x]_2^\tau.$$

LEMMA 4. For every function $f \in L(\delta)$, we have

$$\langle \tilde{U}_1(f)\xi , \eta \rangle = \int_{\mathcal{T}} \langle \tilde{U}_1^\tau(f)\xi(\tau) , \eta(\tau) \rangle \, d\mu(\tau) \qquad (\xi \in E_1^d, \ \eta \in E_2^d).$$

It follows from Lemma 4 that, for every $\alpha = \sum_i \phi_i \otimes f_i \in \mathcal{A}(G)$, $\beta = \sum_j \psi_j \otimes g_j \in \mathcal{A}(G)$,

$$\langle [\alpha]_1 , [\beta]_2 \rangle = \int_{\mathcal{T}} \langle \sum_i [\phi_i(\tau) f_i]_1^\tau , \sum_j [\psi_j(\tau) g_j]_2^\tau \rangle d\mu(\tau).$$

This means that every $[\alpha]_1 \in \mathfrak{H}_1$ can be seen as a function

$$[\alpha]_1(\tau) = \sum_i [\phi_i(\tau) f_i]_1^\tau$$

on $\mathcal{T}$, and that every $[\beta]_2 \in \mathfrak{H}_2$ can be seen as a function

$$[\beta]_2(\tau) = \sum_j [\psi_j(\tau) g_j]_2^\tau.$$

Then it is easy to verify that the pair $\mathfrak{H}_1$, $\mathfrak{H}_2$ is regular, summable, and saturating. Thus the LSR $S = \langle S_1 , S_2 \rangle$ of G on $\mathfrak{H} = \langle \mathfrak{H}_1 , \mathfrak{H}_2 \rangle$ is decomposable in the following way ;

$$\mathfrak{H} = \langle \mathfrak{H}_1 , \mathfrak{H}_2 \rangle = \int_{\mathcal{T}} \langle H_1^\tau , H_2^\tau \rangle d\mu(\tau),$$

$$\langle S_1(x) [\alpha]_1 , [\beta]_2 \rangle = \int_{\mathcal{T}} \langle T_1^\tau(x) [\alpha]_1(\tau) , [\beta]_2(\tau) \rangle d\mu(\tau).$$

Since, as is remarked above, S is equivalent to T', the theorem follows.