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LORENZ PLOTS AND CHAOS

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1. INTRODUCTION

Let X be a smooth vector field on R^n. Let $s(t) = (s_1(t), s_2(t), \ldots, s_n(t))$ be a solution curve for X. Assume that the solution $s(t)$ is positively confined in some compact domain D, i.e. $s(t) \in D$ for any $t \geqslant 0$. We pose the following problem, which may not be very clearly formulated from the mathematical viewpoint but is significant for applications of dynamical systems theory to various disciplines (for example see [1]).

We suppose that the only data concerning the system X is one of the components, say $s_1(t)$, of a solution $s(t)$. The problem is to guess the system X and find what we can conclude from the data available. Of course, it is impossible to determine the system completely. However, we often encounter the situation where the available data of the unknown system is limited and none the less we must guess somehow the dynamics of the system to predict the behavior of the system in the future.

Suppose that the data $s_1(t)$ appears to be irregularly oscilating as in fig.1. What system of ordinary differential equations can produce such a solution? Is the solution $s(t)$

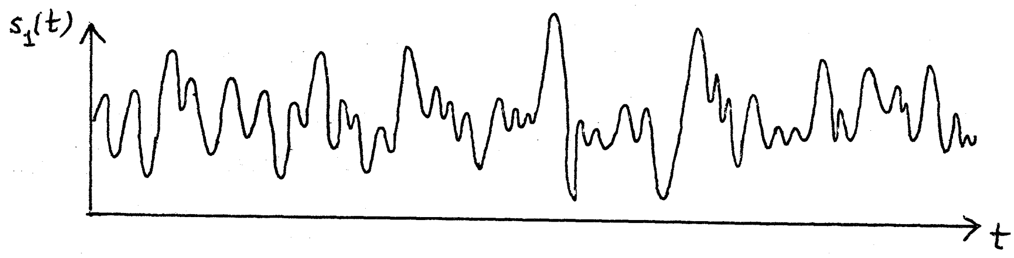


fig.1.

periodic, quasi-periodic or irregular? Can it be produce by a deterministic system? In this paper we shall give an approach to this problem.

2. DIFFERENTIABLE MAPS OF A CIRCLE INTO ITSELF

Let $f : S^1 \rightarrow S^1$ be a differentiable mapping of a circle into itself. Let $g : S^1 \rightarrow I$ be a differentiable map of the circle onto a closed interval I. Let C denote the set of critical points of f and D denote the set of critical points if g ,i.e.

$$C = \{ x \in S^1 \mid df_x = 0 \}$$
$$D = \{ x \in S^1 \mid dg_x = 0 \}.$$

We assume f and g to be generic in the following sense : all the singular points of f and g are non-degenerate; C and D are finite sets; $C \cap D = \emptyset$; $f^{-1}(D) \cap C = \emptyset$. Define a mapping $F : S^1 \rightarrow I \times I$ by $F(x) = (g(x), g(f(x)))$.

PROPOSITION If f and g are generic in the sense stated above, mapping F is an immersion.

Let us examine several examples.

<u>EXAMPLE 1</u> (quasi-periodic rotation)

Let S^1 be the circle $S^1 = R/Z$. Define $g : S^1 \to I=[-1, 1]$ by $g(x)=\cos(2\pi x)$. Let a be an irrational real number. Define $f : S^1 \to S^1$ by $f(x)=x+a \pmod 1$. In this case, f has no critical point, i.e. $C=\emptyset$. The projection g has two non-degenerate critical points at 0 and $1/2$, i.e. $D = \{0,1/2\}$. Mappings f and g are generic in our sense. The image of immersion F is an ellipse. For a point $x \in S^1$, the sequence $\{ f^n(x) \}$ is called the orbit of x. The image $F(f^n(x))$ is given by $(f^n(x), f^{n+1}(x))$. In our case, mapping F is an embedding. So, the dynamics of dynamical system $f : S^1 \to S^1$ is determined completely from the dynamics observed on the ellipse $F(S^1)$ in $I \times I$. Let $\bar{f}$ denote the mapping of $F(S^1)$ into itself defined by $\bar{f}(g(x),g(y)) = (g(f(x)),g(f(y)))$. For each $q_n=(g(f^n(x)),g(f^{n+1}(x)))$ in $F(S^1)$, we have $\bar{f}(q_n) = q_{n+1}$. An example of $F(S^1)$ (for a an irrational real number near 0.2) is given in fig.2.

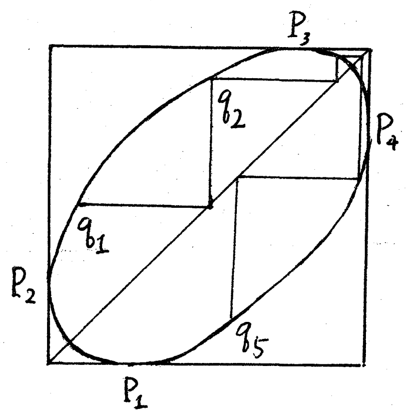


fig.2.

Note that the arc $\widehat{P_3P_4P_1}$ is mapped onto the arc $\widehat{P_4P_1P_2}$ and the arc $\widehat{P_1P_2P_3}$ onto $\widehat{P_2P_3P_4}$.

EXAMPLE 2 (mapping of degree zero)

Let $f : S^1 \to S^1$ be the continuous map defined as follows. Let $p : R^1 \to S^1$ be the universal covering map defined by $p(x) = x \pmod 1) \in R^1/z = S^1$. Let $f : R^1 \to R^1$ be a periodic map of period 1 as in fig.3.

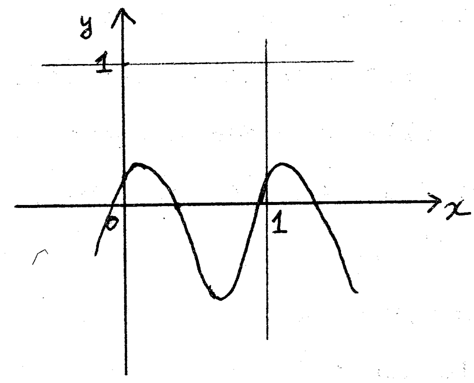


fig.3.

Let $g(x) = \cos(2\pi x)$ for $x \in R^1/Z$. We have $C = \{c_1, c_2\}$, $D = \{d_1, d_2\}$ as in fig.4. Mapping F is defined similarly as in example 1.

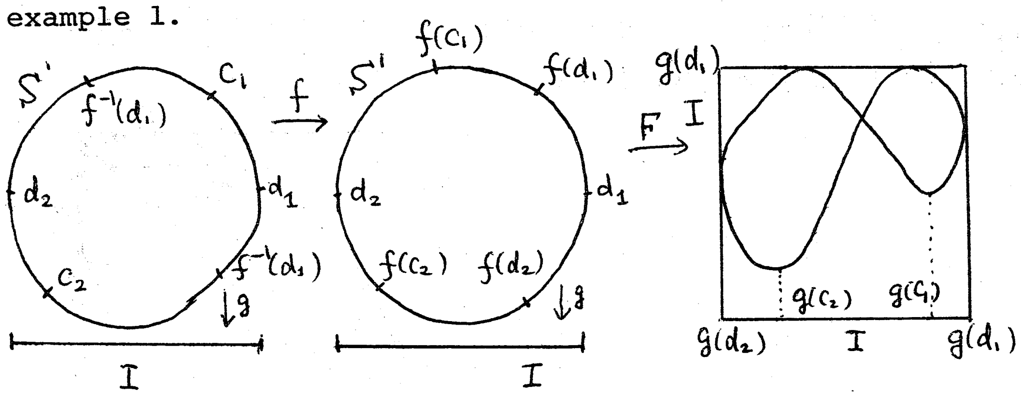
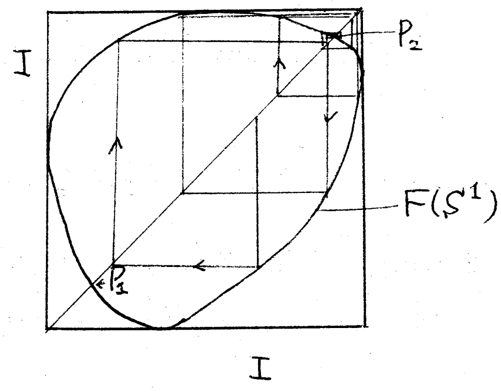


fig.4.

EXAMPLE 3 (orientation reversing case)

Let $f : S^1 \to S^1$ be an orientation reversing diffeomorphism with two fixed points p_1, p_2 one of which is asymptotically stable and the other unstable. In this case the embedded image

$F(S^1)$ is similar to that in example 1, but the dynamics on $F(S^1)$ is different.



EXAMPLE 4 (quasi-periodic with 8-shaped image)

Let $g : S^1 \to I$ be as in example 1. Let $f : S^1 \to S^1$ be an orientation preserving diffeomorphism constructed as follows. Let $h : R^1 \to R^1$ be a differentiable mapping of period 1 satisfying :

1) $\left|\frac{dh}{dx}\right| < 1$,
2) $0 < h(0) < \frac{1}{2}$,
3) $\frac{1}{2} < h(\frac{1}{2}) < 1$.

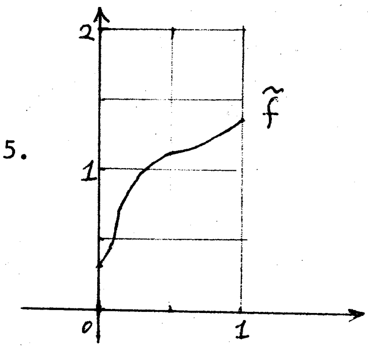


fig.5.

Let $\tilde{f} : R^1 \to R^1$ by $\tilde{f}(x) = x + h(x)$. The map $\tilde{f} : R^1 \to R^1$ defines a map $f : S^1 \to S^1$ via $S^1 = R^1/Z$. In this case the embedding $F : S^1 \to I \times I$ is 8-shaped.

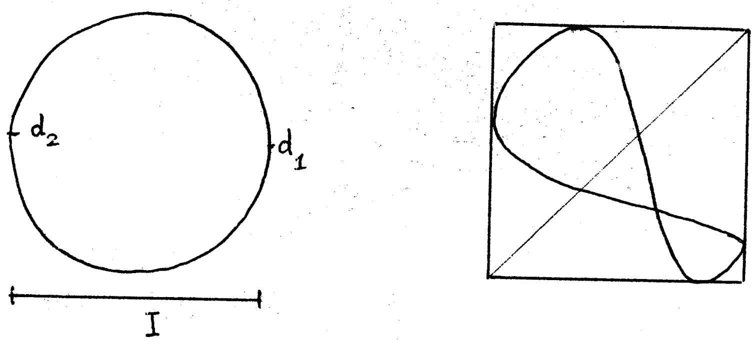


fig.6.

3. POINCARÉ TRANSFORMATION

Let us consider a system of ordinary differential equations on R^n ,

$$\frac{dx}{dt} = X(x) , \quad x \in R^n , \quad X(x) \in R^n ,$$

which has an attractor containing a recurrent orbit. For example, the system of ordinary differential equations studied in Lorenz [2] :

$$(1) \begin{cases} \frac{dx}{dt} = & -\sigma x + \sigma y , \\ \frac{dy}{dt} = -xz & +rx - y , \\ \frac{dz}{dt} = xy & - bz , \end{cases}$$

with parameters $\sigma = 10.0$, $b = \frac{8}{3}$ and $r = 28$. System (1) has recurrent orbits and an attractor as in fig.7.

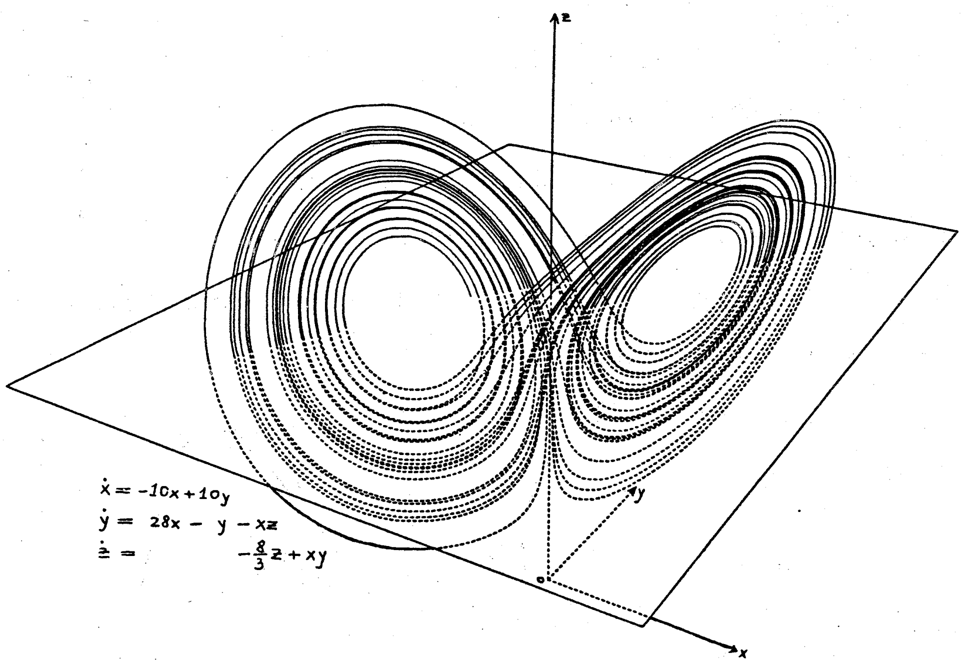


fig.7. (reproduced from [6])

The structure of the attractors of this type, slightly idealized geometrically, is studied in Guckenheimer [3]. In his investigations, he employed the method of Poincaré transformation. Take a portion of hypersurface H transversal to the orbits of the attractor (see fig.8.).

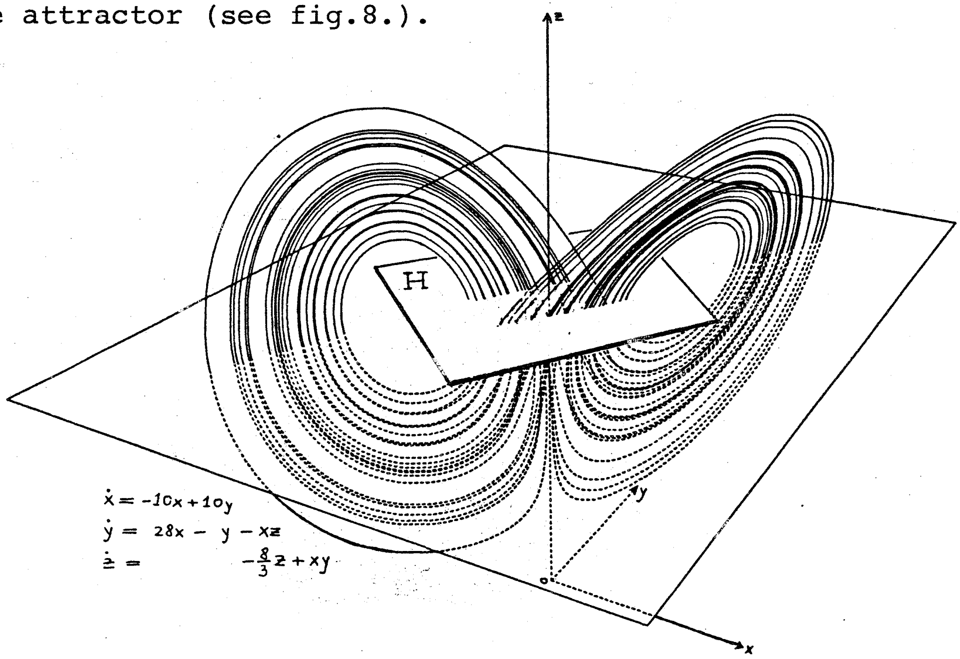


fig.8.

For a point x in H, let s(x) be the first intersection point of the orbit starting x with the hypersurface H (if it exists). In our case, mapping s can be defined on some neighbourhood of the attractor except at the points where the orbit tends asymptotically to the singular point at the origin. Mapping s is called generalized Poincaré transformation. If we have enough information about s, we can derive information about the dynamics on the attractor.

Another example of Poincaré transformation is studied in Rössler [4]. One of the systems of ordinary differential equations studied by Rössler is given by :

$$(2)\quad \begin{cases} \dfrac{dx}{dt} = -y - z\ , \\ \dfrac{dy}{dt} = x + ay\ , \\ \dfrac{dz}{dt} = bx - cz + xz\ , \end{cases}$$

with parameters $a = 0.36$, $b = 0.4$ and $c = 4.5$. System (2) has a Möbius-band-like attractor(see fig.9.). He calls the attractor of this type a walking-stick attractor.

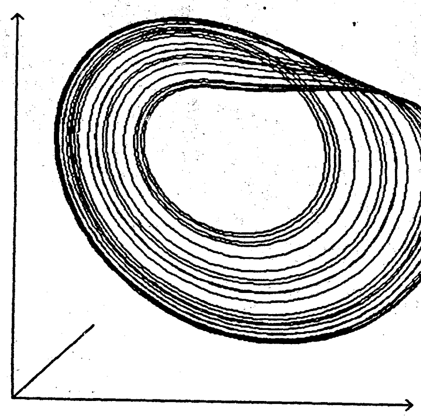


fig.9.

Take a portion of hypersurface H as depicted in fig.10.

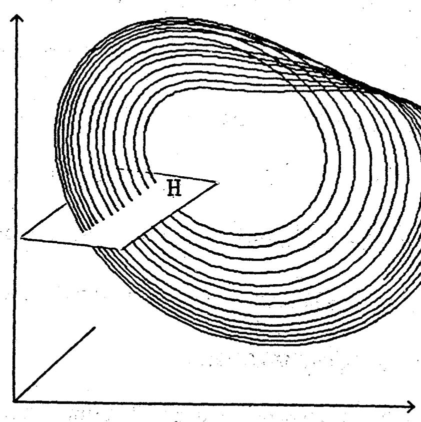


fig.10.

Define the Poincaré transformation $s : H \to H$ as with the system of Lorenz. We can find a rectangular domain in H which is mapped by s into itself. The image is of the form of a walking-stick (fig.11). Some iterated composition of s has horse-shoe type nonwandering sets.

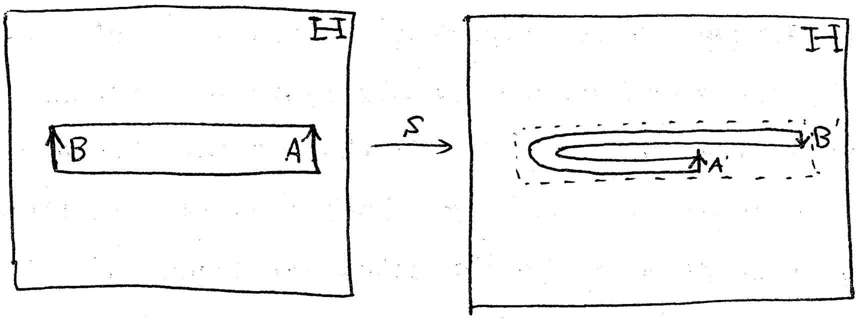


fig.11.

4. PAPER-SHEET MODELS

Williams [5] studied Lorenz attractors and proposed 'paper-sheet models'. He approximates the dynamics on Lorenz attractors by semiflows on branched manifolds. See [5] for details of the construction.

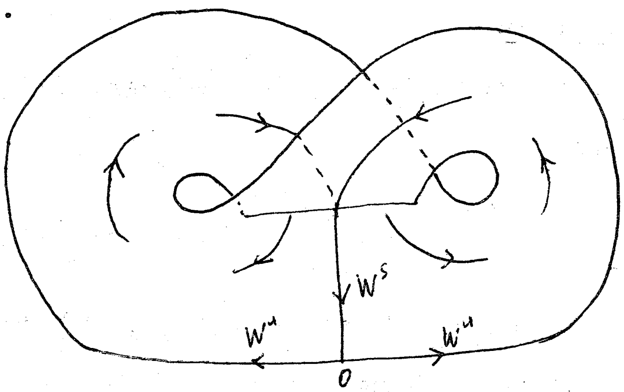


fig.12. paper sheet model for Lorenz attractor

The branched manifold with a semiflow is called a paper model. In the following , we shall consider only systems of dimension three with the obtained branched manifold of dimension two. If the attractor is sufficiently attracting in the normal direction, the attractor will be quite thin and will be well approximated by the paper-sheet model.

In the study of non-hyperbolic attractors, Rössler [7] employed paper-sheet models for his system of ordinary differential equations with "chaos". In these cases, the attractor seems to be contained in a thin sheet locally. In those cases where paper-sheet model approximates the dynamics on the attractor sufficiently well, the Poincaré transformations will be approximated by a mapping of an one-dimensional manifold into itself. For example, the Poincaré transformation for Rössler's walking-stick attractor can be approximated by a mapping of an interval into itself, which may produce "chaos" (see [8] [9]).

5. LORENZ PLOTS

Now consider a system of ordinary differential equations of the form :

$$\frac{dx_i}{dt} = X^i(x), \qquad x = (x_1, x_2, \ldots, x_n) \in R^n.$$

Suppose $\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))$ be a bounded solution for the system. Let p_k be the k-th local maximum value after $t=0$ and t_k be the value of t at that point (see fig.13). Plot the points (p_k, p_{k+1}) in R^2. Lorenz obtained a graph similar to the graph of "baker's transformation". We call this plot a Lorenz plot. The Lorenz plot will give information about the dynamics on the attractor.

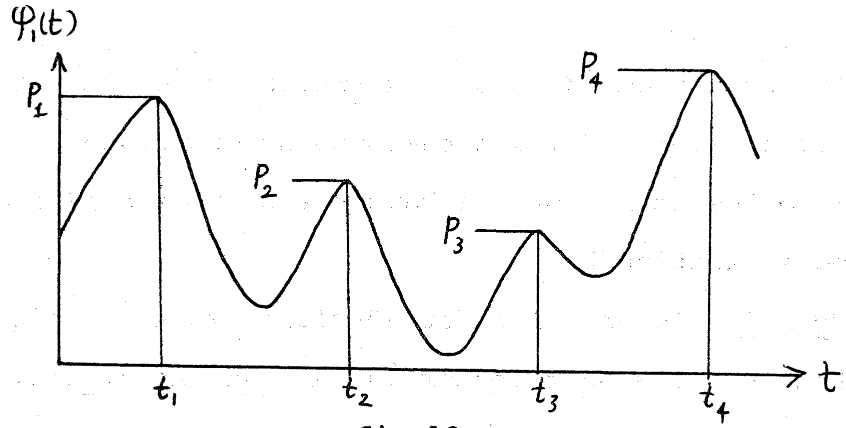


fig.13.

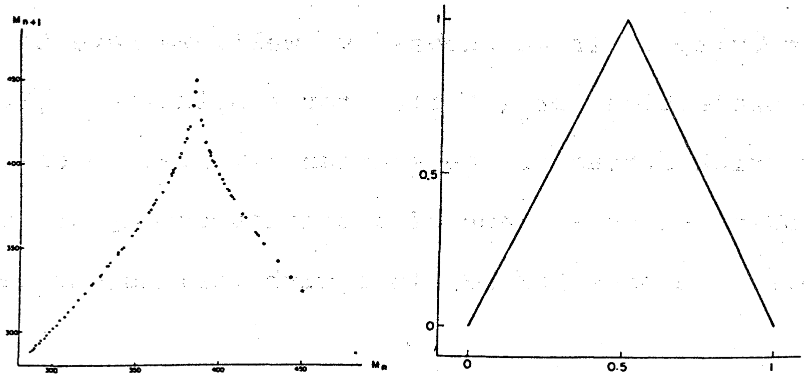


fig.14. Lorenz plot for Lorenz attractor and the graph of Baker's transformation

Let $S = \left\{ x \in R^n \;\middle|\; X^1(x)=0 \right\}$,

$$S^+ = \left\{ x \in S \;\middle|\; \sum_{i=2}^{n} X^i(x) \frac{\partial}{\partial x_i} X^i(x) > 0 \right\} \quad \text{and}$$

$$S^- = \left\{ x \in S \;\middle|\; \sum_{i=2}^{n} X^i(x) \frac{\partial}{\partial x_i} X^i(x) < 0 \right\}.$$

For generic systems X, the sets S, S^+, S^- are regular submanifolds. Suppose that some portion σ of S^+ is transversal to the orbits of X and that the Poincaré transformation map ψ can be defined on σ. Let $pr_1 : R^n \rightarrow R^1$ be the projection to the first coordinate. We have the following diagram :

$$\begin{array}{ccc} \sigma & \xrightarrow{\psi} & \sigma \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle pr_1} \\ R^1 & & R^1 \end{array}$$

If the system X has an attractor which can be approximated by a paper model and that the hypersurface is transversal to the attractor, we have the situation similar to that studied in section 2.

EXAMPLE 5 (Lorenz plot for Rössler's walking stick attractor)

Let v be a unit vector in R^3 at the origin. Let $pr_v : R^3 \to R^1$ denote the projection defined by inner product $pr_v(x) = \langle v,x \rangle$. If we choose v well, we have the Lorenz map via the projection $pr_v(\varphi(t))$ for a solution $\varphi(t)$ of Rössler's walking-stick attractor(see section 3), i.e. Lorenz plot can be approximated by a graph of a continuous map of an interval to itself. We have Lorenz plots with various choice of v.

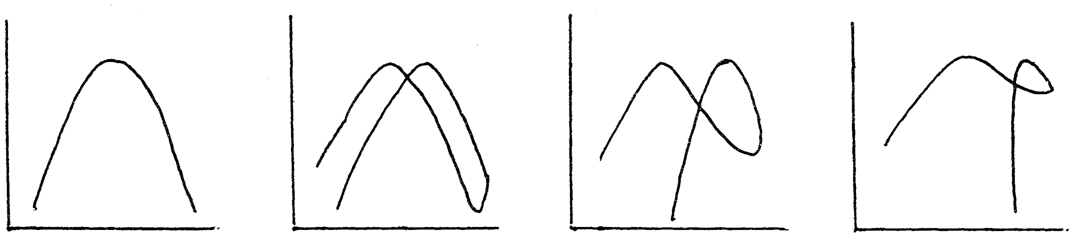


fig.15.

In the case of screw attractors or more complicated attractors in [4], Lorenz plots cannot be approximated by a graph of continuous mapping of an interval any more.

We can try another plot by taking the period $t_k - t_{k-1}$ in the place of $p(n)$. In fact any function on the hypersurface may be taken as a candidate for Lorenz plot. Y.Oono indicated the author that Lorenz plots of Lorenz attractors cannot always be approximated by a graph of univalent mapping if the projection map is modified from the projection $z : R^3 \to R$ taken in [2], or the parameter in Lorenz equation is modified. They produce duplicated "baker's transformation" (see fig.16).

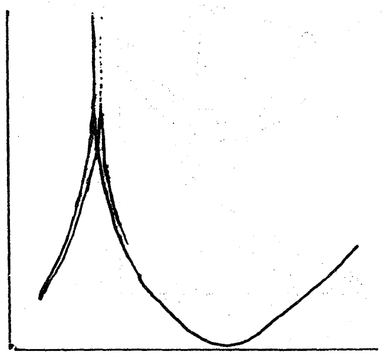


fig.16. (reproduced from [10])

The dynamics on the attractor with such feature of Lorenz plots can be understood by examining the images of each branch.

6. APPLICATION TO DATA ANALYSIS

Let S^1 be an immersed circle in the x-y-plane as depicted in fig.17 and g be the projection to the x-axis.

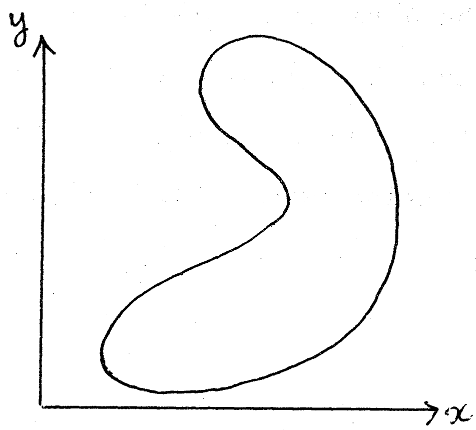


fig.17

Let $f : S^1 \to S^1$ be a 'rotation' on S^1, i.e. an orientation preserving diffeomorphism without fixed points. The immersed circle constructed as in section 2 may be an immersion of degree two as in fig.18.

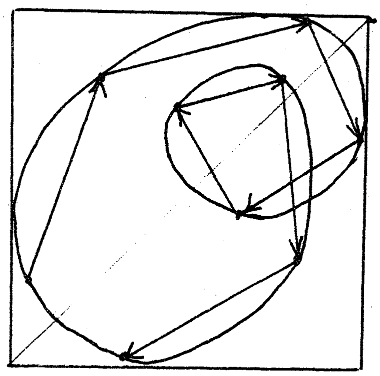


fig.18.

The author [1] studied the average sunspot numbers using the method mentioned above. Suppose that the annual average sunspot number is a function of time, which is one of the coordinates representing the state of the activity of the sun. We suppose that the activity of the sun is regulated by a system of ordinary differential equations. Take Lorenz plots of the average sunspot number function. The data available is not sufficient to determine the structure and the dynamics of the attractor. But we find surprisingly conspicuous features in the plots. Some of them present the "rotating Möbius band" type dynamics (as fig.18). Some plots remind us of the plots for Rössler's walking-stick attractors.

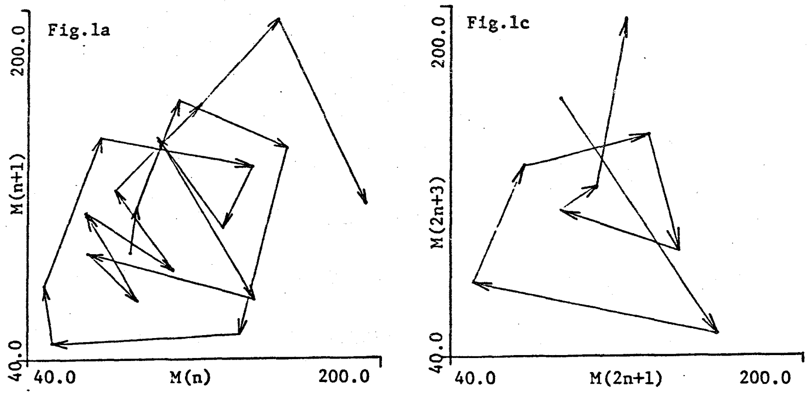


fig.19.
some plots
from [1].

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