数理解析研究所講究録
第 372 巻 1979 年 179-307

P.G.コホモロジー論 と 完備化理論

都立大・理・笹倉頌夫

0. ここには題目にある筆者の原稿の約半分を収める。残りの半分は別の機会に掲載したい希望である。(最終原稿の全ては雑誌に投稿中である。) 表題の理論及びその解析的ド・ラム理論への応用は1971年に開始されたが、現在未だ完全な形では発表されていない。他方詳細を知りたいという希望がかなり多くの人々から寄せられている。ここに収録するものの内容が有益である事を期待する。尚すぐ後の'英文'の部分の簡単な要約をしておこう。

1. 第一章は、筆者の理論に於ける基礎概念、主結果及びその解析的ド・ラム理論への応用を含む。§1.1 で P.G.(= Polynomial growth)-コホモロジー論を'定義'する為の最小概念を与える。

要約すれば、これらの概念中最も主要なものは'growth函数'及び'距離函数(= metric)'である。筆者のコホモロジー論の取扱いはČech理論に基くが、距離函数は"開被覆"のsizeの測定(§1.1)に用い、'生成函数'は"開被覆のsize"及び"Cochainsの増大度の測度に用いる。

基礎概念 ⊕ "additional 基礎概念"

$$(\mathrm{I})\quad \left\{\begin{matrix}\text{閉被覆}\\ \text{連接層}\\ \text{Cochains}\end{matrix}\right\} \oplus \left\{\begin{matrix}\text{growth函数}\\ \text{距離函数}\end{matrix}\right\} \Longrightarrow \left\{\begin{matrix}\text{P.g.コホモロジー}\\ \text{論の定義}\end{matrix}\right\}$$

(詳細は§1.1を参照されたい。) 上記に概述した"定義"は本体の論文中(=原稿中)一貫して用いられる。次に§1.2に於いて, affine (algebraic) variety及びそれに類似性を持つ幾つかのanalytic varieties に対し, これらのvarietyの"自然なP.g.構造"を反映する"growth"及び"距離"函数を定め, P.g.コホモロジー論を定義する。(より正確には, P.g.連接層に対し, P.g.複体を定める。) §1での主結果は, Th.1.1～Th.1.6でそれらは, 我々のP.g.コホモロジー論に於けるH. Cartanの定理A, Bの類似である。§1.3は上記定理の鍵となる事実で, $\mathbb{C}^n$の構造層$\mathcal{O}$に関するP.g.一様評価(定理1.7)が最重要である。§2は筆者の論文の主要結果を含む。まずXを§1.2で用いたvariety, Vをそのsubvarietyとする。Vは有限個の函数$f=(f_j)_{j=1}^{s}\subset\Gamma(X,\mathcal{O})$の零点集合とする。この時まず対$(X, V)$の"a.d.(= algebraic division) 構造"をfを用いて定める。(この構造は要するに, X上の正則函数等のV上の零点位数に関するものである。(Introduction及び§2.1参照)。) 容易に想像される様に"a.d.構造"は完備化理論への応用が主目的である。同時に我々は"P.g.構造"と"a.d."構造を結合した概念として, d.P.(= a.d + p.g.) 構造を対(X, V)に対して定義する。詳

細は特に§2.1に譲るとして，(Ⅰ)と類似の図式を書いておく。

$$(\mathrm{II})\quad \left\{\begin{array}{l}\text{P.g. コホモロジー論}\\ \text{に於ける基礎概念}\end{array}\right\} \oplus \text{a.d.函数} \Rightarrow \left\{\begin{array}{l}\text{d.P. コホモロジー}\\ \text{論 の定義}\end{array}\right\}$$

§2の諸結果(従って論文の主結果)は定理2.1～定理2.6でこれらはd.P.コホモロジー論及び"P.g.コホモロジー論 in completion theory"に於ける定理A, Bの類似と見做されると思われる。(尚上記二理論中，前者は一様評価を含み後者は'評価を落した形での定式化'である。--- §2.1及び§2.2を参照されたい。) §2.3はreducti step：

(Ⅲ)　d.P.コホモロジー論(§2) ⟹ P.g.コホモロジー論(§1)

の鍵となる事実で，それはある種のKoszul複体の"閉写像条件"(§2.1)に関するd.P.一様評価である(Lemma 2.5～Lemma 2.7)。尚"閉写像条件"を満す複体の例については，Part B, §4.1をも参照されたい。§3は定理1.1～定理2.6の解析的ドラム・コホモロジー論の応用であり，その議論は"正則de Rham定理 On Stein variety"のTh A.Bに基く証明と平行して行われる。(§3を参照されたい。) 最後に§4は，層準同型に関する(non cohomological)一様評価及びそのcohomological versionsを与える。亦，かなり前に述べた事実…定理1.7…を基礎にして次のimplicationsを証明する。

(Ⅳ)　Th.1.7 ⟹ P.g.コホモロジー論(Th.1.1～Th.1.6) → d.P.コホモロジー論(Th.2.1～Th.2.6)．

亦 §4のnon cohomological 一様評価の証明は§5で行い，亦Th.1.7の証明は第三章(§6, §7)で行なう。(§5〜§7は別の機会に掲載を希望している)。尚雑誌"数学"に筆者の解析的ドラム理論に於ける'試み'の解説もする予定なので，より詳しくはそちらを参照して頂きたい。

2. 次にこの講究録中の山口・神谷及び野海氏の論説との関連を簡単に記しておく。まず山口氏は筆者の結果(P.g.コホモロジー論を)'局所自由'(=locally free) sheafに拡張した。(筆者のP.g.連接層は大域的resolutionを許す(但し層はlocally freeでなくともよい)が"張り合わせ"は行っていない。)尚同氏は最近我々の方法に基く取扱いとDeligne − Maltsionist (Asterisque'74)のヨ一評価に基くP.g.コホモロジー(より正確には"コホモロジー with modéré")との関連を大略明確にされたと理解する。これらの事実は基本性を有すると思われるので，同氏が結果を近い内にまとめられる事を期待しよう。神谷氏は"P.g.コホモロジー論"に於けるfuctoralな取扱い　　の幾つかを試みている。我々のP.g.コホモロジー論におけるfunctorialな取扱いは非常に望ましいと思われる(筆者の論文のIntroduction参照)ので筆者も同氏の考え方から刺激を受けたのである。(興味を持たれる読者を期待する。)最後に野海氏は"閉写像条件"の代数的構造を(筆者にとって)非常

に満足される形で明確にされた。必用条件をみたす複体の例に関する考察も行いつつあると理解している。（筆者の論文のIntroductionに述べた様に）"閉写像条件を満たす複体の例(典型的な)を知る事は(局所的レベルでも)非常に興味があると思われる。（野海氏の結果にも興味を持たれる読者が多い事を期待する。）

3. 最後に73年の講究録での筆者の原稿(これは今回の論文の原型ではあるが、目を通される事はあまり勧め難い)中で、"岡−Cartan"の基本結果の　　理解を試み且つ analytic de Rham理論への応用を目的とした試みをなす事が主目的であると述べたがそれは現在でも変更はない。(但し完備化理論に関する部分は別の解釈も必要とすると思う。) 他方解析的ド・ラム理論及び今回の論文(Th. A. Bの類似が主結果である)は幾つかの観点から"starting point for something"と見做されるべきであろう。

同時に"Something"への試みを恐らく我々はなすべきであろう。

Cohomology with polynomial growth and completion theory

Nobuo Sasakura

Introduction

1 As the title indicates, the main purpose of this paper is to give a unification of the following two basic throries for coherent sheaves on analytic varieties: (1) a type of cohomology theory, in which

what we call polynomial growth(=p.g.) conditions on cochains and coverings are involved and (2) completion theory along subvarieties of a given analytic variety. Our theory is given to (algebraic) affine varieties and their analytic analogues(n.1,§ 1, 2), which are more general than the affine varieties. The main body of this paper is devoted to certain explicit uniform estimations on p.g. and what we call algberaic division (=a.d.) properties of coherent sheaves on such varieties(n.2). Our main application of such uniform estimations is to give:

(1) an analogue of Th A,B of H Cartan in our unified cohomology theory for the two theories mentioned just above, which we simply call 'cohomology with p.g. in completion theory'(cf. Th.2.5, Th.2.6 and Th.1.5,Th.1.6). We will apply such a result to generalize, to varieties with singularities the well known theorems of A.Grothendieck on algebraic and analytic de Rham theory(cf.[5]). This paper is originally developped*) to provide an analytic base of the generalization as above, in such a manner that the Stein and algebraic properties, which may be the most important properties of the varieties as above, reflect closely in getting the generalization. Our explicit formulations in (1) and in our uniform estimations are so chosen, to certain degrees, that they are convenient for the application to the de Rham theory.

2. Concerning the two quantitative properties mentioned in n.1, the first one 'p.g.' is a synonym for 'rational' or 'meromorphic', when the degree of the cohomology degree is zero(Th.1.5), and such a property concerns most basic properties of algebraic and analytic varieties. Our

---

*) cf.[$]_1$.

When $q \gtrsim 1$, our treatments of the p.g.cohomology(=cohomology with p.g) theory may be a sharpening of purely algebraic treatments of coherent sheaves(§1). By the second one, algebraic division(=a.d.)property, we mean such a property that concerns the degrees of zeros of cochains etc. along (imbedded) subvarieties of an analytic variety. As we learn from the classical Hilbert zero point theorem, such a property concerns basic properties of the imbedded varieties, and is important for investigations of analytic varieties. Now our studies of the a.d.properties will be focussed on what we call open map properties(Def.2.1) of geometric filtered complexes, such as Cech and de Rham complexes of global nature(§2,§3) as well as certain local complexes formed from homomorphisms of coherent sheaves(§4), where the filtered structures are defined by the powers of the ideals of the subvarieties. The open map property concerns that propert of the degree one map in question(§2.1), and implies standard comparison theorems in completion theories(cf.[13]). In particular, it insures:

(2) exactness of cpmplexes → that of the completion of the complexes.

The open map property for the Cech complexes will insure the analogue of Th.A,B mentioned in (1), n.1, while that for the other complexes will concern interesting a.d.properties of analytic varieties(§3, §4 and §5). Our main task in the uniform estimations is to combine those on that open map properties with those on the p.g.properties of the above complexes.

Now letting the type of our cohomology theories be as above, we summarize briefly the content of this paper.

3

3. Chap I contains the basic notions and the main results of this paper. First, in §1.1, we summarize basic notions which are used in our p.g. cohomology theory. In §1.2 we give our main results in the p.g. uniform estimations(Th.1.1 ∼ Th.1.4), and we derive from them an analogue of Th A,B in our p.g. cohomology theory(Th.1.5, Th.1.6). The cohomology theory in §1 concerns the p.g. properties of the complexes but not with the a.d. properties, and Th.1.5, Th.1.6 may be regarded as a prototype of the result mentioned in (1), n.1. Our proof of Th.1.1 ∼ Th.1.6 will be given by using a p.g. version of standard tools for treatments of coherent sheaves, syzygies, imbedding of analytic varieties as well as extensions of cochains, and a p.g. uniform estimation on Cousin integrals(cf. Lemma 1.2 ∼ Lemma 1.4 and Th.1.7 in §1.3).

Cohomology theories with p.g. conditions were studied by P.Deligne-G.Maliotionist [11] and by M.Corbalna-P.A. Griffiths [2] for locally free sheaves over smooth algebraic barieties, by using the $\bar{\partial}$ -estimations. The situation in our p.g. cohomology theory, where we work with what we call 'p.g. coherent sheaves(Def.1.5)' over the analytic varieties as in n.1, is more general than theirs. Our method depending on Cousin integrals differ from theirs.*) Next, in § 2, we generalize the p.g. uniform estimations by combining them with uniform estimations on the a.d. properties of the complexes in question. The main results in this generalization, which we call d.p.(=a.d. + p.g.) uniform estimation, as well as in the uniform estimation of this paper are given in Th.2.1 ∼ Th.2.4. Such results insure the open map properties of the complexes, and our analogue of Th.A,B in the p.g.cohomology in the completions(cf.(1),n.1) is a formal consequence of them.

4. In the first part of Chap.II, we summarize our non cohomologica uniform estimations on homomorphisms of coherent sheaves(cf. § 4.1). We then give a cohomological version of those estimations, and we derive, from such cohomological results, all the lemmas in Chap.I,II which concern the uniform estimations on the sheaf homomorphisms(§4.2). The uniform estimations in § 4.1 contain results on the open map property of certain Koszul complexes, which provide a cohomological generalization of Hilbert zero point theorem and are a non cohomologic version of the main lemma, Lemma 2.5, in the d.p.uniform estimations in §2(cf.Lemma 4.2∼ Lemma 4.4). Such a fact, together with an open map property of the de Rham complex(Lemma 4.7), is our main result on the open map property of the sheaf homomorphisms, and may be worthwhi pointing out in connection with our treatments of completion theories.

5. Finally, in Chap.III, we prove the key theorem, Th.1.7, for the geometric arguments in §1, §2(as was indicated in § 3), which concerns a p.g.uniform estimation of the structure sheaves of the complex euclidean spaces. We prove Th.1.7, by reducing it to rather elementary p.g.estimations on Cousin integrals(on the euclidean line)

Our reduction depends on certain filtrations defined for the sets of the cochains (in question) and some algebraic machinaries for the filtrations, which imply a strong sharpening of the standard degerency theorem in the spectral sequence theory . The algebraic arguments and the p. g. estimations on Cousin integrals in Chap. III may owe their own interests, aparting from the applications to Chap. I, II . (For the content of Chap III indicated soon above, see the beginning of Chap. III We add a brief outline of Chap. I $\sim$ III in the beginning of each chapter. Such an introduction may be useful for understanding of the content of each chapter and of the whole line takin in this paper.)

6. In giving the application of the cohomology theories in §1, §2 to the analytic de Rham theory(§3), we should quote our results on the C^{∞}-de Rham theory for certsin stratified spaces, whose outline was given in $[15]_{2\sim 4}$ and in [17]. The details of $[15]_{2\sim 4}$ and [17] will be published elsewhere in a near future . Except thr part of the application to the de Rham theory in §3, this paper is completely self contained .

The author began the study of the contents of this paper and of the analytic de Rham theory in 1971, and the very early versions of the content of this paper were given in [15]$_1$ and [16].Considerable parts of the explicit computaions in the uniform estimation of the present paper depend on[16].However, the present paper is written entirely newly from [16].Finally, the contents of the present paper seem to deserve to be generalized in more general situations: our p.g.cohomology theory is given in a more or less categorical form. Generalizations of the content of § 1 seems to be very desirable in that line(cf.n.6, §1.2). Assuming the p.g.cohomology theory in §1, the most important facts in giving the d.p.cohomology theory are the open map propeties of the geometric complexes mentioned hitherto. From the scope of the arguments in § 2, the validity of the open property as well as the clarifications of their geometric meanings seems to be desirable for more general classes of geometric complexes. Finally, our explicit p.g.unicorm estimations and the algebraic machinaries in Chap.III seem to deserve to be tried their applicablities for more general types of 'cohomology with growth conditions'. The author hopes that he will try possible genralizations about what are mentioned above. We also hope that the contents of the present paper provide a basis for possible generalizations.

Chapter I. Cohomology with polynomail growth and completion theory

Here, for convenience of reading of Chap I, we illustrate the basic notions and the styles of the formulations in our p.g. uniform estimations. For this we first let C^n be a complex euclidean space and z coordinates of it. We set $g:=|z|+1$, O:=structure sheaf of $\underline{C}^n$, and, for an element $\partial=(\partial_1,\partial_2)\in\underline{R}^{+2}$, we define:

$$(0)_1\ \begin{cases} H^0(\underline{C}^n,\underset{\sim}{O};g)_{\partial} :=\{\varphi\in\Gamma(\underline{C}^n,\underset{\sim}{O});\ |\varphi(P)|<\partial_1 g(P)^{\partial_2}\ \text{in}\ \underline{C}^n\}, \\ H^0(\underline{C}^n,\underset{\sim}{O};g)_{p.g}:=\bigcup_{\partial\in R^{+2}} H^0(\underline{C}^n,\underset{\sim}{O};g)_{\partial}\ . \end{cases}$$

We then recall that a classical consequence of Cauchy integral formula implies:

$$(0)_1'\quad H^0(\underline{C}^n,\underset{\sim}{O};g)_{p.g}\quad \cong\{\text{polynomials in } z\}.$$

Next letting X be a complex space, we may say that

$$(0)_2\quad \begin{Bmatrix}\mathrm{Coh}(X)\\ \mathrm{Cov}(X)\end{Bmatrix}:=\text{collection of all}\begin{Bmatrix}\text{coherent sheaves over } X\\ \text{open coverings of } X\end{Bmatrix}$$

and the cochain map

$$(0)_3\quad C^q:\mathrm{Coh}(X)\times\mathrm{Cov}(X)\ni(\underset{\sim}{F},\underline{A})\longrightarrow C^q(\underline{A},\underset{\sim}{F})$$

constitute the underlying data for the cohomology theories of coherent sheaves over X. Now our first task in §1 is to give p.g. versions of $(0)_2,(0)_3$, which yield a generalization of the sets in $(0)_1$, with respect to the underlying varieties, coherent sheaves and cohomology degrees. Our main results in §1 for the case of the degree =0 are a generalization of $(0)_1'$, while those for degree$\geqq 1$ insures the vanishing property of the cochomology groups*): in §1.1 we introduce spme abstract notions, p.g. filtration, q-structure of abelian sheaves**) and p.g. functions;

---

*) cf. Th.1.5 and Th.1.6 . Also see Introduction.

**) This is an obvious abstraction of the 'absolute value' as in $(0)_1$ to general abelian sheaves(cf. Def.1.4_1).

we see that such notions suffice to generalize the sets as in $(0)_1$ to genral abelain sheaves(Def. 1.4_5) and to give a p.g. version of Coh(X), denote by $\mathrm{Coh}(X)_{p.g}$, to reduced complex spaces. Also, using the p.g. and a 'distance function' of a topological space X, we define for each subset Y of X what we call 'p.g. covering of Y in X' in a concrete p.g. fashion(Def. 1.7).

In the first part of §1.2, we attach, to our analytic vatieties in the main body of Chap.I(cf. n.1, §1.2), a p.g. version of Cov(X), denoted by $\mathrm{Cov}(X)_{p.g}$, by using the arguments in § 1.1 . Next take a p.g. sheaf $\underset{\sim}{H}$ $\in \mathrm{Coh}(X)_{p.g}$. Then, using a p.g. version of the cochain map as in $(0)_3$, we attach to $\underset{\sim}{H}$ what we call 'p.g. cochain collection', $C^*(X,\underset{\sim}{H})_{p.g}$ in symbol. Such a p.g. collection contains all necessary sets of p.g. cochains in our p.g. uniform estimations, and may be the most basic underlying data for our p.g. cohomology theory . We note that the above p.g. collections, $\mathrm{Cov}(X)_{p,}$ and $C^*(X,\underset{\sim}{H})_{p.g}$, are parametrized in a certain concrete fashion, where the main part of the parameter space is a product $\underline{R}^{+s}(s>0)$; we define what we call p.g. estimation maps, which are a concrete transformation of $\underline{R}^{+s}(s>0)$. The main results in §1, Th 1.1 $\sim$ Th 1.6, are given to the p.g. cochain collections . We use the p.g. estimation maps for the explicit estimations in those results;.

9

In § 2 we generalize the p. g. cohomology theory in § 1 to what we called the d. p. cohomology theory(cf. Introduction). We give a generalization of the p. g. cohanin collection, which we call d. p. cochain collection(§ 2. 2). The main uniform estimations, Th. 2.1～ Th. 2. 4, of § 2 as well as of this paper are formulated in terms of the latter cochain collections(§ 2. 2). The content of §2 is more general than the one of § 1, by the introduction of the new factor of what we called the a d estimations(cf. Introduction). However, the algebraic style of the formulations in § 2 will be given parallely to the one in §1.

Terminologies and Notations

Here we summarize some terminologies and notations, which are used throughout the present paper

1 First letting X be a topological space, we set:

$$(1)_1 \quad \left\{\begin{array}{l}\left.\begin{array}{l}\mathrm{Ouv}(X)\\ \mathrm{Ab}(X)\end{array}\right\} := \text{collection of all} \left\{\begin{array}{l}\text{open sets oof } X\\ \text{abelian sheaves over } X\end{array}\right\}\\ \mathrm{Cov}_0(X) := 2^{\mathrm{Ouv}(X)}\end{array}\right.$$

For an element $\underset{=\{A_\mu\}}{\underline{A}} \in \mathrm{Cov}_0(X)$ we set $|\underline{A}| = \cap_\mu A_\mu$. We use the symbol $\underline{N}^q\underline{A}(q \geqq 1)$ for the q-nerve of $\underline{A}$:

$$(1)_2 \quad \underline{N}^q\underline{A} := \{\underline{A}_\mu = (A_1 < \cdots \prec A_q) \subset \underline{A};\ |\underline{A}_\mu| \neq \phi\},$$

2. For a positive number a, we set:

$$(2)_1 \quad \underline{R}^+_a := \{r \in \underline{R}^+;\ r \geqq a\}.$$

We use the set $\underline{R}^+_1$ frequently in the uniform estimations in this paper. Also we use the symbol $\underline{R}^+$, as usual, for the set $\{r \in \underline{R}; r > 0\}$. Moreover, for a subset $\underline{T}$ of $\underline{R}$, we use the set:

$$(2)_2 \quad \underline{T}^+_a := \{t \in \underline{T};\ t \geqq a\}.$$

We use such a set for the case $\underline{T}=\underline{Z}=$set of all integers(cf. Chap II).

Next, for an element $\sigma=(\sigma_1,\sigma_2)\in R_1^{+2}$, we define:

(2)$_3$ $R_\sigma^{+2} := \{(r_1;r_2)\in R^+\times R^+;r_1\geqq\sigma_1,\ r_2\geqq\sigma_2\}$.

When $\sigma=(1,1)$, we use the symbol R_1^{+2} also for R_σ^{+2} .(This symbol is concordant to $R_1^{+2}=R_1^+\times R_1^+$(cf.(2)$_1$). Thirdly, for elements $\alpha=(\alpha_1,\alpha_2)\in R^{+2}$ and $a\in R^+$, we set:

(2)$_4$ $a\cdot\alpha:=(a\alpha_1,\alpha_2)$.

Moreover, for elements $\sigma=(\sigma_1,\sigma_2)$, $\sigma'=(\sigma'_1,\sigma'_2)\in R^{+2}$, we write $\sigma\geqq\sigma'$, if $\sigma_1\geqq\sigma'_1$ and $\sigma_2\geqq\sigma'_2$. When $\sigma_1>\sigma'_1$ or $\sigma_2>\sigma'_2$, we write $\sigma>\sigma'$.

§1. Cohomology with polynomial growth

§1.1. Polynomial growth conditions

1. Cochain collection. First letting X be a topological space and $(\underline{A},\underline{F})$ an elemenet of $\mathrm{Cov}_0(X)\times \mathrm{Ab}(X)$, we make:

Definition 1.1. By q-th cochain(resp. cocycle) collection for $(\underline{A},\underline{F})$, we mean the collection of all subsets of $C^q(\underline{A},\underline{F})$(resp. $Z^q(\underline{A},\underline{F})$). We use the symbols $\underline{C}^q(\underline{A},\underline{F}),\underline{Z}^q(\underline{A},\underline{F})$ for such collections. Here we check:

Proposition 1.1. The following two facts are equivalent:

$(1.1)_1$ $Z^q(\underline{A},\underline{F}) \subset \delta C^{q-1}(\underline{A},\underline{F})$.

$(1.1)_2$ For each $\underline{D} \in \underline{Z}^q(\underline{A},\underline{F})$, there is an element $\underline{D}' \in \underline{C}^{q-1}(\underline{A},\underline{F})$ so that

$(1.1)_2'$ $\underline{D} \subset \delta\underline{D}'$.

Proof. Taking $\underline{D}$ to be $Z^q(\underline{A},\underline{F}) \in \underline{Z}^q(\underline{A},\underline{F})$, $(1.1)_2'$ insures $(1.1)_2$. Conversely setting $\underline{D}'=C^{q-1}(\underline{A},\underline{F}) \in \underline{C}^{q-1}(\underline{A},\underline{F})$, we have $(1.1)_2$ from $(1.1)_1$.

In our later arguments, we do not work with the 'whole spaces' $C^q(\underline{A},\underline{F}),\ldots$ but with what we call p.g.subgroups, $C^q(\underline{A},\underline{F})_{p.g}$ in symbol, ... of $C^q(\underline{A},\underline{F}),\ldots$, which are characterized by concrete p.g.conditions (cf. §1.2.See also n.2 soon below). We will derive corresponding facts to $(1.1)_1$ ('vanishing property' in p.g.cohomology theory) from correspondences of $(1.1)_2$. In the later arguments, the former is also a formal consequence of the latter, but the converse is not true. Our main subject in §1 will be to get similar inclusions to $(1.1)_2$ in our p.g.cohomology, by making explicit similar correspondences to the one: $\underline{D}\to\underline{D}'$ in $(1.1)_2$. Our main results in such a direction are given in Th.1.1 ~ Th.1.6 in §1.2. The remainder of §1.1 will be devoted to define what we call p.g.cochain collection, which is a family of elements of $\underline{C}^q(\underline{A},\underline{F})$(characterized by concrete p.g.cocnditions), by making clear basic notions in the definition of such collections. The arguments will be given in a somewhat abstract fashion.

2. P.g.filtration. We begin n.2 by making a definition, which will play a basic role to define the collection mentioned at the end of n.1.

Definition 1.2_1. (1) By b-(resp.p.g-) filtation *) for an abelian group $\underline{B}$, we mean a map $\theta : \underline{R}^+ \longrightarrow 2^{\underline{B}}$(resp.$\Psi : \underline{R}^{+2} \rightarrow 2^{\underline{B}}$) satisfying:

$(1.2)_1$ $\theta(a) \ni 0$ for each $a \in \underline{R}^+$(resp.$\Psi(\partial) \ni 0$ for each $\partial \in \underline{R}^{+2}$).

$(1.2)_2$ $\theta(a') \supset \theta(a)$ if $a' \geqq a$(resp.$\Psi(\partial') \supset \Psi(\partial)$ if $\partial' \geqq \partial$).

$(1.2)_3$ (Archimedian property) For any $a, a' \in \underline{R}^+$(resp. $\partial, \partial' \in \underline{R}^{+2}$), there is an element $a'' \in \underline{R}^+$(resp.$\partial'' \in \underline{R}^{+2}$) so that

$(1.2)_3'$ $\theta(a'') \supset \{b \pm b' ; (b,b') \in \theta(a) \times \theta(a')\}$(resp.$\Psi(\partial'') \supset \{b \pm b' ; (b,b') \in \Psi(\partial) \times \Psi(\partial')\}$.

(2) We call $\bigcup_{a \in \underline{R}^+} \theta(a)$, $\bigcup_{\partial \in \underline{R}^{+2}} \Psi(\partial)$ respectively θ-bdd and ← *) Ψ-p.g. subgroups of $\underline{B}$.

(When no confusions occur, we drop the terms 'bdd' and 'p.g.' from the terminology just above.)

Definition 1.2_2. We say that b-filtrations $\theta_1, \theta_2 : \underline{R}^+ \longrightarrow 2^{\underline{B}}$ are equivalent, if $\theta_i(\underline{R}^+)(i=1,2)$ are cofinal with respct to the increasing inclusion as in $(1.2)_2$.

Definition 1.2_3. Letting $\underline{B}'$ be a subgroup of $\underline{B}$(resp.$\omega : \underline{B} \rightarrow \underline{B}''$ a homomorphism of abelian group), we call the b-filtration $\theta' : \underline{R}^+ \ni a \longrightarrow 2^{\underline{B}'} \ni \theta(a) \cap \underline{B}'$(resp.$\omega^*\theta : \underline{R}^+ \ni a \longrightarrow 2^{\underline{B}''} \ni \omega\theta(a)$) the one induced from

---

*)'b' := initial of 'bounded' and 'bdd' = abbreviation of 'bounded'.

to B'(resp. to B'' by ω). Moreover, letting $\tilde{\theta}$, $\tilde{\theta}''$ be b-filtrations of B, B'' we say that $\tilde{\theta}$, $\tilde{\theta}''$ are compatible with ω, if, for each $b \in R^+$, we have:

$(1.2)_4$ $\omega\tilde{\theta}(b) \subset \tilde{\theta}''(b')$, with a suitable $b' \in R^+$.

The 'induced filtrations', 'equivalence' and 'compatiblity' as above are defined for p.g. filtrations in the similar manner to Def.1.2_2. Now taking B to be $C^q(A,F)$ in n.1, let Ψ be the p.g. filtration in Def.1.2_1:

Definition 1.3. By q-th Ψ-p.g. cochain and cocycle collections for (A,F), we mean:

$(1.3)_1$ $C^q(A,F;\Psi)_{p.g} := \Psi(R^{+2}) (\subset C^q(A,F))$, $Z^q(A,F;\Psi)_{p.g} := \tilde{\Psi}(R^{+2}) (\subset Z^q(A,F))$,

where $\tilde{\Psi}$ is the induced filtration of Ψ to $Z^q(A,F)$(Def.1.2_2).

Also we set:

$(1.3)'_1$ $C^q(A,F;\Psi)_{p.g} := \Psi$-p.g. subgroup of $C^q(A,F)$(Def.1.2_1).

$(1.3)''_1$ $Z^q(A,F;\Psi)_{p.g} := C^q(A,F;\Psi)_{p.g} \cap Z^q(A,F)$,

Moreover, when A consists of the single element X(i.e. $A = \{X\}$), we set:

$(1.3)'''_1$ $\Gamma(X,F;\Psi)_{p.g} := Z^0(A,F;\Psi)_{p.g}$.

Now we will construct p.g. filtration in a geometric manner. For this we first make a definition, which is a slice abstraction of 'absolute value' for analytic functions etc. in the standard meaning:

Definition 1.4_1.(1) By q-structure of $F \in Ab(X)$, we mean a family $\theta = \{\theta_U; U \in Ouv(X)\}$ of b-filtrations $\theta_U : R^{+2} \longrightarrow 2^B$, $B = \Gamma(U,F)$, satisfying:

$(1.3)_2$ $\rho\theta_{U'}(a) \supset \theta_U(a)$ for any $U' \supset U$ and $a \in R^+$, where ρ = restriction: $\Gamma(U';F) \to \Gamma(U,F)$.

$(1.3)_3$ The stalk θ_P of θ at $P \in X$(i.e. $\theta_P : R^+ \ni a \to \theta_P(a) := \lim\limits_{U \to P} \theta_U(a)$ satisfies:

$(1.3)'_3$ θ_P is a b-filtration for F_P, and $F_P = \bigcup_{a \in R^+} \theta_P(a)$.

(2) For an element $\mathscr{Y}_P \in F_P$ we call $\inf(a \in R^+;\ \theta_P(a) \ni \mathscr{Y}_P) (\in R^+ \cup 0)$ the θ-absolute value of $\mathscr{Y}_P$.

We call the pair $(\underset{\sim}{F},\theta)$ simply q-sheaf. When there is no fear of confusions, we write the symbol '$(\underset{\sim}{F},\theta)$' also as '$\underset{\sim}{F}$'. Letting $(X,\underset{\sim}{O}_X)$ be a reduced complex space, we define a q-structure $\theta_X=\{\theta_U;U\in \mathrm{Ouv}(X)\}$ by

$(1.3)_4$ $\theta_U:\underline{R}^+\ni a\to \Gamma(U,\underset{\sim}{O}_X)_a:=\{\varphi\in\Gamma(U,\underset{\sim}{O}_X);\ |\varphi(p)|<a \text{ on } U\}$, where $|\ |$ denotes the absolute value (in the standard sense).

Definition 1.4_2. We call θ_X the standard q-structure of $\underset{\sim}{O}_X$. One check easily that the absolute value defined by the standard q-structure coincides with that in $(1.3)_4$. For $\underset{\sim}{O}_X^k$ we define the standard q-structure by $\theta_X^k=\{\theta_U^k\}_U$, where θ_U^k assigns to each $a\in\underline{R}^+$ the k-times diruct sum of the subset in $(1.3)_4$(of $\Gamma(U,\underset{\sim}{O}_X)$). We define q-structures for gneral coherent sheaves in n.3. Now returning to the pair $(\underset{\sim}{A},\underset{\sim}{F})$ in Def.1.4_1, take a subsheaf $\underset{\sim}{F}'$ of $\underset{\sim}{F}$ and a homomorphism $\omega:\underset{\sim}{F}\to\underset{\sim}{F}''$.

Definition 1.4_3. By q-strucure for $\underset{\sim}{F}',\underset{\sim}{F}''$ induced from $\underset{\sim}{F},(\underset{\sim}{F},\omega)$, we mean:

$(1.3)_5$ $\theta':=\{\theta'_U:\underline{R}^+\ni a\to\theta'_U(a):=\theta_U(a)\cap\Gamma(U,\underset{\sim}{F}')\}_U,\ \omega^*\theta:=\{\omega^*\theta_U;\underline{R}^+\ni a\to\underset{\substack{\uparrow\\ \omega\theta_U(a)}}{\omega\theta_U}(a)\}_U$,

where θ is the q-structure of $\underset{\sim}{F}$ as in Def.1.4_1.

Next the following simple definition plays basic roles not only in our geometric construction of p.g.filtrations but also in many aspects of later arguments:

Definition 1.4_4. By a defining function of p.g.structure of X (or, simply, a p.g.function), we simply mean a map $g:X\to\underline{R}_1^+$. We call (X,g) simply a p.g.pair. Now, using the p.g.pair (X,g) and the q-sheaf $(\underset{\sim}{F},\theta)$ as in Def.1.4_1, we define p.g.cochain collection for $(\underset{\sim}{A},\underset{\sim}{F})$ in the following fashion:

Definition 1.4_5.(1) We say that an element $\varphi \in C^q(\underline{A},\underset{\sim}{F})$ is (g,θ)-∂-growth($\partial \in \underline{R}^{+2}$), if, for each $\underline{A}' \in \underline{N}^{q+1}\underline{A}$, we have:

$(1.3)_6$ $|\varphi_{\underline{A}'}(Q)|_\theta < \partial g(Q)$ on $|\underline{A}'|$, where $|\ |_\theta = \theta$ -absolute value(Def.1.4_1).

(2) By (g,θ)-p.g.filtration for $C^q(\underline{A},\underset{\sim}{F})$, we mean:

$(1.3)_7$ $\Psi_{g,\theta}:\underline{R}^{+2} \ni \partial \rightarrow$ the subset of $C^q(\underline{A},\underset{\sim}{F})$ consisting of all (g,θ)-∂-growth cochains.

Letting the subsheaf $\underset{\sim}{F}'$ of $\underset{\sim}{F}$ and the homomorphism $\omega:\underset{\sim}{F}\rightarrow\underset{\sim}{F}''$ be as in Def.1.4_1, we use the symbols $\theta', \omega\theta$ for the q-structures for $\underset{\sim}{F}', \underset{\sim}{F}''$ which are induced from θ to $\underset{\sim}{F}'$(resp.to $\underset{\sim}{F}''$ by ω). Then we easily have:

Proposition 1.2. $\Psi_{g,\theta'} = (\Psi_{g,\theta})'$ and $\Psi_{g,\omega\theta} = \omega\Psi_{g,\theta}$, where the right sides are induced from $\Psi_{g,\theta}$ to $C^q(\underline{A},\underset{\sim}{F}')$ and to $C^q(\underline{A},\underset{\sim}{F}'')$ by ω . In n.3 soon below we define a p.g.filtration for certain coherent sheaves in a more explicit manner.

3. P.g.coherent sheaf. Letting $(X,\mathbf{O}_X)$ be a reduced complex space, take a p.g.function g of X(Def.1.4_4). We begin n.3 by giving a p.g.condition on coherent sheaves over X, which is used in the remainder of this paper:

Definition 1.5. By a (g)-p.g.resolution of an $\mathbf{O}_X$-coherent sheaf $\mathbf{K}$, we mean a pair $\underline{K}=(\omega, \{K_j\}_{j=1}^{p-1})$ consisting of an $\mathbf{O}_X$-homomorphism ω and matrices $K_j(1\leqq j\leqq p-1)$ with entries in*) $\Gamma(X,\mathbf{O}_X;\Psi_g)_{p.g}$. The pair $\underline{K}$ must satisfy a resolution as follows:

$$(1.4)_1 \quad 0 \rightarrow \mathbf{O}_X^{k_p} \xrightarrow{K_{p-1}} - - - - - \xrightarrow{K_1} \mathbf{O}_X^{k_1} \xrightarrow{\omega} \mathbf{K} \rightarrow 0.$$

(For later convenience we call $\mathbf{O}_X^{k_1}$ in $(1.4)_1$ the 'first resolution part' of $\mathbf{K}$.)

Convention 1.1. (1) By a p.g.coherent sheaf over X, we mean a pair $\mathbf{H}=(\mathbf{K},\underline{K})$ as above; starting with the sheaf $\mathbf{K}$, our explicit uniform estimations depend not only on $\mathbf{K}$ but also on a resolution like $(1.4)_1$. The terminology 'p.g. sheaf' as above is convenient for later purposes.

(2) When there is no fear of confusions we use the symbol '$\mathbf{H}$' for also '$\mathbf{K}$'.

We arrange here some data which are useful in later arguments: first, writing (X,g) as $\underline{X}$, we set:

$(1.4)_2$ $\mathrm{Coh}(\underline{X})_{p.g}$:=collection of all p.g.coherent sheaves over X.

We define a map(length map):

$(1.4)_3$ $\mathrm{lg}:\mathrm{Coh}(\underline{X})_{p.g}\ni\mathbf{H}\longrightarrow\underline{Z}^+\ni p_{\mathbf{H}}$(=length of the resolution of $\mathbf{H}$)(cf.(1.4

and we also define an increasing filtration of $\mathrm{Coh}(\underline{X})_{p.g}$:

---

*) cf. $(1.3)_1'$

$(1.4)_4$ $\mathrm{Coh}^p(\underline{X})_{p.g}:=\{\underset{\sim}{H}\in \mathrm{Coh}(\underline{X})_{p.g};p_{\underset{\sim}{H}}\leq p\}$.
Our coherent sheaves in later arguments are in $\mathrm{Coh}(\underline{X})_{p.g}$. Next letting the p.g.sheaf $\underset{\sim}{H}$ be as in $(1.4)_1$, we mean by standard q-structure of $\underset{\sim}{H}$ the one induced from $\omega:\underset{\sim}{O}^{k_1}\to\underset{\sim}{H}$(cf.Def.1.$4_2$ and $(1.4)_1$). This q-structure $\theta_{\underset{\sim}{H}}$ is determined by ω, while that of $\underset{\sim}{O}^{k_1}$ is uniquely determined by the analytic structure of X(cf.n.2); we may say that $\theta_{\underset{\sim}{H}}$ is determined 'uniquely' by the analytic structure of $(\underset{\sim}{H},X)$. Now letting $\Psi_{g,\theta_{\underset{\sim}{H}}}$ denote the $(g,\theta_{\underset{\sim}{H}})$-p.g.filtrartion for $C^q(\underline{A},\underset{\sim}{F})$(cf.Def.1.$4_5$), we get $\Psi_{g,\theta_{\underset{\sim}{H}}}$-p.g.cochain collection etc., which are obtained by applying Def.1.3 to $\Psi_{g,\theta_{\underset{\sim}{H}}}$. For later notaional convenience we arrange here some notaion for such collections.(The key point in the arrangement is:(1) to drop the term $\theta_{\underset{\sim}{H}}$ from $\Psi_{g,\theta_{\underset{\sim}{H}}},\ldots$ and (2) to write $\Psi_g:=\Psi_{g,\theta_{\underset{\sim}{H}}}$ simply as 'g', when no notational ambiguities take place.) Thus we have:

$$(1.4)_6\ \begin{Bmatrix}\underline{C}^q(\underline{A},\underset{\sim}{H};g)_{p.g}\\ \underline{Z}^q(\underline{A},\underset{\sim}{H};g)_{p.g}\end{Bmatrix}:=\ \Psi_g\text{-p.g.}\begin{Bmatrix}\text{cochain}\\ \text{cocycle}\end{Bmatrix}\text{collection for }(\underline{A},\underset{\sim}{F})$$

$$(1.4)_7\quad C^q(\underline{A},\underset{\sim}{H};g)_{p.g}:=\Psi_g\text{- p.g.subgroup of }C^q(A,\underset{\sim}{H}).$$

$$(1.4)_8\ \begin{Bmatrix}C^q(\underline{A},\underset{\sim}{H};g)_{\mathfrak{d}}\\ Z^q(\underline{A},\underset{\sim}{H};g)_{\mathfrak{d}}\end{Bmatrix}:=\begin{Bmatrix}\text{set of }(\mathfrak{g},\theta_{\underset{\sim}{H}})\text{-}\mathfrak{d}\text{-growth cochains with value in }\underset{\sim}{H}\\ C^q(\underline{A},\underset{\sim}{H};g)_{\mathfrak{d}}\cap Z^q(\underline{A},\underset{\sim}{H})\ .\end{Bmatrix}$$

(For the above sets, collections and subgroup, see $(1.3)_6$ and $(1.3)_1$.) Also we will abbreviate 'Ψ_g-p.g.' and '$(g,\theta_{\underset{\sim}{H}})$-' in the above terminologies simply as 'g'. The data as in $(1.4)_{6\sim 8}$ will be frequently used in the remainder of §1 (cf. §1.2, §1.3). Using the above notaion we have the following easily from Prop.1.2:

Proposition 1.3. $C^q(\underline{A},\underset{\sim}{H};g)_{\mathfrak{d}}=\omega C^q(\underline{A},\underset{\sim}{O}_X^{k_1};g)_{\mathfrak{d}}$ (cf.Def.1.4).

(The similar relation to the above holds for $C^q(\underline{A},\underset{\sim}{H};g)_{p.g}$ and $C^q(\underline{A},\underset{\sim}{H};g)_{p.g}$.) We finish n.3 by the following remark for convenience of later argumnets(cf. § 1.3).

Remark 1.1. (1) For $\underset{\sim}{O}_X^k(k>0)$ one can attach the p.g.coherent sheaf in the following manner:$0\rightarrow \underset{\sim}{O}_X^{k} \xrightarrow{1\ \ i} \underset{\sim}{O}_X^k \rightarrow 0$, with the identity i.. This trivial realization of $\underset{\sim}{O}_X^k$ as the p.g.coherent sheaf is useful in later arguments. Unless we say otherwise, we mean by the 'p.g. coherent sheaf $\underset{\sim}{O}_X$' the above (trivial) one.

(2) Next we define a subcollection of $\mathrm{Coh}(\underline{X})_{p.g}$:

$(1.4)_9$ $\mathrm{Coh}'(\underline{X})_{p.g} := \{\underset{\sim}{H}\in \mathrm{Coh}(\underline{X})_{p.g}$; where the first resolution ω: $\underset{\sim}{O}_X^k\rightarrow \underline{H}$(cf.Def.1.5) is defined by a matrix K(i.e. $\omega=K$), with entries in $\Gamma(X,\underset{\sim}{O}_X;g)_{p.g}\}$. Note that $\underset{\sim}{H}$ is a subsheaf of $\underset{\sim}{O}_X^{k'}$, with k'=length of colums of K, and we have a p.g.filtration for $\underset{\sim}{H}$ by means of the incolusion:$\underset{\sim}{H}\hookrightarrow \underset{\sim}{O}_X^{k'}$(Def.$1.4_2$ and Prop.1.3). Writing this filtration as Ψ'_g, the set of Ψ'_g-λ-growth cochains with value in $\underset{\sim}{H}$(cf.$(1.4)_8$) is explicitly as follows:

$$(1.4)_{10}\quad C^q(\underline{A},\underset{\sim}{H};\Psi'_g)_\lambda = C^q(\underline{A},\underset{\sim}{O}_X^k;g)_\lambda \cap C^q(\underline{A},\underset{\sim}{H})(\text{cf.}(1.4)_5).$$

In §1.3 we give a comparison of Ψ' and the standard p.g.filtration Ψ_g, which is a key fact in our p.g.$\overset{\leftarrow g}{\text{uniform}}$ estimations(Lemma 1.2).

4. P.g.covering. Here we assume that the p.g.pair (X,g) is as in n.2, and we fix a map $d:X\times X\rightarrow[0,\infty]$, to which we impose the single condition*):d=0 on the diagonal Δ_X of X. We define a type of p.g.covering, which is used in the main body of this paper. For this lettin

P be a point of X we use the symbol $\widetilde{U}_r(P):=\{Q\in X ; d(P,Q)< r\}$. Then taking a subset Y of X and an element $\sigma\in\underline{R}_1^{+2}$ we make:

Definition 1.6$_1$. By g-p.g.covering of Y of size σ in X, we mean the following collection*) of elements of 2^X:

(1.5)$_1$ $\underline{A}_\sigma(Y;g):=\{\widetilde{U}_\sigma(P;g);\ P\in Y\}$, where $\widetilde{U}_\sigma(Q;g):=\widetilde{U}_r(P)$, with $r=\{\sigma g(P)\}^{-1}$

Next take subsets X',Y' of X satisfying $Y'\subset Y\cap X'$ and an element $\sigma'\in\underline{R}_1^{+2}$ satisfying $\sigma'\geq\sigma$.

Definition 1.6$_2$. We call the map:

(1.5)$_2$ $s:\underline{B}_{\sigma'}(Y';g)\ni\widetilde{U}'_{\sigma}(P;g)\longrightarrow\underline{A}_\sigma(Y;g)\ni\widetilde{U}_\sigma(P;g)$

p.g.refining map(from the left side to right side). Here the left side denotes the g-p.g.covering of Y' of size σ in X', and $\widetilde{U}'_\sigma(P;g):=\widetilde{U}_\sigma(P;g)\cap X'$. Fixing Y(resp.$X'$,$Y'$), the p.g.covering in (1.5)$_1$(resp.the p.g.map in (1.5)$_2$) is determined uniquely by σ(resp. σ,σ'). This fact will be useful to fix our ideas and to simplify argumetns in later explicit uniform estimations(cf. §1.3 and § 4). Also such coverings and maps are suitable for our geometric applications of the uniform estimations to geometric situations(cf. §1.2, §2.2 and § 3). The coverings and the refining maps in the main body of this paper will be the ones in (1.5)$_{1,2}$.

Now, by Def.1.6$_{1,2}$ we have introduced all necessary basic notions to define what we call 'p.g.cohomology theory for analytic varieties'; the first basic datum is the p.g.function $g:\underline{X}\to R_1^+$ which is used to measure the p.g.properties of cochains and coverings. The q-structure for abelain sheaves is used to define the p.g.condition on cochains (cf.Def.1.4$_1$). Finally 'distance function d' is used to define the p.g.condition of the coverings. As was checked in n.3, the q-structure for coherent sheaves may be regard as determined by the underlyin analytic structure of the varieties; we may regard the p.g.function g and the

---

*) In our later examples of the varieties X, $\underline{A}_\sigma(Y;g)$ is a collection of open sets in X .

'distance function' d are most basic 'additional data' to the analytic varieties, which are used to define what we call p.g.cohomology for those varieties. In order to emphasize this, we will sometimes call (X,g) and (X,g,d) as just above 'p.g.pair' and 'p.g.triple'.

5. Finally we arrange here certain concrete maps, which are used in later explicit estimations: first, by a positive monomial, we mean: $M(t)=at^b (a,b>0)$, where t is a variable. We call a map $\underline{L}:\underline{R}^{+2}\ni(\sigma_1,\sigma_2)\longrightarrow\underline{R}^{+2}\ni(\tilde{\sigma}_1,\tilde{\sigma}_2)$ to be of 'exponential linear type' (or, simply, 'el-map'), if $\tilde{\sigma}_1=M(\sigma_1)\exp M'(\sigma_2)$, $\tilde{\sigma}_2=L(\sigma_2)$, with a positive monomial M, a finite sum M' of positive monomials and a linear function $L(t)=ct;c>0$. It is easily checked that a composition of el-maps is also an el-map. Next making a notational convention:

$(1.6)'_0$ $a\cdot(\delta_1,\delta_2)=(a\delta_1,\delta_2)$ for any a,δ_1 and $\delta_2\in\underline{R}^+$,

we set:

$(1.6)'_1$ $\underline{E}'_{p.g.}:=\{E':\underline{R}^+\times\underline{R}^{+2}\ni(r,\sigma)\rightarrow\underline{R}^+\times\underline{R}^{+2}\ni(r',\sigma')\}$, where $r'=M_1(r)$, $\sigma'=M_2(r^{-1})\cdot\underline{L}_1(\sigma)$,

$(1.6)_2$ $\underline{E}_{p.g}:=\{E:(\underline{R}^+\times\underline{R}^{+2})\times\underline{R}^+\ni(r,\sigma;\delta)\rightarrow(\underline{R}^+\times\underline{R}^{+2})\times\underline{R}^{+2}\ni(r',\sigma';\delta')\}$, where $(r',\sigma')=E'(r,\sigma)$, with an element $E'\in\underline{E}'_{p.g}$, and $\delta'=M_3(r^{-1})\cdot\underline{L}_2(\delta+\sigma)$.

(Here M_i, $\underline{L}_j$ are positive monomials and el-maps.)

We write the correspondence: $\underline{E}_{p.g}\ni E\longrightarrow\underline{E}'_{p.g}\ni E'$ as π, where E,E' are as in $(1.6)_2$ (cf.Fig.I).

Fig.I

$$\begin{tikzcd}(\underline{R}^+\times\underline{R}^{+2})\times\underline{R}^{+2} \arrow[r, "\pi"] \arrow[d, "E"'] \arrow[dr, phantom, "\circlearrowright"] & (\underline{R}^+\times\underline{R}^{+2}) \arrow[d, "E'"] \\ (\underline{R}^+\times\underline{R}^{+2})\times\underline{R}^{+2} \arrow[r, "\pi"] & (\underline{R}^+\times\underline{R}^{+2})\end{tikzcd}$$

Definition 1.7. We call an element $E\in\underline{E}_{p.g}$ an estimation map for p.g.cohomology (or, simply, p.g.c.map) and $E'=\pi(E)\in\underline{E}'_{p.g}$ its first part.

In the later estimations, the map E' concerns those on coverings, while the map E concerns those on both coverings and cochains (§ 1.2). letting the element $(r;\sigma;\partial)$ be as in $(1.6)_2$, the estimation:$(r;\sigma)\rightarrow(r';\sigma')=E'(r;\sigma)$ concerns that of coverings, while the element ∂ concerns that of cochains. Note that the factorization in Fig.I insures that the term '∂' has no influences on the correspondence:$(r;\sigma)\longrightarrow(r';\sigma')$. We use this fact in § 1.2. Next take p.g.c.maps $E_1,E_2 \in \underline{E}_{p.g}$. Then the composition $E_2\circ E_1$ is not, in genral, in $E_{p.g.}$. However, define an order in $\underline{R}^{+}\times\underline{R}^{+2}\times\underline{R}^{+2}$ by

$(1.6)_3'$ $\quad (r;\sigma;\partial) \succ (r';\sigma';\partial') \Longleftrightarrow r<r'\ ,\ \sigma>\sigma'$ and $\partial>\partial'$.

Then the set $\underline{E}_{p.g.}$ is closed under the compisition in the sense that there is a p.g.c.map $E_3 \in \underline{E}_{p.g.}$ satisfying

$(1.6)_3$ $\quad E_3(r;\sigma;\partial) \succ E_2\circ E_1(r;\sigma;\partial)$ for each $(r;\sigma;\partial)\in(0,1]\times\underline{R}_1^{+2}\times\underline{R}_1^{+}$

We use this fact in later p.g.estimations frequently, without mentioning it explicitly(cf.§ 1.3). Also we use the symbol: $E_3 \succ E_2\circ E_1$ to indicate the inequality in $(1.6)_3$.

Finally, the p.g.c.maps as above will be used in our main results in the p.g.uniform estimations(cf.Th.1.1 and Th.1.2). Our explicit form of the p.g.c.maps are chosen in such a manner that (1) the p.g.estimations obtained by such maps insure our p.g.analogues of Th.A,B of H.Cartan and (2) the p.g.c.maps are concordant to more elementary p.g.estimations on sheaf homomorphisms and on Cousin integrals(cf. §4 and § 6). Fixing the explicit forms of the p.g.c.maps as above, considerable parts of the arguments will be reduced to those of p.g.c.maps, which are essentially algebraic and elementary(cf. §1.3. Also see, in particular, § 4.2).

§1.2. Main results

Here we summarize our main results on the p.g.cohomology theory in §1:in n.1∼n.4 we introduce some basic data, which will underlie the arguments in the cohomology theory in the remainder of Chap.I. Using such data, we give our main results on the uniform estimations in the p.g.cohomology in Th.1.1∼Th.1.4(cf.n.4,n.5). Also dropping the explicit estimations in these results, we give analogues of Th.A.B of H.Cartan in our cohomology theory in §1(cf.Th.1.5,Th.1.6 in n.6).

1. Geometric data. As was mentioned*), our analytic varieties in Chap.I will be Stein varieties with suitable algebraicity(and will have similarities to affine varieties). Here we introduce such varieties.

(i) First, by a coordinated complex euclidean space, we mean a pair $(\underline{C}^n,z)$ of a complex euclidean space $\underline{C}^n$ and its coordinates z. When there is no fear of confusions, we use the terminology 'complex euclidean space $\underline{C}^n(z)$'(or, simply, 'euclidean space $\underline{C}^n(z)$') as a synonym for 'Coordina euclidean space $(\underline{C}^n,z)$'. We then introduce a geometric datum:

$(1.7)'_0$ $\underline{\tilde{X}}:=(\underline{C}^n(z)\times\underline{C}^{n'}(z'),\tilde{X}=\underline{C}^n(z)\times U'_0,P'_0)$, where $\underline{C}^n(z),\underline{C}^{n'}(z')$ are euclidean spaces and $U'_0(\ni P'_0)$ is an open set of $\underline{C}^{n'}$,

and we set:

$(1.7)_0$ $\underline{An}_0$:=collection of all geometric data as in $(1.7)'_0$.

The underlying variety of $\underline{\tilde{X}}$ will be $\tilde{X}=\underline{C}^n\times U'$. We regard $\underline{C}^n\times\underline{C}^{n'}$ as the ambient space of $\tilde{X}=\underline{C}^n\times U'_0$ and the point P'_0 as the 'center' of X. As we

---

*) cf. Introduction.

will see in later arguments, the uniform estimations, which are given to varieties in $\underline{An}_0$, are most basic among the ones in §1.2.(In §1.2 we introduce two another types of varieties(cf.$(1.8)_0$ and $(1.11)_0$). The p.g.estimations for such varieites will be derived from the ones for varieties in $\underline{An}_\theta$, by using explicit relations of the former varieties to the latter;see Cor.1.4 and Lemma 1.3 in §1.3.)

(ii) Next, by a (smooth) local analytic variety of affine type, we mean a geometric datum:

$(1.8)_0'$ $\underline{X}=(\underline{C}^n(z),U_0,X_0,h,P_0,H_{\underline{X}})$, where $X_0(\ni P_0)$ is an analytic variety in an open set U_0 of $\underline{C}^n$, and h is an element of $\Gamma(U_0,\underset{\sim}{O}_{U_0})$, $\underset{\sim}{O}_{U_0}$ being the structure sheaf of U_0.

Moreover, setting

$(1.8)_0''$ D_0=divisor of h(in U_0), $D=X_0\cap D_0$, and $X=X_0-D$,

the final datum $H_{\underline{X}}$ in $(1.8)_0''$ is a $|h^{-1}|$-p.g.resolution of $\underset{\sim}{O}_X$ over U_0-D_0 (cf.$(1.4)_1$), where $\underset{\sim}{O}_X$ is the structure sheaf of X and the first term of $H_{\underline{X}}$ is of the form:$\underset{\sim}{O}_{U_0-D_0}\xrightarrow{\omega}\underset{\sim}{O}_X$(cf.$(1.4)_1$), with the natural homomorphism ω .(Here $\underset{\sim}{O}_{U_0-D_0}$ is the structure sheaf of U_0-D_0.)

The datum $\underline{X}$ must satisfy:

$(1.8)_1$ $D\ni P_0$, and X is smooth,

$(1.8)_2$ the germs of X_0,D at P_0 have no common irreducible components.

We set:

$(1.8)_0$ $\underline{An}_{1a}$:=collection of all smooth analytic varieties of affine type (cf.$(1.8)_0''$).

同じ高さ

The underlying variety of X will be $X=X_0-D$. We regard U_0-D_0 as the ambient space of X and the point $P\in X_0$ as the 'center' of X_0,.... The p.g.uniform estimation for $\underline{X}\in\underline{An}_{1a}$ will play basic roles in semi-global estimations in later arguments(cf. § 2). (Note that we inclu the p.g. resolution $H_{\underline{X}}$ of the structure sheaf $\underset{\sim}{O}_X$ in $(1.8)_0$. The resolution $H_{\underline{X}}$ is used to give an explicit uniform estimations for the sheaf $\underset{\sim}{O}_X$;see §1.3.)

In the remainder of §1.2 we will fix geometric data $\underline{\tilde{X}}\in\underline{An}_0$ and $\underline{X}\in\underline{An}_{1a}$ of the form in $(1.7)'_0$ and $(1.8)'_0$. In (iii) soon below we fix some additional data and notation for such varieties.

(iii) First, to $\underline{\tilde{X}},\underline{X}$, we attach the following p.g. and distance functions:

$$(1.9)_0 \quad \begin{Bmatrix} g_{\underline{\tilde{X}}} \\ g_{\underline{X}} \end{Bmatrix} := \begin{Bmatrix} |\tilde{z}|+1 \\ |z|+1 \end{Bmatrix}, \qquad \begin{Bmatrix} d_{\underline{\tilde{X}}} \\ d_{\underline{X}} \end{Bmatrix} := \text{natural distance }^{*)} \text{ of } \begin{Bmatrix} \underline{C}^n(z)\times\underline{C}^{n'}(z') \\ \underline{C}^n(z) \end{Bmatrix}$$

where $\tilde{z}=(z,z')$. When there is no fear of confusions, we write $d_{\underline{X}},d_{\underline{\tilde{X}}}$ als as $d_z,d_{\tilde{z}}$. In our frame work, p.g. and distance functions have basic meanings to define what we call 'p.g.cohomology theory' for analytic varieties(cf.n.4, § 1.1). The p.g. and distance functions for $\underline{\tilde{X}},\underline{X}$ as ab may reflect closely the analytic structures of $\underline{\tilde{X}},\underline{X}$ and may be natural ones for studies of p.g. properties of coherent sheaves over $\underline{\tilde{X}},\underline{X}$. Our p.g. and distance functions for $\underline{\tilde{X}}\in\underline{An}_0,X\in\underline{An}_{1a}$ in the remainder of Chap.I will be the ones in $(1.9)_0$(cf.also Remark 1.2 ,n.6, §1.2).

Next, $g_{\underline{X}},d_{\underline{X}}$ as above are determined by $\underline{X}$. We will use the symbol $\underline{X}$ also for the p.g. pair and triple:$(\underline{X},g_{\underline{X}})$ and $(\underline{X},g_{\underline{X}},d_{\underline{X}})$(cf.n.4, § 1.1). A when there is no fear of confuaiosn, we use $\underline{X}$ for its underlying var

---

*) natural distance of $C^n(z)$ is defined to be $d_z:= |z-\tilde{z}|$ for z, $\tilde{z}\in\underline{C}^n$,

X For the variety $\underline{\tilde{X}}\in\underline{An}_0$ we use the similar notational convenience to the above.

3. P.g parametrization. Take analytic varieties $\underline{\tilde{X}}\in\underline{An}_0$ and $\underline{X}\in\underline{An}_{1a}$. We then attach to $\underline{\tilde{X}},\underline{X}$ what we call 'p.g. cochain collection', which will contain all necessary sets of cochains in our p.g. uniform estimations for $\underline{\tilde{X}},\underline{X}$ (cf. Introduction of Chap. I). The arguments are given parallely to $\underline{\tilde{X}}\in\underline{An}_0$ and $\underline{X}\in\underline{An}_{1a}$. For notational simplification, we set:

$(1.9)_0'$ $(\underline{X}^*,X^*,P_0^*):=(\underline{\tilde{X}},\tilde{X},P_0')$ or $(\underline{X}',X,P_0)$ (cf $(1.7)_0'$, $(1.8)_0'$), and $D_{0,\underline{X}^*}:=U_0'$ or D.

We construct the p.g. cochain collection in the following three steps.

First we define a parametrization of open submanifolds of $\underline{X}^*$ by:

$$(1.9)_1 \quad v_{\underline{X}^*}:\mathcal{V}_{\underline{X}^*}:=D_{0,\underline{X}^*}\times\underline{R}^+\ni\nu=\begin{Bmatrix}(P';r)\\(P;r)\end{Bmatrix}\to\begin{Bmatrix}\underline{\tilde{X}}_r(P):=\underline{C}^n\times U_r(P')\\ \underline{X}_r(P):=U_r(P)\cap\underline{X}\end{Bmatrix},$$

where $U_r(P')$, $U_r(P)$ are the discs in $C^{n'}$, C^n of center P', P and radius r, and we set:*)

$(1.9)_1'$ $\quad \mathrm{Ouv}'(\underline{X}^*)_{p.g.}:=v_{\underline{X}^*}(\mathcal{V}_{\underline{X}^*})$

Next setting $\mathcal{U}_{\underline{X}^*}:=\mathcal{V}_{\underline{X}^*}\times\underline{R}_1^{+2}$, we define a parametrization of elements of $\mathrm{Cov}_0(\underline{X}^*)$ by:

$(1.9)_2$ $u_{\underline{X}^*}:\mathcal{U}_{\underline{X}^*}=\mathcal{V}_{\underline{X}^*}\times\underline{R}_1^{+2}\ni(\nu;\sigma)\longrightarrow\underline{A}_\sigma(\underline{X}_r^*(P^*))$, where $\underline{X}_r^*(P^*)=\underline{\tilde{X}}_r(P')$ or $\underline{X}_r(P)$ as in $(1.9)_1$, and

$(1.9)_2'$ $\underline{A}_\sigma(\underline{X}_r^*(P^*)):=g_{\underline{X}^*}$-p.g. covering of $\underline{X}_r^*(P^*)$ of size σ in $\underline{X}^*$ (cf. $(1.6)_1$).

We then set:

$(1.9)_2''$ $\quad \mathrm{Cov}_0(\underline{X}^*)_{p.g}:=u_{\underline{X}^*}(\mathcal{U}_{\underline{X}^*})$

Thirdly, taking an element $\underset{\sim}{H}\in\mathrm{Coh}(\underline{X}^*)_{p.g}$ (Def. 1.5), we define a parametrization of sets of 'p.g. cochains with value in $\underset{\sim}{H}$' by:

---

*) Note that an manifold $\underline{X}_r(P^*)$ in $\mathrm{Ouv}'(\underline{X}^*)_{p\,g}$ does not share particular p.g. properties. However, the suffix 'p.g' just above makes the notation concordant to the ones in $(1.9)_2$, $(1.9)_3$.

$(1.9)_3$ $\quad C^q_{\underset{\sim}{H}}: \lambda_{\underline{X}^*} = \mu_{\underline{X}^*} \times \underline{R}_1^{+2} \ni \lambda = (\mu;\partial) \longrightarrow C^q(\underline{A}_\sigma(\underline{X}^*_r(P^*)), \underset{\sim}{H})_\partial$:=set of all ∂-growth cochains with value in $\underset{\sim}{H}$(cf.$(1.4)_7$),

and we set:

$(1.9)'_3$ $\quad C^q(\underline{X}^*, \underset{\sim}{H})_{p.g} := C^q_{\underset{\sim}{H}}(\lambda_{\underline{X}^*})$.

We define a parametrization $Z^q_{\underset{\sim}{H}}$ by changing 'C^q' to 'Z^q'.

P.g. parametrization table

$$\lambda_{\underline{X}^*} = \mu_{\underline{X}^*} \times \underline{R}_1^{+2} \ni \lambda = (\mu;\partial) \xrightarrow{C^q_{\underset{\sim}{H}}} C^q(\underline{X}^*, \underset{\sim}{H})_{p.g.} \ni C^q(\underline{A}_\sigma(\underline{X}^*_r(P^*)), \underset{\sim}{H})_\partial ,$$

$$\mu_{\underline{X}^*} = \nu_{\underline{X}^*} \times \underline{R}_1^{+2} \ni \mu = (\nu;\sigma) \xrightarrow{u_{\underline{X}^*}} Cov_0(\underline{X}^*)_{p.g.} \ni \underline{A}_\sigma(\underline{X}^*_r(P^*))_\partial ,$$

$$\nu_{\underline{X}^*} = D_{0,\underline{X}^*} \times \underline{R}^+ \ni \nu = (P^*;r) \xrightarrow{v_{\underline{X}^*}} Ouv'(\underline{X}^*)_{p.g.} \ni \underline{X}^*_r(P^*) .$$

The manifolds , their coverings and the sets of the cochains in our p.g. uniform estimations for $\underline{X}^*$ will be taken from $Ouv'(\underline{X}^*)_{p.g}$, $Cov_0(\underline{X}^*)_{p.g}$ and $C^q(\underline{X}^*, \underset{\sim}{H})_{p.g}$. The last collection contains all necessary data in our p.g. uniform estimations . In order to emphasize the role of such a collection in our uniform estimation, we make:

Definition 1.8. We call $C^q(\underline{X}^*, \underset{\sim}{H})_{p.g}$ q-th p.g. cochain collection for $\underset{\sim}{H}$.

3 . Estimation data . In the uniform estimations in n.4, we use the p.g.c. estimation map $E \in \underline{E}'_{p.g}$ and itd first part $E' \in \underline{E}'_{p.g}$(Def.1.7). Next note that our uniform estimation does not work for all elements of $\lambda_{\underline{X}^*}$(cf.$(1.9)_3$) but for elements of a suitable subset of $\lambda_{\underline{X}^*}$. More precisely, take subsets $U'_1(\ni P'_0)$ of U'_0 and $U_1(\ni P_0)$ of U_0(cf $(1.7)_0$, $(1.8)_0$), and we set $D_{1,\underline{X}^*} = U'_1$ or $(D \cap U_1)$

Fig.I.

$$\begin{tikzcd}
(R^+ \times R^{+2} \times R^{+2}) \arrow[r] \arrow[d, "E"] & (R^+ \times R^+) \arrow[d, "E'"] \\
(R^+ \times R^{+2} \times R^{+2}) \arrow[r] & (R^+ \times R^{+2})
\end{tikzcd}$$

$(\subset D_{0,\underline{X}^*})$ Also taking elements $\tilde{r}=\tilde{r}_{\underline{X}^*}\in \underline{R}^+$ and $\tilde{\sigma}=\tilde{\sigma}_{\underline{X}^*}\in \underline{R}_1^{+2}$, we set:*)

$(1.9)_4$ $\mathcal{V}'_{\underline{X}^*} := D_{1,\underline{X}^*}\times(0,\tilde{r})$, $\mathcal{M}'_{\underline{X}^*} := \mathcal{V}'_{\underline{X}^*}\times \underline{R}^{+2}_{\tilde{\sigma}}$ and $\lambda'_{\underline{X}^*}=\mathcal{M}'_{\underline{X}^*}\times \underline{R}_1^{+2}$.

We fix this restricted parameter=space in the remainder of §1 . Our p.g. uniform estimations for X^* in §1 will work for all elements of $\lambda'_{\underline{X}^*}$.

4. Main results . Now, using the sets of the coverings and the cochains in $(1.9)_{1\sim 3}$ and the estimation maps as in n.3, we will give our main results on the p g uniform estimations for $\underline{X}^*=\underset{(\in \underline{An}_0)}{\tilde{\underline{X}}}$ or $\underline{X}\in \underline{An}_{1a}$.

First we will be concerned with Cech coboundary operator:

Theorem 1.1. (P.g uniform estimation for Cech operator $\delta=\delta_{\underline{X}^*}$) .

There is a map $\mathcal{E}_\delta : \mathrm{Coh}(\underline{X}^*)_{p.g} \ni \underset{\sim}{H} \rightarrow \underline{E}_{p.g}\ni E_{\underset{\sim}{H}}\,(q>0)$, with which we have:

$(1.10)_1$ $s^*Z^q(\underline{A}_\sigma(\underline{X}^*_r(P^*),\underset{\sim}{H})_\partial \subset \delta C^{q-1}(\underline{A}_{\sigma'}(\underline{X}^*_{r'}(P^*)),\underset{\sim}{H})_{\partial'}$, with $(r';\sigma';\partial')=E_{\underset{\sim}{H}}(r;\sigma;\partial$

Here the parameter $(P;r;\sigma;\partial)$ is in $\Lambda'_{\underline{X}^*}(\subset D_{1,\underline{X}^*}\times\underline{R}^+\times\underline{R}_1^{+2}\times\underline{R}_1^{+2})$. Moreover, s=p.g. refining map**): $\underline{A}_{\sigma'}(\underline{X}^*_{r'}(P^*))\hookrightarrow \underline{A}_\sigma(\underline{X}^*_r(P^*))$(cf. Def. 1.$6_2$).

---

*) $\underline{R}^{+2}_{\tilde{\sigma}} := \{\sigma\in\underline{R}^{+2};\ \sigma\geqq\tilde{\sigma}\}$ (cf. the end of Introduction of Chap. I).

**) For the p.g. refining maps , see n.4, §1.1. In the later arguments when there is no fear of confusions, we use the symbol 's' for the p. g. refining map in question (without mentioning it) .

Next we will*) be concerned with the rrsolution of $\underset{\sim}{H}\in\mathrm{Coh}(\underline{X}^*)_{p.g}$(cf.also $(1.4)_2$):

Theorem 1.2. (P.g.uniform estimation for resolution).

There is a map $\mathcal{E}_{\underline{X}^*}:\mathrm{Coh}(\underline{X}^*)_{p.g}\ni\underset{\sim}{H}\to\underline{E}_{p.g}\ni E_{\underset{\sim}{H}}(q\geqq 0)$, with which we hav
$(1.10)_2$ $s^*Z^q(\underline{A}_{\sigma}(\underline{X}^*_r(\overset{*}{P})),\underset{\sim}{H})_{\partial}\subset\omega_{\underset{\sim}{H}}Z^q(\underline{A}_{\sigma'}(\underline{X}^*_{r'}(\overset{*}{P})),\underline{O}^k_{\underline{X}^*})_{\partial'}$,with $(r';\sigma';\partial')=E_{\underset{\sim}{H}}(r;\sigma;\partial)$, where the parameter $(P;r;\sigma;\partial)$ is as in Th.1.1. Moreover, $\omega_{\underset{\sim}{H}}:\underline{O}^k_{\underline{X}^*}\to\underset{\sim}{H}$ is the first resolution part of $\underset{\sim}{H}$(Def.1.5).

For the proof of Th.1.1,Th.1.2, see §1.3. Also we give applications of Th.1.1,Th.1.2 in n.6, §1.2 and in §3. Here we add the following

Corolally 1.1. There is a map $\mathcal{E}'_{\delta}:\underline{Z}^+\to\underline{E}'_{p.g}$, which satisfies the factorization in Fig.II.(In Fig.II 'lg' denotes the length map(cf.$(1.4)_2$), and the projection π is as in Fig.I,n.6, §1.1).

Fig.II

$$\begin{tikzcd}
\mathrm{Coh}(\underline{X}^*)_{p.g} \arrow[r, "\mathcal{E}_{\delta}"] \arrow[d, "\mathrm{lg}"] & \underline{E}_{p.g} \arrow[d, "\pi"] \\
\underline{Z}^+ \arrow[r] & \underline{E}_{p.g}
\end{tikzcd}$$

The similar factorization to Fig.II holds slso for the map $\mathcal{E}_{\underline{X}^*}$in Th.1.2. (Cor.1.1 is not a consequence of Th.1.1. But the proof of the latter will also insures the former;see §1.3.)

Now, in an accordance to the parametrization table in n.2, we rewrit Th.1.1 in the following diagram:

---

*) Strictly, the map $\mathcal{E}_{\delta}$ in Th.1.1 depends not only operator δ but also the cohomology degree q . But the influence of the degree q in determining the map $\mathcal{E}_{\delta}$ is quite small(cf. § 1.3). Here and in the remaind of this paper, we use the phrase 'there is a map $\mathcal{E}_{\delta}$'(q>0) ,..., as a syno for 'there is a map $\mathcal{E}_{\delta,q}$(q>0)',.... The similar remarks also holds for th map $\mathcal{E}_{\underline{X}^*}$ in Th.1.2.(See also Remark 1.2_1 at the end of n.1 on the depen--dence of the maps $\mathcal{E}_{\delta},\mathcal{E}_{\underline{X}^*}$,.. on the other geometric data like $C^*(\underline{X}^*,\underset{\sim}{H})_{p.}$,...)

Fig.III. P.g.uniform estimation for Čech operator

$$\begin{tikzcd}
\Lambda_{\underline{X}^*} \ni \lambda \arrow[rr, "Z^q_{\underset{\sim}{H}}"] \arrow[dr, "E_{\underset{\sim}{H}}"] \arrow[dd] & & Z^q(\underline{X}^*,\underset{\sim}{H})_{p.g} \ni Z^q_{\underset{\sim}{H}}(\lambda) \arrow[dd, dashed] \arrow[drrr, "s^*"] & & & \\
& \Lambda_{\underline{X}^*} \ni E_{\underset{\sim}{H}}(\lambda) \arrow[rr, "C^{q-1}_{\underset{\sim}{H}}"] \arrow[dd, "E'_H"] & & C^{q-1}(\underline{X}^*,\underset{\sim}{H})_{p.g} \arrow[r, "\delta"] \arrow[dd] & \delta C^{q-1}(E_{\underset{\sim}{H}}(\lambda)) \arrow[r, phantom, "\supset"] & s^* Z^q_{\underset{\sim}{H}}(\lambda) \\
\mathcal{M}_{\underline{X}^*} \arrow[rr, "u_{\underline{X}^*}"] \arrow[dr] & & \mathrm{Cov}_0(\underline{X}^*)_{p.g} \arrow[dr, "s"] & & & \\
& \widetilde{\mathcal{M}}_{\underline{X}^*} \arrow[rr, "u_{\underline{X}^*}"] & & \mathrm{Cov}_0(\underline{X}^*)_{p.g} & &
\end{tikzcd}$$

(The similar diagram also holds for Th.1.2.) Next we assume that the variety $\underline{X}^*$ in Th.1.1, Th.1.2 is in*) $\underline{An}_{1a}$: $\underline{X}^*=\underline{X}\in\underline{An}_{1a}$ (cf.$(1.8)_0$), and take a point $P\in D_{1,\underline{X}}=(U_1\cap D)$. We then set $\widetilde{\mathcal{M}}_{\underline{X}}:=(0,\hat{r})\times\underline{R}^{+2}_1$ (cf.$(1.9)_4$). For an element $\mu=(r;\sigma)\in\widetilde{\mathcal{M}}_{\underline{X}}(=(0,\hat{r})\times\underline{R}^{+2}_1)$, we write the p.g.covering $\underline{A}_\sigma(\underline{X}_r(P))$ (cf.$(1.9)_2$) as $\underline{A}_\mu(P)$. Moreover, we set:

$(1.10)_3$ $C^q(\underline{A}_\mu(P),\underset{\sim}{H})_{p.g}:=\bigcup^{P}_{\alpha\in\underline{R}^{+2}_1} C^q(\underline{A}_\mu(P),\underset{\sim}{H})_\alpha$ (=$g_{\underline{X}}$-p.g.subgroup**) of $C^q(\underline{A}_\mu,\underset{\sim}{H})$)

Then, from the explicit formulations of Th.1.1, Th.1.2 and from the factorization in Fig.I, n.3, § 1.2 (of the p.g.c.maps $E_{\underset{\sim}{H}}\in\underline{E}_{p.g}$), we easily have:

Corollarly 1.2. We have the inclusions:

$$(1.10)_4\quad\begin{cases} s^*Z^q(\underline{A}_\mu(P),\underset{\sim}{H})_{p.g}\subset\delta C^{q-1}(\underline{A}_{\mu'}(P),\underset{\sim}{H})_{p.g}(q\geqq 1),\\ s^*Z^q(\underline{A}_\mu(P),\underset{\sim}{H})_{p.g}\subset\omega_{\underset{\sim}{H}}Z^q(\underline{A}_{\mu'}(P),\underset{\sim}{H})_{p.g}(q\geqq 0),\end{cases}$$

where μ' is a suitable element of $\widetilde{\mathcal{M}}_{\underline{X}}$.

Cor.1.2 is given in terms of the p.g.subgroups as in $(1.10)_4$, and may be more suitable for geometric applications than Th.1.1, Th.1.2, where we used the sets of the cochains, $C^q(\underline{A}_\mu(P),\underset{\sim}{H})_\alpha$, ... We use Cor.1.2 in n.6.

---

*) The similar fact to Cor.L.2 holds also for $\underline{X}\in\underline{An}_0$. But we do not use such a fact (cf.n.6, §1.2).

**) cf.$(1.4)_7$.

Here we make a remark on the explicit estimation in Th.1.1*).

Remark 1.2_1. As may be clear from its formulation, the estimation map $E_H \in E_{p.g}$ is taken independently from the point P^*, which is the origin of the manifold $X^*_r(P)$ in question When the variety $X^*=X\in An_{1a}$, the divisor $D_{1,X^*}(\ni P^*)$ has, in general, singularieites , and the above independence is never of obvious nature. As we will see in §1.3, § 4.2 and in § 5.1, this independence is insured by certain uniform estimations on Weierstrass polynomials and the coherency theorem of K. Oka (or, more precisely, the structure of the proof of his theorem) . From its formulatio we may regard that the coherency theorem insures a uniform structure of the coherent sheaves with respect to the points on the analytic varieties. The independence mentioned just above plays a very basic role in our treatments of the cohomology theoires in this paper(cf § 2,§ 3) . As in the case of theories of cohernet sheaves, where no explicit estimations are involved, the coherency theorem of K Oka will play the basic role in our cohomology theory in this paper . Next Cor.1.1 and Cor.1,3 concern a type of uniform estimations with respect ot the p.g.sheaves on the analytic varieties in question . Though we do not use those results in this paper, the factorizations in Cor.1.1 and Cor 1.3 may be useful, when ome concerns a family of p.g. coherent sheaves .

Remark 1.2_2. The remark here is of technical nature for the proof of Th.1.1 . Letting X^* be an varietyy in An_{1a} or An_0, we use the phrase 'Th.1.1 holds for O_{X^*}' as a synonym for thst $(1.10)_1$ holds for O_{X^*}, with a suitabl p.g.c map E_{X^*} .Next, letting C be a collection of p.g. coherent sheaves over X^*, we use the phrase 'Th.1.1 holds for C' as a synonym for '$(1.10)_1$ holds for each $H \in C$' (by changing Coh (X^*) in Th.1.1 by C) . When we use this terminology, we assume that the factorization in Cor.1.1 holds for C . We use the similar terminplogy for Th.1.2 and Th.1.3, Th.1.4 .

---

*) The similar remark also holds for Th.1.1.

5. An affine analogue. Here we give an analogue of the results in n.4 to affine varieties. The content here is chiefly given for purpose of geometric application(cf.n.6 soon below and § 3). We do not give corresponding explicit estimations to Th.1.1,Th.1.2 to affine varieties*). Our results here will given in a similar form to Cor.1.2. In order to formulate such results, we first mean by a smooth imbedded affine variety a datum $\underline{X}'$ as follows:

(1.11)$_0$ $\underline{X}'=(\underline{C}^n(z),X',\underset{\sim}{H}_{X'})$, where X' is a smooth affine variety in a euclidean space $\underline{C}^n(z)$(cf.n.1, § 1.2) and $\underset{\sim}{H}_{X'}$ is a $(|z|+1)$-p.g.resolution of the structure sheaf $\underset{\sim}{O}_{X'}$ of X' over $\underline{C}^n$.

We then set:

(1.11)$_0$ Aff:=collection of all smooth imbedded affine varieties.

Letting an element $\underline{X}'\in$ Aff be of the form in (1.11)$_0$, the underlying analytic variety is the affine variety X'. In this paper, unless we say otherwise, we regard**) $(X',\underset{\sim}{O}_{X'})$ as the analytic variety. When we regard it as the algebraic variety, we write it as $(X'_{alg},\underset{\sim}{O}_{X',alg})$, where the underlying topology is that of Zariski. The p.g.and distance functions for X' will be:$g_{\underline{X}'} = |z| + 1$, $d_{\underline{X}'}$:= induced distance from the natural one d_z of $\underline{C}^n(z)$. We write the p.g.pair $(X',g_{\underline{X}'})$ and triple $(X',g_{\underline{X}'},d_{\underline{X}'})$ also as $\underline{X}'$(cf.also n.1,§ 1.2).

Next setting $\mu_{\underline{X}'}:=\underline{R}_1^{+2}$, our coverings will be taken from the family**):

(1.11)$_1$ $\mathrm{Cov}_0(\underline{X}')_{p.g}:=\{\underline{A}_\sigma(\underline{X}');\sigma\in\mu_{\underline{X}'}\}$, where $\underline{A}_\sigma(\underline{X}')$:=$g_{\underline{X}'}$-p.g.covering of $\underline{X}'$ of size σ in $\underline{X}'$(Def.1.6$_1$).

Taking an element $\underset{\sim}{H}\in\mathrm{Coh}(\underline{X}')_{p.g}$, our underlying datum for the p.g.

---

*) For the explicit estimations for Th 1.3, Th 1.4, see §1.3 and §4.2, where the proof of these theorsms is given .

**) As in n.3, we use the symbol $\underline{X}'$ also for its underlying variety X'.

cohomology(given to $\underset{\sim}{H}'$) will be the following p.g.group:

$(1.11)_2$ $C^q(\underline{A}_\sigma(\underline{X}'),\underset{\sim}{H}')_{p.g}:=g_{\underline{X}'}$-subgroup of $C^q(\underline{A}_\sigma(\underline{X}'),\underset{\sim}{H}')_{p.g}$(Def.1.3 and $(1.4)_\eta$).

Thirdly we will use el-maps $\underline{L}:\underline{R}^{+2}\longrightarrow\underline{R}^{+2}$(cf.n.5, 1.1) in our estimations soon below. We set:

$(1.11)_3$ $\underline{L}=$ collection of all el-maps.

For each $q\in\underline{Z}^+\cup 0$ we fix a restricted parameter space $\mu_{\underline{X}'}=\underline{R}^{+2}_{\tilde{\sigma}}$, with an element $\tilde{\sigma}=\tilde{\sigma}_{\underline{X}'}\in\underline{R}^{+2}_1$; our uniform estimations will work for $\mu_{\underline{X}'}$(cf. also n.3, § 1.2). Now we give an analogue of Th.1.1,Th.1.2 to $\underline{X}'$ in the following fashion:

Theorem 1.3.(P.g.uniform estimation for Cech operator $\delta=\delta_{\underline{X}'}$).

There is a map $\mathcal{E}_\delta:\mathrm{Coh}(\underline{X}')_{p.g}\ni\underset{\sim}{H}'\longrightarrow\underline{L}\ni\underline{L}_{\underset{\sim}{H}'}(q>0)$, with which we have:

$(1.12)_1$ $s^*Z^q(\underline{A}_\sigma(\underline{X}'),\underset{\sim}{H}')_{p.g}\subset\delta C^{q-1}(\underline{A}_{\sigma'}(\underline{X}'),\underset{\sim}{H}')_{p.g}$, with $\sigma'=\underline{L}_{\underset{\sim}{H}'}(\sigma)$.

Theorem 1.4.(P.g.uniform estimation for resolution).

There is a map $\mathcal{E}_{\underline{X}'}:\mathrm{Coh}(\underline{X}')_{p.g}\ni\underset{\sim}{H}'\longrightarrow\underline{L}\ni\underline{L}_{\underset{\sim}{H}}(q\geqq 0)$, with which we have:

$(1.12)_2$ $s^*Z^q(\underline{A}_\sigma(\underline{X}'),\underset{\sim}{H}')_{p.g}\subset\omega_{\underset{\sim}{H}'}Z^q(\underline{A}_{\sigma'}(\underline{X}'),\underset{\sim}{H}')_{p.g}$, with $\sigma'=\underline{L}_{\underset{\sim}{H}'}(\sigma)$.

where $\omega_{\underset{\sim}{H}'}:\underset{\sim}{O}^k_{X'}\longrightarrow\underset{\sim}{H}'$ is the first resolution of $\underset{\sim}{H}$(cf.Def.1.5).

(In the above σ' is in the restricted parameter space $\mu'_{\underline{X}'}$.) Also, corresponding to Cor.1.2, we have:

Corollary 1.3. We have the following factorization:

$(1.12)_3$
$$\begin{tikzcd}
\mathrm{Coh}(\underline{X}')_{p.g} \arrow[d, "lg\ \circlearrowleft"] \arrow[dr, "\tilde{\mathcal{E}}"] & \\
\underline{Z}^+ \arrow[r] & \underline{L}
\end{tikzcd}$$

(Here 'lg'=length map(cf.$(1.4)_2$), and $\tilde{\mathcal{E}}=\mathcal{E}_\delta$ or $\mathcal{E}_{\underline{X}'}$.)

The proof of the above results is given in §1.3(cf.also § 4).

6. P.g.complexes. Here we give our analogue of Th.A,B of H.Cartan to the p.g.cohomology theory. For this we fix the following data as in Th.1.1 ∼ Th.1.4(cf.also Cor.1.2):

$$(1.13)_0 \quad \left\{ \begin{array}{l} \text{the local analytic variety } \underline{X} \in \underline{An}_{1a}, \text{ the point } P \in D, \\ \text{and the parameter space } \mathcal{M}_{\underline{X}} = (0, \tilde{r}) \times \underline{R}^{+2}_{\tilde{\sigma}}, \\ \text{the affine variety } \underline{X}' \text{ and the parameter space } \underline{\mathcal{M}_{X'}}; \end{array} \right.$$

and we set:

$(1.13)_1$ $(\underline{X}^*, \mathcal{M}_{\underline{X}^*}) = (\underline{X}, \mathcal{M}_{\underline{X}})$ or $(\underline{X}', \underline{\mathcal{M}_{X'}})$, and $\underline{A}_{\mu} = \underline{A}_{\sigma}(\underline{X}_r(P))$ or $\underline{A}_{\sigma}(\underline{X}')$ for each $\mu = (r;\sigma) \in \mathcal{M}_{\underline{X}} = (0, \tilde{r}) \times \underline{R}^{+2}_{\tilde{\sigma}}$ or $= \sigma \in \mathcal{M}_{\underline{X}'} = \underline{R}^{+2}_{\tilde{\sigma}}$,

where the point $P \in D$ is as in Cor.1.2.

Letting the p.g.sheaf $\underset{\sim}{H}^* = \underset{\sim}{H}$ or $\underset{\sim}{H}'$ be as in Th.1.1, or Th.1.3, We make:

Definition 1.9_1. By p.g.Cech complex for $(\underline{A}, \underset{\sim}{H}^*)$, we mean:

$$(1.13)_2 \quad 0 \to C^0(\underline{A}_{\mu}, \underset{\sim}{H}^*)_{p.g} \to \cdots \to C^q(\underline{A}_{\mu}, \underset{\sim}{H}^*)_{p.g} \to$$

We write this complex as $C^*(\underline{A}_{\mu}, \underset{\sim}{H})_{p.g}$.

Next, assuming that $\underset{\sim}{H}^*$ is of the form in $(1.4)_1$, we call the following complex q-th p.g.resolution complex for $\underset{\sim}{H}^*$:

$$(1.13)_3 \quad 0 \to Z^q(\underline{A}_{\mu}, \underset{\sim}{O}^{k_p}_{\underline{X}^*})_{p.g} \xrightarrow{K_{p-1}} - - \xrightarrow{K_1} Z^q(\underline{A}_{\mu}, \underset{\sim}{O}^{k_1}_{\underline{X}^*})_{p.g} \xrightarrow{\omega_H} Z^q(\underline{A}_{\mu}, \underset{\sim}{H})_{p.g} \to 0$$

Thirdly, we regard $\mathcal{M}_{\underline{X}^*} = \mathcal{M}_{\underline{X}}$ or $\mathcal{M}_{\underline{X}'}$ as the ordered set in the following manner(cf.$(1.13)_0$):

$$(1.13)_4 \quad \mu = (r;\sigma) \succ \mu' = (r';\sigma') \Leftrightarrow r < r', \sigma > \sigma', \text{ and } \mu = \sigma \succ \mu' = \sigma' .$$

Then letting $\underline{X}_P$ denote the germ of $\underline{X}$ at P we make:

Definition 1.9_2. By p.g.Cech complex for $(\underline{X}_P, \underset{\sim}{H})$ or $(\underline{X}', \underset{\sim}{H}')$, we mean:

$$(1.13)_5 \quad C^*(\underline{X}_P, \underset{\sim}{H})_{p.g} := \varinjlim_{\mu} C^*(\underline{A}_{\mu}, \underset{\sim}{H})_{p.g} \quad , \quad C^*(\underline{X}', \underset{\sim}{H}')_{p.g} := \varinjlim_{\mu} C^*(\underline{A}_{\mu}, \underset{\sim}{H}')_{p.g} .$$

We define 'p.g.resolution complex for $(\underline{X}_P, \underset{\sim}{H})$ or $(\underline{X}', \underset{\sim}{H}')$' by operating the similar limit procedure to $(1.13)_5$ to the complex in $(1.13)_3$.

Writing the q-th cohomology groups of the p.g.Cech complexes in Def.1.9_3 as $H^q(X_P,H)_{p.g}$, $H^q(X',H')_{p.g}$, we have the following directly from Cor.1.2 and Th.1.3:

Theorem 1.5. $H^q(X_P,H)_{p.g} \cong 0$ and $H^q(X',H')_{p.g} \cong 0 (q>0)$.

Next applying the standard syzygy arguments to Cor.1.2 and Th.1.4, we easily have:

Lemma 1.1. The q-th p.g.resolution complex($q \geqq 0$) for (X_P,H) and (X',H') are exact(Def.1.9_2):

$$(1.13)_6 \quad 0 \rightarrow \varinjlim_{\mu} Z^q(A_\mu(P), O_X^{k_p}) \xrightarrow{K_{p-1}} \cdots \xrightarrow{K_1} \varinjlim_{\mu} Z^q(A_\mu(P), O_X^{k_1})_{p.g} \xrightarrow{\omega_H} \varinjlim_{\mu} Z^q(A_\mu(P),H)_{p.g},$$

where we set $A_\mu(P) := A_\sigma(X_r(P))$,

(and the similar exact sequence for (X',H').)

Now, in order to determine $H^0(X_P,H)_{p.g}$, $H^0(X',H')_{p.g}$, we let $\theta_P, \theta_{X'}$ denote the natural homomorphisms from the algebraic object to the analytic one:

$$(1.13)_7 \quad \theta_P : O_{X_0}(*D)_P \rightarrow H^0(X_P,O_X)_{p.g} \ , \ \theta_{X'} : \Gamma(X'_{alg}, O_{X',alg}) \rightarrow H^0(X',O_{X'})_{p.g},$$

where

$(1.13)'_7$ $O_{X_0}(*D)$:=sheaf(over X_0) of meromorphic functions with the pole D.

Then we have:

Theorem 1.6_1. The homomorphisms θ_P and $\theta_{X'}$ are isomorphic.

If $X'=C^n$ then Th.1.6_1 is a classically well known consequence of Cauchy integral formula.*) If X_0 is smooth, then we get Th.1.6_1 also easily from Hartogus theorem on removable singularities(in the codimension one case). For general X,X' we derive Th.1.6_1 from what are mentioned just abvoe (cf.n.4,§ 1.3). Finally, applying the standard syzygy arguments to Th.1.6_1 and Lemma 1.1, we easily have:**)

**) cf. also Lemma 1.2 in § 1.3,

*) cf. $(0)_1$ in the introduction of Chap.I,

arguments to Th 1.6_1, we easily have:

Theorem 1.6_2. The following complexes are exact:

$$(1.13)_8 \begin{cases} \mathcal{O}_X^{k_2} \xrightarrow{K_1} \mathcal{O}_X^{k_1} \xrightarrow{\omega_H} H^0(X_p, H)_{p.g} \longrightarrow 0, \\ \Gamma(X'_{alg}, \mathcal{O}_{X',alg}^{k_2}) \xrightarrow{K'_1} \Gamma(X'_{alg}, \mathcal{O}_{X',alg}^{k_1}) \xrightarrow{\omega_H} H^0(X', H')_{p.g} \longrightarrow 0. \end{cases}$$

Th.1.5,Th 1.6 are our p.g.analogues of Th.A,B of H.Cartan( ). Applications of Th.1.5 and Th.1.6 will be given in § 3.

Remark 1.3_1. In [ ], H. Yamaguchi showed an analogue of Th.1.5, Th.1.6 to algebraic locally free coherent sheaves over affine varieties, by using Th.1.5 and Th.1.6. Next, note that, in Th.1.1 ∼ Th.1.6, we gave a more or less categorical treatments of the p.g.uniform estimations. At present, we lack the notion of 'p.g. maps'. In this direction, S.Kamiya ([ ]) gave some funcotial treatments of our p.g.cohomology theory. It seems to be quite desirable to give a suitable functorial generalization of our p.g.cohomology theory in §1.

Remark 1.3_2. As was mentioned,*) cohomology theories with p.g.condition were studied by P.Deligne-G.Maliotiost([11]) and by M.Corbalna-P.A.Griffiths([2]) for locally free algebraic coherent sheaves over smooth algberaic varieties. Our results for p.g.coherent sheaves over the analytic varieties as in §1.2, together with the result of H.Yamaguchi([21]), are more genral than theirs. In particular, the independence assertions mentioned in Remark 1.2 are not found in [2],[11]. Also, our proof of Th.1.1. ∼ Th.1.6 depends on p.g.estimations on the uniform estimations on homomorphisms on coherent sheaves in §1.3 an~~d on Cousin integ~~rals (cf. Chap.III), and is entirely different from in [2], [11], which use the $\bar{\partial}$- estimations.

---

*) cf.Introduction.

§1.3. A key theorem and key lemmas

Here we will reduce Th.1.1~Th.1.4 in §1.2 to a key theorem, Th.1.7, and key lemmas, Lemma 1.2~1.4. The latter is proven in Chap.III and § 4.

1. A key theorem. First setting

$(1.14)_0$ $\underline{\mathrm{Euc}}'$:= collection of all products $\underline{C}^n(z)\times\underline{C}^{n'}(z')$ of coordinated euclidean spaces(n.1, § 1.2),

we define a map(dimension map) dim: $\underline{\mathrm{Euc}}'\ni\underline{\tilde{X}}=\underline{C}^n(z)\times\underline{C}^{n'}(z')\to\underline{Z}^+\times\underline{Z}^+\ni(n,n')$. The p.g. and distance functions for $\underline{\tilde{X}}$ are: $g_{\underline{\tilde{X}}}=|\tilde{z}|+1$, with $\tilde{z}=(z,z')$, and $d_{\underline{\tilde{X}}}:=d_{\tilde{z}}$(=natural distance of $\underline{\tilde{X}}$)(cf.n.1, § 1.2). Taking an element $(P;r,\sigma;\partial)$ $\in\Lambda_{\underline{\tilde{X}}}:=\underline{C}^{n'}\times(0,1]\times\underline{R}_1^{+2}\times\underline{R}_1^{+2}$, the set of the cochains in Th.1.7 soon below is as follows:

$(1.14)_1'$ $C^q(\underline{A}_\sigma(\underline{\tilde{X}}_r(P')),\mathcal{O}_{\underline{\tilde{X}}})_\partial$:=set of all $g_{\underline{\tilde{X}}}$-$\partial$-growth cochains with value in the structure sheaf $\mathcal{O}_{\underline{\tilde{X}}}$ of $\underline{\tilde{X}}$(cf.$(1.4)_8$), where

$(1.14)_2'$ $\underline{A}_\sigma(\underline{\tilde{X}}_r(P'))$:= $g_{\underline{\tilde{X}}}$ -p.g.covering of $\underline{\tilde{X}}_r(P'):=\underline{C}^n\times U_r(P')$ of size σ in $\underline{C}^n\times\underline{C}^n$, with*) $U_r(P'):=\{Q'\in\underline{C}^{n'}(z');d_{z'}(Q',P')<r\}$(Def.$1.6_1$).

Then the following theorem is most basic among the results in § 1.3:

Theorem 1.7. (P.g.uniform estimation for $\mathcal{O}_{\underline{\tilde{X}}};\underline{\tilde{X}}=\underline{C}^n\times\underline{C}^{n'}$).

There is a map $\mathcal{E}_\delta:\underline{\mathrm{Euc}}'\ni\underline{\tilde{X}}\to\underline{E}_{p.g}\ni E_{\underline{\tilde{X}}}(q>0)$, which is factored as in Fig.I, and with which we have the following for each $\underline{\tilde{X}}\in\underline{\mathrm{Euc}}'$:

$(1.14)_1$ $s^*Z^q(\underline{A}_\sigma(\underline{\tilde{X}}_r(P')),\mathcal{O}_{\underline{\tilde{X}}})_\partial\subset\delta C^{q-1}(\underline{A}_{\alpha'}(\underline{\tilde{X}}_{r'}(P)),\mathcal{O}_{\underline{\tilde{X}}})_{\partial'}$, with $(r',\alpha';\partial')=E_{\underline{\tilde{X}}}(r,\sigma;\partial)$, where $(P';r,\sigma;\partial)$ is in $\Lambda_{\underline{\tilde{X}}}=\underline{C}^{n'}\times\cdots\times\underline{R}_1^{+2}$.

Fig.I

$$\begin{tikzcd}
\underline{\mathrm{Euc}} \arrow[d, "\mathrm{dim}"] \arrow[drr, "\mathcal{E}_\delta"] \arrow[dr, phantom, "\circlearrowleft"] & & \\
\underline{Z}^+\times\underline{Z}^+ \arrow[rr] & & \underline{E}_{p.g}
\end{tikzcd}$$

---

*) $d_{z'}$ =natrural distance in $\underline{C}^{n'}(z')$.

As may be clear from its formulation, Th.1.7 will be most basic for the proof of Th.1.1~Th.1.6(among the p.g.uniform estimations givin in §1.3). The proof of Th.1.7 will be given in Chap.III in an independe manner$^{*)}$ from the contents of Chap.I,II. Here we derive a consequence of Th.1.7. For this we first set:

$(1.14)_3$ $\underline{\text{Euc}}$:=collection of all euclidean spaces $\underline{X}'=\underline{C}^n(z)$.

We denote by '$\dim_0$' the map:Euc$\ni\underline{X}'\longrightarrow Z^+\ni n$, and we also define a map:

$(1.14)_4$ $I':\underline{\text{Euc}}\ni\underline{X}'=\underline{C}^n(z)\longrightarrow\underline{\text{Aff}}\ni(\underline{C}^n(z),X'=\underline{C}^n,\underset{\sim}{H}_{\underline{X}'})$, where $\underset{\sim}{H}_{\underline{X}'}$ denotes the trivial resolution of $\underset{\sim}{O}_{\underline{X}'}$:$0\rightarrow\underset{\sim}{O}_{\underline{X}'}\rightarrow\underset{\sim}{O}_{\underline{X}'}\rightarrow0$(cf.Remark 1.1).

By means of I' we regard an elemenet $\underline{X}'=\underline{C}^n(z)\in\underline{\text{Euc}}$ as the elemenet of $\underline{\text{Aff}}$; we use the terminology for $\underline{\text{Aff}}$(n.5,§1.2) for $\underline{X}'$. In particular, the p.g. and distance functions for $\underline{X}'$ are:$g_{\underline{X}'}=|z|+1$ and $d_{\underline{X}'}$=natural distance of $\underline{C}^n(z)$(n.4, §1.2). Next taking an elemenet $(\sigma;\lambda)\in\Lambda_{\underline{X}'}:=\underline{R}_1^{+2}\times\underline{R}_1^{+2}$ we set:

$(1.14)'_4$ $C^q(\underline{A}_\sigma(\underline{X}'),\underset{\sim}{O}_{\underline{X}'})_\lambda$:= set of $\underset{g_{\underline{X}'}}{\lambda}$-growth cochains with value in $\underset{\sim}{H}$, where $\underline{A}_\sigma(\underline{X}')$:=$g_{\underline{X}'}$-p.g.covering of $\underline{X}'$ (in $\underline{X}'$) of size σ (Def.1.6_1).

We use el-maps for the estimations in Cor.1.4 soon below. We set:

$(1.14)''_4$ $\underline{\tilde{L}}=\underline{L}\times\underline{L}$, with $\underline{L}$=collection of all el-maps(cf.$(1.11)_3$).

To an element $\underline{\tilde{L}}=(\underline{L}_1,\underline{L}_2)\in\underline{\tilde{L}}=\underline{L}\times\underline{L}$ we attach a map:

$(1.14)'''_4$ $\underline{\tilde{L}}:\underline{R}^{+2}\times\underline{R}^{+2}\ni(\sigma;\lambda)\longrightarrow\underline{R}^{+2}\times\underline{R}^{+2}\ni(\underline{L}_1(\sigma),\underline{L}_2(\lambda+\sigma))$.

We then have:

Corollary 1.4 . There is a map $\mathcal{E}:\mathrm{Euc}\ni X'=C^n(z) \to \tilde{L}\ni L_{X'}(q>0)$, which satisfies Fig I, and with which we have the following for each $X'\in\mathrm{Euc}$:

Fig, II

$$\begin{tikzcd}
\mathrm{Euc} \arrow[d, "\dim_0"'] \arrow[dr, "\mathcal{E}"] & \\
Z^+ \arrow[r] & \tilde{L}
\end{tikzcd}$$

$$(1.14)_5 \quad s^*Z^q(A_\sigma(X'),O_{X'})_\partial \subset \delta\, C^{q-1}(A_{\sigma'}(X'),O_{X'})_{\partial'} ,$$

with $(\sigma';\partial')=L_{X'}(\sigma;\partial)$, where $(\sigma;\partial)$ is in $\Lambda_{X'}(=R_1^{+2}\times R_1^{+2})$.

Proof. Letting U_1' denote the disc in the euclid line $C(w)$ with the center 0(=origin of C) and radus r=1, we identify C^n with $C^n\times 0\subset C^n\times U_1'$. We write the projection: $C^n\times U_1'\to C^n$ as $\pi_{X'}$. Then, writing the left side of $(1.14)_5$ as Z^q, we have: $\pi_X^* Z^q\subset Z^q(A_\sigma(C^n\times U_1'),0)_\partial$, where 0:=structure sheaf of $C^n\times U_1'$.

Apply Th 1.7 to the right

side of the inclusion just above, and we restrict the resulting inclusion of the form in $(1.14)_1$ to $\underline{C}^n \times 0'(\simeq \underline{C}^n)$. Then, comparing the explicit estimations in Th.1.7 and Cor.1.4(cf.also the explicit forms of the p.g.c.maps and el-maps in n.5, §1.1), we get easily $(1.14)_5$. q.e.d.

Comparing $(1.14)_5$ with $(1.12)_1$,Th.1.3, we easily have the implication

<u>Corollary 1.4'</u>. <u>Th.1.7 $\longrightarrow$ Th.1.3 for each*) $\underset{\sim}{O}_{\underline{X}'}$; $\underline{X}' \in \underline{Euc}$.</u>

The right side will be our starting point of the proof of Th.1.3,Th.1.4 (cf.n.2 $\sim$ n.5 soon below).

2. <u>Sheaf homomorphisms</u>. Letting $\underline{X}^*$ be one of $\tilde{\underline{X}} \in \underline{An}_0$, $\underline{X} \in \underline{An}_{1a}$ or $\underline{X}' \in \underline{Aff}$ as in Th.1.2 or Th.1.4, we will give here a lemma on the title of n.2, which will be most basic in deriving Th.1.1$\sim$Th.1.4 from Th.1.7. For this taking an element $\underset{\sim}{H} \in \mathrm{Coh}'(\underline{X}^*)_{p.g}$(cf.$(1.4)_8$), we recall that such a sheaf $\underset{\sim}{H}$ is endowed with <u>two 'natural' p.g.filtrations</u>: the first one, Ψ , is induced from the (first) resolution $K: \underset{\sim}{O}_{\underline{X}^*}^{k} \longrightarrow \underset{\sim}{H}$ of $\underset{\sim}{H}$, where K is a matrix with entries in $\Gamma(\underline{X}^*, \underset{\sim}{O}_{\underline{X}^*}; g_{\underline{X}^*})_{p.g}$(cf.$(1.4)_8$)**), and has been used hitherto in §1. The second one, Ψ' in symbol, is induced from the inclusion: $\underset{\sim}{H} \hookrightarrow \underset{\sim}{O}_{\underline{X}^*}^{\tilde{k}}$, $\tilde{k}$=length of colums of K,and letting the parameter $(P;r,\sigma;\alpha)$ or $(\sigma;\alpha) \in \Lambda'_{\underline{X}^*}$ have***) the similar meaning to Th.1.2 or Th.1.4, the set of α-growth cochains defined by Ψ' is as follows(cf.$(1.4)_9$):

$$(1.15)'_1 \quad C^q(\underline{A}_\sigma(\underline{Y}^*), \underset{\sim}{H}; \Psi')_\alpha := C^q(\underline{A}_\sigma(\underline{Y}^*), \underset{\sim}{H}) \cap C^q(\underline{A}_\sigma(\underline{Y}^*), \underset{\sim}{O}_{\underline{X}^*}^{\tilde{k}})_\alpha \ .$$

(Here $\underline{Y}^*$ denotes the manifold $\underline{X}_r^*(P^*)$ or $\underline{X}'$ as in Th.1.2,Th.1.4.)

We write the corresponding set to Ψ explicitly as follows(cf.$(1.4)_8$):

$$(1.15)'_2 \quad C^q(\underline{A}_\sigma(\underline{Y}^*), \underset{\sim}{H}; \Psi)_\alpha := KC^q(\underline{A}_\sigma(\underline{Y}^*), \underset{\sim}{O}_{\underline{X}^*}^{k})_\alpha \ .$$

Now we give the key lemma, mentioned soon above, in terms of a <u>comparsion of the sets of the cochains in $(1.15)'_{1,2}$</u>:

---

*) cf.Remark 1.2_2.

**) $g_{\underline{X}^*}$ =p.g.function for $\underline{X}^*$: $=|\tilde{z}|+1$, $|h_{\underline{X}}^{-1}|$ or $|z|+1$(cf.Th.1.2,1.4).

***) For the sets of the parameter $\Lambda'_{\underline{X}^*}$, see n.2 and n.5, §1.2.

Lemma 1.2. (P.g. uniform estimations for $O_{\underset{\sim}{X}^*}$-homomorphisms)*)

There are maps $\mathcal{E}_{\underset{\sim}{X}^*}:\mathrm{Coh}'(\underline{X}^*)_{p.g}\ni \underset{\sim}{H}\rightarrow\underline{\tilde{L}}\ni\underline{\tilde{L}}_{\underset{\sim}{H}}$ and $\mathcal{E}'_{\underline{X}^*}:\underline{Z}^+\longrightarrow\underline{L}$, which is factored in Fig.III, and with which we have:

Fig.III

$$\begin{tikzcd} \mathrm{Coh}'(\underline{X}^*)_{p.g} \arrow[r, "\mathcal{E}_{X^*}"] \arrow[d, "lg"] & \underline{\tilde{L}} \arrow[d] \\ \underline{Z}^+ \arrow[r] & \underline{L} \end{tikzcd}$$

$(1.15)_1$ $s^*C^q(\underline{A}_{\sigma}(\underline{Y}^*),\underset{\sim}{H};\Psi')_{\delta}\subset C^q(\underline{A}_{\sigma'}(\underline{Y}^*),\underset{\sim}{H};\Psi)_{\delta'}$, with $(\sigma';\delta')=\underline{\tilde{L}}_{\underset{\sim}{H}}(\sigma;\delta)$, where the parameter $(\sigma;\delta)$ is as in $(1.15)'_1$.

For the proof of Lemma 1.2, see §4. (See also Remark 1.4 at the end of §1.3

3. Consequences of Lemma 1.2. First we prove the implication:

Corollary 1.5. Th.1.7 + Lemma 1.2 $\longrightarrow$ Th.1.2, Th.1.1 for $\underline{An}_0$.

Proof. Take an element $\underline{\tilde{X}}\in\underline{\ddot{An}}_0$. Then, applying Th.1.7 to the right side of $(1.11)_2$, Th.1.2, we easily have:

$(1.15)_2$ Th.1.2 for $\underline{An}_0\longrightarrow$ Th.1.1 for $\underline{An}_0$ (cf. n.4, § 1.2).

We prove the left side inductively on $\mathrm{Coh}^p(\underline{\tilde{X}})_{p.g}$ $(p=1,2,..)$ (cf. $(1.4)_2$), using the standard syzygy arguments: if $p=1$ then $\underset{\sim}{H}\cong O^k_{\underset{\sim}{\tilde{X}}}(k>0)$, and we have directly Th.1.2 from Th.1.7 (cf. §1.2). Assume that (1) $p\geqq 2$, (2) Th.1.2 holds**) for $\mathrm{Coh}^{p-1}(\underline{\tilde{X}})_{p.g}$ and (3) $\underset{\sim}{H}\in\mathrm{Coh}^p(\underline{\tilde{X}})_{p.g}$. Writing $\underset{\sim}{H}$ as: $\dashrightarrow O^{k'}_{\underset{\sim}{\tilde{X}}}\xrightarrow{K_1}O^k_{\underset{\sim}{\tilde{X}}}\rightarrow\underset{\sim}{H}\rightarrow 0$, we define an element $\underset{\sim}{H}_1\in\mathrm{Coh}^{p-1}(\underline{\tilde{X}})_{p.g}\cap\mathrm{Coh}'(\underline{\tilde{X}})_{p.g}$ to be: $\dashrightarrow O^k_{\underset{\sim}{\tilde{X}}}\xrightarrow{K_1}\underset{\sim}{H}_1(\subset O^k_{\underset{\sim}{\tilde{X}}})\rightarrow 0$. Now taking a parameter $(P^U;r,\sigma:\delta)\subset\Lambda'_{\underline{\tilde{X}}}(\subset U'_0\times\underline{R}^+\times\underline{R}^{+2}\times\underline{R}^+$ (cf. Th.1.2), we set:

(a) $Z'^q:=\{\varphi\in C^q(\underline{A}_{\sigma}(P;r),O^k_{\underset{\sim}{\tilde{X}}})_{\delta}\ ;w\delta\varphi=0\}$, where we set $\underline{A}_{\sigma}(P;r)=\underline{A}_{\sigma}(\underline{\tilde{X}}_r(P))$.

Then letting the p.g. filtrations Ψ'_1,Ψ_1 for $\underset{\sim}{H}_1$ have the similar meanings to Ψ,Ψ' for $\underset{\sim}{H}$ (as in Lemma 1.2), we have: $\delta Z'^q\subset C^{q+1}(\underline{A}_{\sigma}(P;r),\underset{\sim}{H}_1;\Psi'_1)_{\delta}$.

Applying Lemma 1.2 and Th.1.1***) to the right side, we get:

(b) $s^*\delta Z'^q\subset Z^{q+1}(\underline{A}_{\sigma'}(P;r),\underset{\sim}{H}_1;\Psi_1)_{\delta'}\subset\delta C^q(\underline{A}_{\sigma''}(P;r''),\underset{\sim}{H}_1;\Psi_1)_{\delta''}$, where $(\sigma';\delta')=\underline{\tilde{L}}_{\underset{\sim}{H}}(\sigma;\delta$ and $(r'',\sigma'';\delta'')=E_{\underset{\sim}{H}_1}(r,\sigma';\delta')$ are defined as in Lemma 1.2, Th.1.2.

---

*) For the sets $\underline{\tilde{L}},\underline{L}$ of estimation maps, see $(1.14)''_4$. Also the set $\underline{L}$ in Fig.III is the first component of $\underline{\tilde{L}}=\underline{L}\times\underline{L}$, and lg=length map (cf. $(1.4)_2$

**) For this terminology, see Remark 1.2_2 and Remark 1.3.

***) By the induction hypothesis we have Th.1.2 for $\mathrm{Coh}^{p-1}(\underline{\tilde{X}})_{p.g}$; by $(1.15)_2$ we have Th.1.1 for $\mathrm{Coh}^{p-1}(\underline{\tilde{X}})_{p.g}$.

41

It is clear that (b) insures:

(c) $s^*Z'^q \subset Z^q(A_{\alpha''}, O^k_{\tilde{X}})_{\alpha''} + C^q(A_{\alpha''}, H_1)_{\alpha''}$, where $A_{\alpha''} = A_{\alpha}(P; r^{\prime\prime})$.

Finally operating the homomorphism ω to the both dides of (c), we get the desired inclusion $(1.11)_1$, Th.1.2. Q.e.d.

For later qotations, we rewrite Cor.1.5 in the form:

$(1.15)_3$ Th.1.7 $\xrightarrow{\text{syzygy(Lemma 1.2)}}$ Th.1.2 for $\underline{An}_0 \xrightarrow{T}$ Th.1.1 for $\underline{An}_0$.

Next take subsets of $\underline{A}, \underline{A}'$ of $\underline{An}_{1a}$, $\underline{Aff}$(n.1,n.5, § 1.2). Then the similar syzygy arguments to Cor.1.5 insure:

Proposition 1.4. If Th 1.1 holds for each $O_{\underset{\sim}{X}}$; $\underline{X} \in \underline{A}$ (resp. Th.1.3 holds for each $O_{\underset{\sim}{X}'}$; $\underline{X}' \in \underline{A}'$), then we have Th.1.1, Th.1.2 for $\underline{A}$ (resp. Th.1.3, Th.1.4 for $\underline{A}'$).

Taking $\underline{A}'$ to be $\underline{Euc}$(n.1,§1.3), Prop.1.4 and Cor.1.4 insure:

Corollary 1.6. Th.1.3 $\xrightarrow{}$ Th.1.7 for each $O_{\underset{\sim}{X}'}$; $\underline{X}' \in \underline{Euc}$ $\xrightarrow{\text{syzygay(Lemma 1.2)}}$ Th.1.3, Th.1.4 for $\underline{Euc}$.

We give here an analogue of Cor.1.6 to $\underline{An}_{1a}$. For this we define a sub-collection $\underline{An}^0_{1a}$ of $\underline{An}_{1a}$ as follows:

$(1.15)_4$ $\underline{An}^0_{1a} := \{\underline{X} \in \underline{An}_{1a};\ \underline{X}$ is of the form: $(\underline{C}^n(z), U_0, X_0, h_{\underline{X}}, \underline{H}_{\underline{X}}, P_0)\}$

(cf. $(1.8)_0$), where X_0 coincides with the ambient space U_0. Moreover, $\underline{H}_{\underline{X}}$ is the trivial resolution of $O_{\underset{\sim}{X}}: 0 \to O_{\underset{\sim}{X}} \to O_{\underset{\sim}{X}} \to 0$ (Remark 1.1), where $\underline{X} := X_0 - D$, with the divisor D of $h_{\underline{X}}$.

Thus $\underline{X}$ coincides with the ambient space $U_0 - D$. This paroperty is similar to the one of $\underline{X}' = \underline{C}^n(z) \in \underline{Euc}(\subset \underline{Aff})$ (cf. $(1.14)_3$), and $\underline{An}^0_{1a} \subset \underline{An}_{1a}$ has a similar role to $\underline{Euc} \subset \underline{Aff}$:

Corollary 1.7. Th 1.1, Th 1.2 for $\underline{An}_0 \to$ Th 1.1 for $O_{\underset{\sim}{X}}$; $\underline{X} \in \underline{An}^0_{1a}$ $\longrightarrow$ Th.1.1, Th.1.2 for $\underline{An}^0_{1a}$.

The second inplication follows from Prop.1.4. The first one is proven in n.4, by imbedding varieties $\underline{X}\in\underline{An}^{0}_{1a}$ to higher dimensional euclidean spaces(cf.Lemma 1.3 in n.4 soon below).

Diagram I.

$$\begin{tikzcd}
\text{Th.1.7}(O_{\underline{X}'};\underline{X}'\in\underline{\text{Euc}}') \arrow[rr, dashed, "\text{syzygy}", "\text{(Lemma 1.2)}"'] \arrow[d] & & \text{Th.1.1,Th.1.2 for } \underline{An}_{0} \arrow[d] \\
\text{Cor.1.4}(\underset{\sim}{O}_{\underline{X}'};\underline{X}'\in\underline{\text{Euc}}) \arrow[rr, dashed, leftrightarrow, "\text{imbedding}", "\text{(Lemma 1.3)}"'] \arrow[d] & & \text{Cor.1.7}(\underset{\sim}{O}_{\underline{X}};\underline{X}\in An^{0}_{1a}) \arrow[d] \\
\text{Th.1.3,Th.1.4 for } \underline{\text{Euc}} \arrow[rr, dashed, leftrightarrow, "\text{syzygy}", "\text{(Lemma 1.2+Prop.1.4)}"'] & & \text{Th.1.1,Th.1.2 for } \underline{An}^{0}_{1a}
\end{tikzcd}$$

(The theorems at the bottom will be the starting points of the final part of the proof of Th.1.1~ Th.1.4(cf.n.5).)

4. Imbedding. First taking a euclidean line $\underline{C}(w)$, we define a map:

$(1.16)_0$ $I:\underline{An}^{0}_{1a}\ni\underline{X}\longrightarrow\underline{An}_0\ni\tilde{\underline{X}}=(\underline{C}(w)\times\underline{C}^{n}(z),\ \tilde{\underline{X}}=\underline{C}\times U_0,P_0)$(cf. n.1, § 1.2), where $\underline{X}=X_0-D$, with $X_0=U_0$, is as in $(1.15)_4$.

Setting[*] $h_{\tilde{\underline{X}}}:=1-h_{\underline{X}}w$, $S:=S_{\underline{X}}:=$locus of $h_{\tilde{\underline{X}}}$ in $\tilde{\underline{X}}$, and $\pi_{\underline{X}}:=$biregular map:$S\longrightarrow\underline{X}=U_0-D$, we will use the imbedding $\pi_{\underline{X}}^{-1}:X=U_0-D\hookrightarrow S$ and the p. g. sheaf $\underset{\sim}{H}_S$: $0\to\underset{\sim}{O}_{\tilde{\underline{X}}}\xrightarrow{h_{\tilde{\underline{X}}}}\underset{\sim}{O}_{\tilde{\underline{X}}}\to\underset{\sim}{O}_S\to 0$ over $\tilde{\underline{X}}$ for the proof of Lemma 1.3 soon below.(Here $\underset{\sim}{O}_{\tilde{\underline{X}}}$ and $\underset{\sim}{O}_S$ are the structure sheaves of $\tilde{\underline{X}}$,S.) Namely, letting the parameter $(P;r;\sigma;\delta)\in\Lambda'_{\underline{X}}(\subset U'_0\times(0,\tilde{r})\times\underline{R}^{+2}_{1}\times\underline{R}^{+2}_{1})$ be as in Th.1.2, we will compare the following two sets of δ-growth cochains in Lemma 1.3 soon below:

$$(1.16)_1\ \left\{\begin{matrix} C^q(\underline{A}_\sigma(\underline{X}_r(P)),O_X)_\delta \\ C^q(\underline{A}_\sigma(\tilde{\underline{X}}_r(P)),H_S)_\delta \end{matrix}\right. :=\text{set of } g:=\left\{\begin{matrix}|h^{-1}| \\ |z|+|w|+1\end{matrix}\right\}\text{-}\delta\text{-growth cochains with}$$

value in $\left\{\begin{matrix}\underset{\sim}{O}_{\underline{X}} \\ \underset{\sim}{H}_S\end{matrix}\right\}$ (cf.$(1.4)_7$ and Def.1.4_5) ,

where

$$(1.16)_2\ \left\{\begin{matrix}\underline{A}_\sigma(\underline{X}_r(P)) \\ \underline{A}_\sigma(\tilde{\underline{X}}_r(P))\end{matrix}\right\}:=\text{g-p. g.}-\text{covering of }\left\{\begin{matrix}\underline{X}_r(P) \\ \tilde{\underline{X}}_r(P)\end{matrix}\right\}\text{of size } \sigma \text{ in }\left\{\begin{matrix}U_0-D \\ \tilde{\underline{X}}=C\times U_0\end{matrix}\right\}.$$

(Here $\underline{X}_r(P)=\underline{X}\cap U_r(P)$, $\tilde{\underline{X}}_r(P):=\underline{C}\times U_r(P)$, with the disc in $U_r(P)\subset\underline{C}$, are as in $(1.9)_2$, § 1.2).

---

[*] The element $h_{\underline{X}}$ is also as in $(1.15)_4$.

Lemma 1.3 (P. g. uniform estimation for imbedding).

There is an element $\underline{E}_{\underline{X}} \in \underline{E}_{p.g}$ $(q \geqq 0)$, with which we have:

$$(1.16)_3 \begin{cases} C^q(\underline{A}_{\sigma'}(\underline{X}_{r'}(P)),\underset{\sim}{O}_X)_{\partial'} \supset s_X^*(\pi_X^{-1})^*\,\omega_S^* C^q(\underline{A}_{\sigma}(\tilde{\underline{X}}_r(P)),\underset{\sim}{H}_S)_{\partial} \\ s_S^*\pi_X^* Z^q(\underline{A}_{\sigma}(\underline{X}_r(P)),\underset{\sim}{O}_X)_{\partial} \subset \omega_S^* Z^q(\underline{A}_{\sigma'}(\tilde{\underline{X}}_{r'}(P)),\underset{\sim}{H}_S)_{\alpha'} \end{cases},$$

with $(r';\sigma';\partial')=\underline{E}_{\underline{X}}(r;\sigma;\partial)$, where**) ω_S^*=natural homomorphism*): $\underset{\sim}{H}_S \to \underset{\sim}{Q}_S$ and $s_{\underline{X}}, s_S$ are the 'p.g.refining maps' in $\underline{X}$, S.

We check Lemma 1.3 in §4.2. Cor. 1.7 follows from Lemma 1.3 as follows: apply Th. 1.1 for $\underline{An}_0$ to $\underset{\sim}{H}_S$, which is a p. g. coherent sheaf over $\underline{C}\times U_0$. Then we have the inclusion of the form $(1.10)_1$ for the right side of the second inclusion in $(1.16)_3$. Using the first inclusion in $(1.16)_3$, we pull back this inclusion to $\underline{X}$ (by means of $\pi_{\underline{X}}$). Then we have the desired inclusion in $(1\ 10)_1$ for $\underset{\sim}{O}_{\underline{X}}$.

Finally, we will complete Diagram I in the following fashion

Diagram II

$$\begin{tikzcd}
\text{Th.1.3, Th.1.4 for Euc} \arrow[dd] & & \text{Th.1.1, Th.1.2 for } \underline{An}^0_{1a} \arrow[dd] \\
& \text{extension (Cor 1.8 + Lemma 1.4)} \arrow[l, dashed] \arrow[r, dashed] & \\
\text{Th.1.3}(\underset{\sim}{O}_{X'};\underline{X}'\in\underline{Aff}) \arrow[dd, leftrightarrow] & & \text{Th.1.1}(\underset{\sim}{O}_X;\underline{X}\in\underline{An}_{1a}) \arrow[dd] \\
& \text{syzygy (Lemma 1.2)} \arrow[l, dashed] \arrow[r, dashed] & \\
\text{Th.1.3, Th.14 for } \underline{Aff} & & \text{Th.1.1, Th.1.2 for } \underline{An}_{1a}
\end{tikzcd}$$

(The second implication is insured by Prop 1.4; Cor 1.8 and Lemma 1.4 will be given soon below.) By Diagram I, II, the remaining task for the proof of Th.1.1 ~ Th 1.4 is to prove:

Corollary 1.8. We have the following implications:

$$(1.17)_1 \left.\begin{cases} \text{Th.1.1, Th.1.2 for } \underline{An}^0_{1a} \\ \text{Th.1.3, Th.1.4 for } \underline{Euc} \end{cases}\right\} \longrightarrow \begin{cases} \text{Th.1.1, for each } \underset{\sim}{O}_{\underline{X}};\underline{X}\in\underline{An}_{1a} \\ \text{Th.1.3 for each } \underset{\sim}{O}_{\underline{X}'};\underline{X}'\in\underline{Aff} \end{cases}$$

For the proof of Cor. 1.8, we attach to elements of $\underline{An}_{1a}$, $\underline{Aff}$ their ambient

---

*) Note that $\underset{\sim}{H}_S, \underset{\sim}{Q}_S$ are obtained by regarding the structure sheaf of S as the sheaves over $\underline{X}$, S.

**) For the precise form of the refining maps $s_S, s_{\underline{X}}$, see $(1.13)_2$ in Prop 4.7_2.

spaces:

$$(1.17)_2 \quad J\ :\begin{cases} An_{1a}\ni\underline{X}=(\underline{C}^n(z),U_0,X_0,h,\underline{\underset{\sim}{H}}_{\underline{X}},P_0)\rightarrow \underline{An}^0_{1a}\ni\underline{Y}=(\underline{C}^n(z),U_0,Y=U_0-D_0,h,\underline{\underset{\sim}{H}}_{\underline{Y}},P_0), \\ Aff\ni\underline{X}'=(\underline{C}^n(z),\underline{X}',\underset{\sim}{H}_{\underline{X}'})\longrightarrow \underline{Euc}\ni\underline{Y}'=(\underline{C}^n(z),Y'=\underline{C}^n,\underset{\sim}{H}_{\underline{Y}'}). \end{cases}$$

(For the above notaion, see $(1.8)_0$ and $(1.11)_0$. In particular, $\underset{\sim}{H}_{\underline{X}}$ is the p.g. resolution of the structure sheaf $\underset{\sim}{O}_{\underline{X}}$ of $\underline{X}=X_0-D$, with D=locus of g on X_0, and $\underset{\sim}{H}_{\underline{Y}}$ is the trivial resolution of $\underline{Y}=U_0-D_0:0\rightarrow\underset{\sim}{O}_{\underline{Y}}\longrightarrow\underset{\sim}{O}_{\underline{Y}}\rightarrow 0$, with the pole D_0 (= divisor) of h on $\overline{U}_0$.) We will prove Cor.1.8 by extending cochains on $\underline{X}$ to its ambient space $\underline{Y}$.

5. Extensions of cochains. Letting the variety $\underline{X}\in\underline{An}_{1a}$ and $\underline{Y}=J(\underline{X})$ be as in $(1.17)_2$ and letting the parameter $(P;r;\sigma;\partial)\subset\Lambda'_{\underline{X}}$ have the similar meaning to Lemma 1.3, we compare the following sets of the cochains*):

$$(1.17)'_1\begin{Bmatrix} \underline{C}^q(A_\sigma(\underline{X}_r(P)),\underset{\sim}{O}_{\underline{X}})_\partial \\ \underline{C}^q(A_\sigma(\underline{Y}_r(P)),\underset{\sim}{H}_{\underline{X}})_\partial \end{Bmatrix}:=\text{set of all } g(=|h^{-1}|)\text{-}\partial\text{-growth cochains with value in}\begin{Bmatrix}\underset{\sim}{O}_{\underline{X}}\\ \underset{\sim}{H}_{\underline{X}}\end{Bmatrix}$$

where

$$(1.17)''_1\begin{Bmatrix} \underline{A}_\sigma(X_r(P)) \\ \underline{A}_\sigma(Y_r(P)) \end{Bmatrix}:=g\text{-p.g.-covering of}\begin{Bmatrix} \underline{X}_r(P):=\underline{X}\cap U_r(P) \\ \underline{Y}_r(P):=\underline{Y}\cap U_r(P) \end{Bmatrix}^{**)}\text{of size }\sigma\text{ in}\begin{Bmatrix} \underline{X}=X_0-D \\ \underline{Y}=U_0-D_0 \end{Bmatrix}.$$

Also letting the affine variety $\underline{X}'\in\underline{Aff}$ and $\underline{Y}'=\underline{C}^n(z)$ be as in $(1.17)_2$, we will compare the following sets*):

$$(1.17)'_2\begin{Bmatrix} C^q(\underline{A}_\sigma(\underline{X}'),\underset{\sim}{O}_{\underline{X}'})_\partial \\ C^q(\underline{A}_\sigma(\underline{Y}'),\underset{\sim}{H}_{\underline{X}'})_\partial \end{Bmatrix}:=\text{set of all } g(=|z|+1)\text{-}\partial\text{-growth cochains with value in}\begin{Bmatrix}\underset{\sim}{O}_{\underline{X}'}\\ \underset{\sim}{H}_{\underline{X}'}\end{Bmatrix}$$

where

$$(1.17)''_2\begin{Bmatrix} \underline{A}_\sigma(\underline{X}') \\ \underline{A}_\sigma(\underline{Y}') \end{Bmatrix}:=\text{ g-p.g. coverings of}\begin{Bmatrix} \underline{X}' \\ \underline{Y}' \end{Bmatrix}\text{ of size }\sigma\text{ in }\begin{Bmatrix} \underline{X}' \\ \underline{Y}' \end{Bmatrix}.$$

(In the above, the parameter $(\sigma;\partial)$ is in $\Lambda'_{\underline{X}'}(\subset\underline{R}^{+2}_1\times\underline{R}^{+2}_1)$ (cf. Th. 1.3). Then we have:

---

*) cf $(1.4)_8$ and $(1.3)''_6$.

**) $U_r(P)$:=disc in $\underline{C}^n$ as in $(1.9)_1$, §1.2.

Lemma 1.4.(P.g.uniform estimation for extension).

Take suitable $E_{\underline{X}} \in \underline{\mathbb{E}}_{p.g}$ and $\underline{L}_{\underline{X}} \in \underline{\tilde{L}}(q \gtrsim 0)$. Then we have:

$$(1.17)_3 \begin{cases} s^*Z^q(\underline{A_\sigma(X_r}(P)), \underset{\sim}{O}_{\underline{X}})_\partial \subset \omega_{\underline{X}} Z^q(\underline{A_{\sigma'}(Y_{r'}}(P)), \underset{\sim}{H}_{\underline{X}})_{\partial'} \\ s^*Z^q(\underline{A_\sigma}(\underline{X}'), \underset{\sim}{O}_{\underline{X}'})_\partial \subset \omega_{\underline{X}'} Z^q(\underline{A_{\sigma'}}(Y'), \underset{\sim}{H}_{\underline{X}'})_{\partial'} \;, \end{cases} \underline{\text{where}} \begin{cases} (r',\sigma';\partial')=E_{\underline{X}}(r,\sigma;\partial \\ (\sigma';\partial')=\underline{\tilde{L}}_{\underline{X}'}(\sigma;\partial). \end{cases}$$

Here $\omega_{\underline{X}}$, $\omega_{\underline{X}'}$ are the natural homomorphisms: $\underset{\sim}{H}_{\underline{X}} \longrightarrow \underset{\sim}{O}_{\underline{X}}$ and: $\underset{\sim}{H}_{\underline{X}'} \longrightarrow \underset{\sim}{O}_{\underline{X}'}$.
(We prove Lemma 1.4, by extending cochains on $\underline{X}$ to $U_0 - D_0$,..(cf. §4.2).)

Now Cor.1.8 is derived from Lemma 1.4 as follows: first applying the Th.1.1(for $\underset{\sim}{H}_{\underline{X}}$) to the first inclusion in $(1.17)_3$, we get the inclusion of the form $(1.10)_1$ for the right side of the former inclusion. Then, operating $\omega_{\underline{X}}$ to that inclusion(of the form $(1.10)_1$), we have the desired inclusion of the form $(1.10)_1$ for $O_{\underline{X}}$. This insures the checked first implication in $(1.17)_1$. The second implication is checked in the similar manner to the above. Thus we have Cor.1.8, and we also finish the proof of Th.1.1 ~ Th.1.4(cf.also the remark soon below Cor.1.8).*)

6. Proof of Th.1.6_1. First recall that we checked the comparison of 'meromorphic' and 'p.g.' in Th.1.6_1 for elements of $\underline{An}^0_{1a}$ and $\underline{Euc}$(cf. n.6,§1.2 Then, using the extension of the cochains in Lemma 1.4 for $\underline{An}_{1a}$,$\underline{Aff}$, we get Th.1.6_1 for $\underline{An}_{1a}$,$\underline{Aff}$ from the corresponding facts**) for $\underline{An}^0_{1a}$,$\underline{Euc}$. Thus we have shown that, for the proof of Th.1.1 ~ Th.1.6, it suffices to prove the key theorem, Th.1.7 and the key lemmas, Lemma 1.2 ~ Lemma 1.4.

---

*) See also Diagrams I,II.

**) Also, in this step, we use Th.1.2 , Th.1.4 for $\underline{An}^0_{1a}$,$\underline{Euc}$. This follows from Lemma 1.2(cf. n.3 §1.3), and our use of those theorems is legitimate(cf also Diagram III at the end of §1.3).

For convenience of understanding of the logical structure of §1.3, we summarize Diagrm I,II and the content of n.6, §1.3 as follows:

Diagram III

$$\begin{tikzcd}
\begin{array}{c}\text{Th.1.7}\\ (O_{\tilde{X}};\tilde{X}\in\text{Euc}')\end{array} \arrow[rr, "\text{Lemma 1.2}", "(\text{for } X\in\text{An}_0)"'] \arrow[dd] & & \begin{array}{c}\text{Th.1.1,1.2}\\ (\text{for } \underline{X}\in\underline{\text{An}}_0)\end{array} \arrow[dd] \\
{} & \begin{array}{c}\text{Lemma 1.3}\\ (\text{for } X\in\text{Euc}, X\in\text{An}^0_{1a})\end{array} \arrow[l, dashed] & \\
\begin{array}{c}\text{Cor.1.4}\\ (O_X;X\in\text{Euc})\\ \text{Th.1.6}_1 \text{ for } X\in\text{Euc}\end{array} \arrow[dd] & & \begin{array}{c}\text{Cor.1.7}\\ (O_X;X\in\text{An}^0_{1a})\\ \text{Th.1.6}_1 \text{ for } X\in\text{An}^0_{1a}\end{array} \arrow[dd] \\
{} & \begin{array}{c}\text{Lemma 1.2}\\ (\text{for } X\in\text{Euc}, X\in\text{An}^0_{1a})\end{array} \arrow[l, dashed] \arrow[r, dashed] & {} \\
\begin{array}{c}\text{Th.1.3,1.4}\\ (\text{for } X\in\text{Euc})\end{array} \arrow[dd] & & \begin{array}{c}\text{Th.1.1,1.2}\\ (\text{for } X\in\text{An}^0_{1a})\end{array} \arrow[dd] \\
{} & \begin{array}{c}\text{Lemma 1.4}\\ (\text{for } X\in\text{Aff}, X\in\text{An}_{1a})\end{array} \arrow[l, dashed] \arrow[r, dashed] & {} \\
\begin{array}{c}\text{Th.1.3}\\ (O_X;X\in\text{Aff})\\ \text{Th.1.6}_1 \text{ for } X\in\text{Aff}\end{array} \arrow[dd] & & \begin{array}{c}\text{Th.1.1}\\ (O_X;X\in\text{An}_{1a})\\ \text{Th.1.6}_1 \text{ for } X\in\text{An}_{1a}\end{array} \arrow[dd] \\
{} & \begin{array}{c}\text{Lemma 1.2}\\ (\text{for } X\in\text{Aff}, X\in\text{An}_{1a})\end{array} \arrow[l, dashed] \arrow[r, dashed] & {} \\
\begin{array}{c}\text{Th.1.3,1.4}\\ (\text{for } X\in\text{Aff})\end{array} & & \begin{array}{c}\text{Th.1.1,Th.1.2}\\ (\text{for } X\in\text{An}_{1a})\end{array}
\end{tikzcd}$$

We will finish §1.3 by a technical remark for the proof of Lemma 1.2:

Remark 1.4. Letting the varietyes $X\in\text{An}_{1a}$, $X'\in$ Aff be as in Lemma 1.2, we define the following subcollections of $\text{Coh}'(X')_{p.g}$, $\text{Coh}'(\underline{X}')_{p.g}$(cf.n.2, §1.3 and $(1.4)'_9$):

$(1.18)_1$ $\text{Coh}''(X^*)_{p.g} := \{ H \in \text{Coh}'(X^*)_{p.g}$ (cf.$(1.4)'_9\}$, where $X^* = X\in\text{An}_{1a}$ or $=X'\in$ Aff, and writing H explicitly in the form of $(1.4)_1$:

$(1.18)'_1$ $H: 0 \to O_X^{k_p} \xrightarrow{K_{p-1}} O_X^{k_{p-1}} \longrightarrow - - - \xrightarrow{K_1} O_X^{k_1} \xrightarrow{K_0} H \to 0$,

the element $\underset{\sim}{H}$ must satisfy:

$(1.18)''_1$ the entries of $K_j (0 \leqq j < p)$ are in $\Gamma(X_0, O_{X_0}(*D))$ or $\Gamma(X'_{alg}, O_{X'alg})$

according as $X^*=X \in An_{1a}$ or$=X' \in$ Aff
(Recall that, for an element $H \in Coh'(X^*)_{p.g}$, the corresppnding condition to $(1.18)_1''$ is 'the entries of K_j are p.g. with respect to the p.g. function $g_{X^*}=|\tilde{z}|+1$ or $|z|+1$ '.(cf.$(1.4)_9$).

Now recall that Lemma 1.2 was given to $Coh'(X^*)_{p.g}$. Here we check: $(1.18)_2$ one can replace '$Coh'(X^*)_{p.g}$' by '$Coh''(X^*)_{p.g}$' in Lemma 1.2. First, if $X \in An^0_{1a}$ or $X' \in$ Euc , then the comparison of 'p.g' and 'meromorphi (or, rational)' in Th.1.6_1 is a well known fact(cf. n.6, §1.2), and $(1.18)_2$ is legitimate. On the otherhand, Diagram III insures that 'Lemma 1.2 for An^0_{1a}, Euc as well as the extension of cochains in Lemma 1.4 ' imply Th.1.6_1 for genral $X \in An_{1a}$ and $X' \in$ Aff . Thus we have $(1.18)_2$

---

*)Note that Lemma 1 4 follows from am estimation on local parametrization of analytic varieties, and is independent from Lemma 1 2 (n.5, §4.2). Also we use Lemma 1.3 in Diagram III. This lemma is also proven independently from Lemma 1.2(n.6, §4.2).

§ 2. Cohomology with algebraic division and polynomial growth

This section contains the main results of this paper: first, in § 2.1, we summarize some algebraic notions concerning the a.d.and p.g.*) properties of certain complexes. Using such notions we give our main results of § 2 as well as of this paper in § 2.2. In § 2.3 we reduce the results of § 2.2. to those in § 1, by using**) some uniform estimation on the a.d.and p.g.properties of coherent sheaves.

§ 2.1. Algebraic division conditions

1. Open map property. We begin § 2.1 by arranging some terminology which will be used in later arguments: first, a filtered group is, as usual, a decreasing sequence $\underline{B}=\{B(m)\}_{m=0}^{\infty}$ of abelian groups $B(m)$. When there is no fear of confusions we write $B(0)$ also as $\dot{B}$. By a filtered complex we mean such a one:

$(2.1)_1$ $0 \to E \xrightarrow{e} \underline{C}^0 \dashrightarrow \underline{C}^q \xrightarrow{d_q}$, where $d_q (q \geqq 0)$ is a homomorphism of filtered groups and the augumentation e is that of abelian groups. Letting $\underline{C}'^{*}: 0 \to E' \xrightarrow{e'} \underline{C}'^0 \to \cdots \underline{C}'^q \to$ be an another filtered complex, a homomorphism $\omega: \underline{C}^* \to \underline{C}'^*$ is a collection $\omega=\{\omega', \{\omega_q\}_{q=0}^{\infty}\}$ of homomorphisms $\omega_q: \underline{C}^q \to \underline{C}'^q$ (of filtered groups) and that of abelian groups $\omega': E \to E'$ satisfying the standard commutativity condition. Next let $\underline{C}=\{\underline{C}_{\mu}^{*}\ ; \mu \in \mathcal{M}\}$ be a direct system of filtered complexes. Writing $\underline{C}_{\mu}^{*}$ as: $0 \to E_{\mu} \xrightarrow{e_{\mu}} \underline{C}_{\mu}^{0} \to \dashrightarrow \underline{C}_{\mu}^{q} \xrightarrow{d_{\mu}}$ (cf. $(2.1)_1$) and $\underline{C}_{\mu}^{q}$ as $\{C_{\mu}^{q}(m)\}_{m=0}^{\infty}$, we make:

---

*) 'a.d.'='algebraic division' and 'p.g'='polynomial growth'(cf. Introduction).

**) cf. also Introduction.

Definition 2.1_1. We say that C has open map property(resp.is $\mathcal{M}$-exact) if, for each $q \in \underline{Z}^+ \cup 0$ and $\mu \in \mathcal{M}$, there is an element $\mu' \geq \mu$, with which $(2.1)_2$(resp.$(2.1)_3$) below holds:

$(2.1)_2$ there is a map $a: \underline{Z}^+ \longrightarrow \underline{Z}^+$ so that $d_{\mu'} C^q_{\mu'}(m) \supset \rho_{\mu'\mu} Z^{q+1}(C^*_{\mu}(a(m)))$.

$(2.1)_3$ $d_{\mu'} C^{q-1}_{\mu'} \supset \rho_{\mu'\mu} Z^q(C^*_{\mu})(q \geqq 1)$, $e_{\mu'} E_{\mu'} \supset \rho_{\mu'\mu} Z^0(C^*_{\mu})(q=0)$, where $C^{q-1}_{\mu} = C^{q-1}_{\mu}$ and $C^*_{\mu} = \sum_{q \geqq 0} C^q_{\mu}$.)

(If $\# \mathcal{M} = 1$ then $(2.1)_2$ is equivalent to say that $d_{\mu}: C^q_{\mu} \longrightarrow Z^{q+1}(C^*_{\mu})$ is an open map, with respect to the topology determined by $C^q_{\mu}, \ldots$) The following equivalent condition to $(2.1)_2$ is useful in later arguments:

$(2.1)'_2$ there is a map $b: \underline{Z}^+ \longrightarrow \underline{Z}^+$ satisfying $\lim\limits_{m \longrightarrow} b(m) = \infty$ and $d_{\mu'} C^q_{\mu'}(b(m)) \supset \rho_{\mu'\mu} Z^{q+1}(C^*_{\mu}(m))$ for $m \gg 0$.

The open map property is important because we have:*)

Proposition 2.1_1. If C satisfies the open map property and is $\mathcal{M}$-exact, then we have:

$$(2.1)_4 \begin{cases} \varinjlim_{\mu} H^q(\varprojlim_m C^*_{\mu}/C^*_{\mu}(m)) \cong \varinjlim_{\mu}(\varprojlim_m H^q(C^*_{\mu}/C^*_{\mu}(m))) \cong 0(q \geqq 1), \\ \varinjlim_{\mu} Z^0(\varprojlim_m C^*_{\mu}/C_{\mu}(m)) \cong \varinjlim_{\mu}(\varprojlim_m Z^0(C^*_{\mu}/C^*_{\mu}(m))) \cong \varinjlim_{\mu}(\varprojlim_m \theta_{\mu m} e_{\mu} E_{\mu} \end{cases}$$

where $\theta_{\mu m}$ is the natural homomorphism: $C^0_{\mu} \longrightarrow C^0_{\mu}/C^0_{\mu}(m)$.

Remark 2.1. Take a Noetherian ring $\underset{\sim}{O}$, an ideal $\underset{\sim}{I}$ of $\underset{\sim}{O}$ and a complex C^* of O-modules:**) $0 \rightarrow C^0 \rightarrow \cdots \rightarrow C^q \rightarrow \cdots$. We set $\underline{C}^q := \{\underset{\sim}{I}^m C^q\}_{m=0}^{\infty}$. Then Artin-Rees theorem insures that if C^* is exact then $\underline{C}^*$ satisfies the

---

*) For the proof of Prop. 2.1_1 and for roles of the open map property in other standard comparison theorems in the completion theory, see M.Nomi [13].

**) We understand that the augumentation map $e: E \rightarrow \underline{C}^0$ is of the form: E=0 and e=zero map. We use the similar notations in later arguments (cf.Def.2.5_2).

open map property. Also it is well known that the above theorem insures the exactness of the completion of C^*(cf. [12]). In spite of this basic character of the open map property(involved in the above result), it seems that such a property has not been taken up in general siturations. (The author knows no other examples of complexes of general nature, where that property is emphasized explicitly.) As was mentioned*), the conjecture of S. Lubkin for such a propety for local de Rham complex is our starting point of the studies of the contents in § 2. The open map properties will be given**) for some complexes in § 2, §3 and part B, § 4.2. As we will see in the course of § 2, the most substantial part of § 2 concerns that property for certain Čech complexes of global nature.

---

*)**) cf. Introduction.

Finally, letting $\underline{C}^* = \{\underline{C}^*_\mu ; \mu \in \mathcal{M}\}$ be as in Def 2.1_1 , take an another direct system $\underline{C}' = \{\underline{C}'^*_\mu ; \mu \in \mathcal{M}\}$ of filtered complexes. Writing $\underline{C}^*_\mu , \underline{C}'^*_\mu$ as $0 \to E_\mu \to \underline{C}^0_\mu \to \cdots \to \underline{C}^q_\mu \to$ and $0 \to E'_\mu \to \underline{C}'^0_\mu \to \cdots \to \underline{C}'^q_\mu \to \cdots$, we assume that the complexes (of abelian groips): $0 \to E_\mu \to C^0_\mu \to \cdots \to C^q_\mu$ and $0 \to E'_\mu \longrightarrow C'^0_\mu \to \cdots \to C'^q_\mu \to$ coincide. (Here $C^q = C^q(0), \ldots$)

<u>Definition 2.1_2</u>. We sat that $\underline{C}, \underline{C}'$ are <u>equivalent</u>, if for each $q \in \underline{Z}^+ \cup 0$ and $\mu \in \mathcal{M}$, there is an element $\mu' > \mu$, with which we have the following for each $m \geqslant 0$:

$(2.1)_5$ $\rho_{\mu'\mu} C^q_\mu(m) \subset C'^q_{\mu'}(m')$, with an element $m' \in \underline{Z}^+$ satisfying $\lim_{m \to} m' = \infty$, and if the converse relation to this holds.

<u>Proposition 2.1_2</u>. <u>Assume that $\underline{C}, \underline{C}'$ are equivalent. If $\underline{C}$ satisfies the open map property</u> , <u>then $\underline{C}'$ satisfies that condition</u>.

<u>2</u>. <u>A.d. filtration</u>. Let X be a topological space , $\underset{\sim}{O}$ a sheaf of ring over X, $\underset{\sim}{K}$ an $\underset{\sim}{O}$-module and $\underline{f} = (f_j)_{j=1}^s$ a subset of $\Gamma(X, \underset{\sim}{O})$. We write $\{f_j^m\}_{j=1}^s$ as $\underline{f}^m$. By <u>m-th standard homomorphism for $\underline{f}$</u>, we mean the homomorphism F^m: $\underset{\sim}{O}^s \ni \varphi = (\varphi_j) \to \underset{\sim}{O} \ni \sum_j f_j^m \varphi_j (1 \leqq j \leqq s)$, and we write the image $F^m \underset{\sim}{O}^s \subset \underset{\sim}{O}$ also as $\underline{f}^m \underset{\sim}{O}$. We use the symbol $\underline{f}^m \underset{\sim}{K}$ for the $\underset{\sim}{O}$-submodule of $\underset{\sim}{K}$, which is spanned by elements $\varphi_m \cdot \psi$, with $\varphi_m \in \underline{f}^m \underset{\sim}{O}$ and $\psi \in \underset{\sim}{K}$.

Next take an element $\underline{A} \in \mathrm{Cov}_0(X)$. We then make the following definition for later terminological convenience:

<u>Definition 2.2.</u> By <u>f-a.d. filtered group of</u> $C^q(\underline{A}, \underset{\sim}{K})$(or, <u>q-th</u> <u>f- a.d.-filtered cochain group for</u> $(\underline{A}, \underset{\sim}{K})$), we mean the following:

(2.2) $\{C^q(\underline{A}, \underline{f}^m \underset{\sim}{K})\}_{m=0}^{\infty}$, with $\underline{f}^0 \underset{\sim}{K} = \underset{\sim}{K}$.

In n.3 soon brlow we will combine Def 2.2 with the p.g. filtration in Def. 1.2_1.

---

*) $\mathrm{Cov}_0(X) = 2^{\mathrm{Ouv}(X)}$ (cf. the end of the introduction of Chap. I).

3. D.p.filtration. First taking an abelian group $\underline{B}$ and a map $\Phi:(\underline{Z}^{+}\cup 0)\times\underline{R}^{+2}\ni(m,\alpha)\longrightarrow 2^{\underline{B}}$, we denote by Φ_m the restriction of Φ to $m\times\underline{R}^{+2}(\simeq\underline{R}^{+2})$. Setting $\underline{B}(m;\Phi)_{p.g}:=\bigcup_{\alpha\in\underline{R}^{+2}}\Phi_m(\alpha)(\subset\underline{B})$, we make:

Definition 2.3_1. We say that Φ is a d.p.filtration*) of $\underline{B}$, if we have:

$(2.3)_1$ $\Phi_m:\underline{R}^{+2}\longrightarrow 2^{\underline{B}}$ is a p.g.filtration for each $m\in\underline{Z}^{+}\cup 0$(Def.$1.2_1$).

$(2.3)_2$ $\underline{B}(m;\Phi)_{p.g}\subset\underline{B}(m';\Phi)_{p.g}$ for any $m\geqq m'$, and $\Phi_m,\Phi_{m'}$ are compatible with the inclusion:$\underline{B}(m;\Phi)_{p.g}\hookrightarrow\underline{B}(m';\Phi)_{p.g}$(Def.$1.2_3$).

Next letting the geometric datum $(X,\underset{\sim}{O},\underset{\sim}{K},\underline{f})$ be as in n.2, we take a p.g.function $g:X\longrightarrow\underline{R}_1^{+}$(Def.$1.4_4$). We will define d.p.filtrations for $C^q(A,\underset{\sim}{K})$(cf.n.2), by means of $(\underline{f},g)$. For this we assume that $\underset{\sim}{K}$ is a homomorphic image of $\underset{\sim}{O}^k(k>0):\underset{\sim}{O}^k\xrightarrow{\omega}\underset{\sim}{K}\rightarrow 0$, We assume that $\underset{\sim}{O}$ is endowed with a q-structure θ(Def.1.4_1), and we endow $\underset{\sim}{K}$ with the induced q-structure, θ_K in symbol, from (θ,ω)(Def.1.4_3). Recall that $(g,\theta),(g,\theta_{\underset{\sim}{K}})$ define p.g.filtrations for $C^q(\underline{A},\underset{\sim}{O}^k),C^q(\underline{A},\underset{\sim}{K})$(Def.$1.4_5$). For an element $\alpha\in\underline{R}_1^{+2}$ we set

$$(2.3)_3'\ \left\{\begin{matrix}C^q(\underline{A},\underset{\sim}{O}^k;g)_\alpha\\ C^q(\underline{A},\underset{\sim}{K};g)_\alpha\end{matrix}\right\}:=\text{ set of}\left\{\begin{matrix}(g,\theta)\\ (g,\theta_{\underset{\sim}{K}})\end{matrix}\right\}-\alpha\text{-growth cochains with value in}\left\{\begin{matrix}\underset{\sim}{O}^k\\ \underset{\sim}{K}\end{matrix}\right\}$$

(cf.$(1.3)_{6,7}$)**). Also recall that Prop.1.3 implies:

$$(2.3)_3''\quad C^q(\underline{A},\underset{\sim}{K};g)_\alpha=\omega C^q(\underline{A},\underset{\sim}{O}^k;g)_\alpha\ .$$

Next we use the symbol F^m(=m-th standard homomorphism for $\underline{f}$)$:\underset{\sim}{O}^s\longrightarrow\underset{\sim}{O}$(cf.n.2) for its k-times direct sum:$\underset{\sim}{O}^{sk}:=\underset{\sim}{O}^s+\cdot\cdot+\underset{\sim}{O}^s\xrightarrow{F^m+\cdot\cdot+F^m}\underset{\boxed{O^k=}}{\underset{\sim}{O}+\cdot\cdot+\underset{\sim}{O}}$. Then assuming that $\underline{f}\subset\Gamma(X,\underset{\sim}{O};g)_{p.g}$ we make:

Definition 2.3_2. By left and right $(\underline{f},g)$-d.p.filtrations of $\underline{B}=C^q(\underline{A},\underset{\sim}{K})$, we mean the maps in $(2.3)_3$ soon below:

Fig.I

$$\begin{tikzcd}
C^q(\underline{A},\underset{\sim}{O}^{sk};g)_\alpha \arrow[r, "F^m"] \arrow[dr, "G"'] & C_1^q(\underline{A},\underline{f}^m\underset{\sim}{O}^k;g)_\alpha \arrow[d, "\omega"] \\
 & C_1^q(\underline{A},\underline{f}^m\underset{\sim}{K};g)_\alpha
\end{tikzcd}$$

$$(2.3)_3\ \left\{\begin{matrix}{}_l\Psi_{\underline{f},g}\\ {}_r\Psi_{\underline{f},g}\end{matrix}\right\}:(\underline{Z}^{+}\cup 0)\times\underline{R}^{+2}\ni(m;\alpha)\longrightarrow 2^{\underline{B}}\left\{\begin{matrix}\ni\omega F^m C^q(\underline{A},\underset{\sim}{O}^{sk};g)_\alpha\\ \ni C^q(\underline{A},\underset{\sim}{K};g)_\alpha\cap C^q(\underline{A},\underline{f}^m\underset{\sim}{K})\end{matrix}\right.\ .$$

---

*) 'D.p.' = 'a.d.' + 'p.g.', where 'a.d.'='algebraic division' and 'p.g.' ='polynomial growth'(cf.n.2, §2.1 and §1.1).

**) cf.also $(1.4)_8$.

We use the following notation for the first set in the right side of (2.3

$(2.3)_4 \quad C^q_1(\underline{A},\underline{f}^m\underset{\sim}{K};g)_\partial :={}_1\Phi_{f,g}(m;\partial),\ Z^q_1(\underline{A},\underline{f}^m\underset{\sim}{K};g)_\partial := {}_1\Phi_{\underline{f},g}(m;\partial)\cap\delta^{-1}(0).$

Note that $(2.3)_3$ implies:

$(2.3)_5 \quad C^q_1(\underline{A},\underline{f}^m\underset{\sim}{K};g)_\partial :=\omega C^q_1(\underline{A},\underline{f}^m\underset{\sim}{K};g)_\partial$ (cf. also Fig. I) .

(Here and in Fig. I, we regard $\underset{\sim}{O}^k$ as the trivial p.g. sheaf: $0\to\underset{\sim}{O}^k\xrightarrow{i}\underset{\sim}{O}^k\to 0$, with the identity i(Remark 1.1).) We use the notaion $C^q_r(\underline{A},\underline{f}^m\underset{\sim}{K};g)_\partial,\cdots$ for the second set of $(2.3)_3$. Moreover, we set:

$(2\ 3)_6 \quad C^q_{\tilde{c}}(\underline{A},\underline{f}^m\underset{\sim}{K};g)_{p.g} := \bigcup_{\partial\in \underline{R}_1^{+2}} C^q_{\tilde{c}}(\underline{A},\underline{f}^m\underset{\sim}{K};g)_\partial$ (cf. $(2.3)_4$), where the symbol $\tilde{c}$ indicates the symbol '1' or 'r'

The left filtration ${}_1\Phi_{\underline{f},g}$ makes use of informations of the left side of the homomorphism $\omega\cdot F^m:\underset{\sim}{O}^{sk}\longrightarrow\underset{\sim}{H}$, and the definition of the sets in $(2.3)_4$ is concordant to the similar sets in the p.g. cohomology theory in §1 (cf. $(1.4)_8$). The left filtration is suitable for later explicit uniform estimation(cf. §2.2). The right filtration ${}_r\Phi_{\underline{f},g}$ is, as we will see soon later, suitable for applications to the completion theory:

<u>Definition 2.3_3</u>. By <u>g-p.g subgroup of</u> $C^q(\underline{A},\underset{\sim}{K}^{\wedge\underline{f}})$, $\underset{\sim}{K}^{\wedge\underline{f}}:=\varprojlim_m\underset{\sim}{K}/\underline{f}^m\underset{\sim}{K}$, we mean:

$(2.3)_7 \quad C^q(\underline{A},\underset{\sim}{K}^{\wedge\underline{f}};g)_{p.g}:= \varprojlim_m C^q(\underline{A},\underset{\sim}{K},g)_{p.g}/C^q_r(\underline{A},\underline{f}^m\underset{\sim}{K};g)_{p.g}$

The word 'subgroup' is justified by the following:

<u>Proposition 2.2_1</u>. <u>There is a natural injection:</u>

$(2.3)_8 \quad i:C^q(\underline{A},\underset{\sim}{K}^{\wedge\underline{f}};g)_{p\ g}\hookrightarrow C^q(\underline{A},\underset{\sim}{K}^{\wedge\underline{f}})$

<u>Proof</u>. First, from $(2.3)_3$, we have the exact sequence:

(a) $0\longrightarrow C^q_r(\underline{A},\underline{f}^m\underset{\sim}{K};g)_{p.g}\longrightarrow C^q(\underline{A},\underset{\sim}{K};g)_{p.g}\xrightarrow{\mu_m\cdot i} C^q(\underline{A},\underset{\sim}{K})/C^q(\underline{A},\underline{f}^m\underset{\sim}{K})$,

where the homomorphism μ_m is induced from the natural one: $\underset{\sim}{K}\to\underset{\sim}{K}/\underline{f}^m\underset{\sim}{K}$ and i=inclusion: $C^q(\underline{A},\underset{\sim}{K};g)_{p\ g}\hookrightarrow C^q(\underline{A},\underset{\sim}{K})$. (Thus we use the information of the right side of (a) in the definition of the right filtration ${}_r\Phi_{\underline{f},g}$).

It is easy to get $(2.3)_8$ from (a) . q.e.d .

Concerning the right and left filtrations, we remark:*)

Proposition 2.2$_2$. $C^q_r(\underline{A},\underline{f}^m\underset{\sim}{K};g)_{p.g} \supset C^q_l(\underline{A},\underline{f}^m\underset{\sim}{K};g)_{p.g}$.
This follows directly from the definition of the both sides;see $(2.3)'_6$. In Lemma 2.3 , § 2.2, we show that the above two filtrations are 'equivalent' for the varieties of the type in § 1 .

4. D. p. c. estimation map. Finally we introduce an estimation map, which will be used in the main estimations in § 2(cf. Th.2.1~Th 2.2$_2$,§ 2.2).

Definition 2.4. By a d. p. c. estimation map, we mean a collection $E=(\tilde{E},\exp M,L)$, where $\tilde{E}$ is a p. g. c estimation map $\in \underline{E}_{p.g}$(Def.1.5), M is a positive monomial(n.5, § 1.1) and L is a linear map:$\underline{R}^+ \ni m \longrightarrow \underline{R}^+ \ni cm; c>0$
Recall that $\tilde{E}$ is a map:$\underline{D}:=(\underline{R}^+\times\underline{R}^{+2}\times\underline{R}^{+2}) \ni (r;\sigma;\partial) \rightarrow \underline{D}=(\underline{R}^+\times\underline{R}^{+2}\times\underline{R}^{+2}) \ni (r';\sigma';\partial')$.
We regard E as the map:**)

$(2.4)_1$ $E:\underline{D}\times(\underline{Z}^+\cup 0) \ni (r;\sigma;\partial)\times m \longrightarrow \underline{D}\times(\underline{Z}^+\cup 0) \ni (r';\sigma';a\cdot\partial')\times[L(m)]$, where $a'=\exp M(m)$.

(In the later estimations, we write the parameter space $\underline{D}\times(\underline{Z}^+\cup 0)$ as $(\underline{R}^+\times\underline{R}^{+2})\times(\underline{Z}^+\cup 0)\times\underline{R}^{+2}$; see §2.2) Note that the correspondence:$(r;\sigma)\rightarrow(r';\sigma')$ is given by the first part $\tilde{E}'$ of $\tilde{E}(\in \underline{E}'_{p\ g})$(cf.Def.1.5). We call $\tilde{E}'$ also the first part of E. The correspondence:$(\underline{Z}^+\cup 0) \ni m \rightarrow (\underline{Z}^+\cup 0) \ni [L(m)]$ will concern the 'a.d part' of the cochains (cf.§2-2). We call this correspondence 'the a.d.part of E'. The map E is factored as follows:

$(2.4)'_1$
$$\begin{tikzcd}
(\underline{R}^+\times\underline{R}^{+2})\times(\underline{Z}^+\cup 0)\times\underline{R}^{+2} \arrow[r, "E"] \arrow[d] & (\underline{R}^+\times\underline{R}^{+2})\times(\underline{Z}^+\cup 0)\times\underline{R}^{+2} \arrow[d] \\
(\underline{R}^+\times\underline{R}^{+2})\times(\underline{Z}^+\cup 0) \arrow[r, "\tilde{E}'\times L"] & (\underline{R}^+\times\underline{R}^{+2})\times(\underline{Z}^+\cup 0)
\end{tikzcd}$$

(Here 'L ' denotes the a.d.part of E.)

---

*) If we replace the symbol 'C^q_r' in $(2.3)_7$ by 'C^q_l', then the corresponding fact to $(2.3)_8$ fails in genral; the right filtration is more suitable than the left one for application to the completion theory.

**) Writing $\partial'\in\underline{R}^{+2}_1$ as $(\partial'_1,\partial'_2)$, we set $a\cdot\partial':=(a\partial'_1,\partial'_2)$(cf.$(1.6)'_0$).

55

§ 2.2. Main results

1. Case of local variety. Letting the local variety $\underline{X}=(\underline{C}^n(z),U_0,X_0,h,..$ be as in Th. 1. 1(cf. also $(1.8)_0$), we fix a finite subset $\underline{f}=(f_j)_{j=1}^{s}\subset\Gamma(X_0,\underset{\sim}{O}_{X}$ satisfying (1) $f_j(P_0)=0(1\leqq j\leqq s)$, and (2) $f_j\not\equiv 0(X_{0},_{P_0},_{\nu})(1\leqq j\leqq s)$ for each irreducible component $X_{0},_{P_0},_{\nu}$ of the germ of X_0 at P_0. Here we generalize Th.1.1, Th. 1. 2 to the present d.p. cohomology theory , which is given to the pair $(\underline{X},\underline{f})$. As in §1, the underlying variety for the arguments here is $X=X_0-D$, where D =locus of h, The p.g. properties of cochains etc. are measured, as in §1, by the p. g. function $g_{\underline{X}}:=|h^{-1}|$, while the a. d. properties of the cochains will be measured by $\underline{f}$. As in §1 we use the symbol $\underline{X}$ also for X and $(\underline{X},g_{\underline{X}})$. When there is no fear of confusion, we use '$\underline{X}$' also for $(\underline{X},g_{\underline{X}},\underline{f})$ and $(\underline{X},\underline{f})$.

(i) D. p. parametrization. Here we generalize the p. g. parametrization in n.2, §1.2 to the present d.p. cohomology theory, First the parametrizatio of the coverings here is same as that in n.3, §1.2(cf.$(1.8)_2$):

$(2.5)_1$ $u_{\underline{X}}:\mathcal{M}_{\underline{X}}:=D_{\underline{X}}\times\underline{R}^{+}\times\underline{R}^{+2}\ni\mu=(P;r;\tau)\longrightarrow \mathrm{Cov}_0(\underline{X})_{p.g}\ni \underline{A}_{\mu}:=\underline{A}_{\tau}(\underline{X}_r(P))$, where we write D as $D_{\underline{X}}$. Also the manifold $\underline{X}_r(P)$ and its p. g. covering $\underline{A}_{\mu}$ are as in $(1.9)_2$.

Next, we form a product $\mathcal{T}_{\underline{X}}:=\mathcal{M}_{\underline{X}}\times(\underline{Z}^{+}\cup 0)\times\underline{R}_{1}^{+2}$, and, for an element $\underset{\sim}{H}\in\mathrm{Coh}(\underline{X})_{p}$ (Def. 1. 5), we define the following parametrization of sets of cochains:

$(2.5)_2$ $\mathcal{T}_{\underline{X}}:=\mathcal{M}_{\underline{X}}\times(\underline{Z}^{+}\cup 0)\times\underline{R}^{+2}\ni\tau=(\mu;m;\delta)\xrightarrow{C^q_{\underset{\sim}{H}}}C^q_1(\underline{A}_{\mu},\underline{f}^m\underset{\sim}{H};g)_{\delta}$ (cf. $(2.3)_4$).

(We define a parametrization $Z^q_{\underset{\sim}{H}}$ by changing 'C^q_1' to 'Z^q_1'(cf.$(2.3)_4$).) Then, we generalize the p. g. cochain collection in $(1.9)_3'$ as follows:

$(2.5)_3$ $\underline{C}^q_1(\underline{X},\underset{\sim}{H})^d_{p.g}:=C^q_{\underset{\sim}{H}}(\mathcal{T}_{\underline{X}})=\{C^q_1(\underline{A}_{\mu},\underline{f}^m\underset{\sim}{H};g)_{\delta};\ (\mu;m;\delta)\in\mathcal{T}_{\underline{X}}=\mathcal{M}_{\underline{X}}\times(\underline{Z}^{+}\cup 0)\times\underline{R}_{1}^{+2}\}$.

We call $\underline{C}_1^q(\underline{X},\underset{\sim}{H})_{p.g}^d$ the q-th $(\underline{f},g)$-d.p.cochain collection for $\underset{\sim}{H}$. We define $(\underline{f},g)$-d.p.cocycle collection $\underline{Z}_1^q(\underline{X},\underset{\sim}{H})_{p.g}^d$ by changing the symbol $C_{\underset{\sim}{H}}^q$ in $(2.5)_3$ to $Z_{\underset{\sim}{H}}^q$. Such collections contain all necessary sets of cochains in the d.p.uniform estimations in n.1. We will fix the p.g. sheaf $\underset{\sim}{H}$ as above in the remainder of § 2.

D.p. parametrization table

$$\begin{tikzcd}
\tau_{\underline{X}}=\mu_{\underline{X}}\times(\underline{Z}^+\cup 0)\times\underline{R}_1^{+2}\ni\tau=(\mu;m;\partial) \arrow[r, "C_{\underset{\sim}{H}}^q"] & C_1^q(\underline{X},\underset{\sim}{H})_{p.g}^d\ni C_1^q(\underline{A}_\sigma(\underline{X}_r(P)),\underline{f}^m\underset{\sim}{H};g)_\partial \\
\mu_{\underline{X}}=(D_{\underline{X}}\times\underline{R}^+\times\underline{R}_1^{+2})\ni\mu=(P;\sigma,r) \arrow[r, "u_{\underline{X}}"] & \mathrm{Cov}_0(\underline{X})_{p.g}\ni\underline{A}_\sigma(\underline{X}_r(P)).
\end{tikzcd}$$

(ii) Estimation data. We will use the d.p.c.estimation maps $E\in\underline{E}_{d.p}$ (Def.2.4) for the uniform estimations in n.1. As in §1.2 our uniform estimations will work for a subset of the parameter space $\tau_{\underline{X}}$:letting $D_{1,\underline{X}}$ be an open subset of $D_{\underline{X}}$, which contains P_0(=origin of $D_{\underline{X}},\underline{X},$,(cf.n.1,§1.2) we take an element $(\tilde{r},\tilde{\sigma},\tilde{m})\in\underline{R}^+\times\underline{R}_1^{+2}\times\underline{Z}^+$. We then form a subset $\mu_{\underset{\sim}{H}}=D_{1,\underline{X}}\times(0,\tilde{r})\times\underline{R}_{\tilde{\sigma}}^{+2}$ of $\mu_{\underline{X}}=D_{\underline{X}}\times\underline{R}^+\times\underline{R}_1^{+2}$ and$^{*)}$ $\tau'_{\underset{\sim}{H}}=\mu_{\underset{\sim}{H}}\times\underline{Z}_{\tilde{m}}^+\times\underline{R}_1^{+2}$ of $\tau_{\underline{X}}=\mu_{\underline{X}}\times(\underline{Z}^+\cup 0)\times\underline{R}_1^{+2}$. As in § 1.2 we call $\tau'_{\underset{\sim}{H}}$ restricted parameter space for $\underset{\sim}{H}$. We fix $\tau'_{\underset{\sim}{H}}$ in the remainder of § 2.

(ii) Now, using the sets of the cochains as in the table soon above (cf.also $(2.5)_2$), we generalize Th.1.1,Th.1.2 to the d.p.cohomology theory:

Theorem 2.1. (D.p. uniform estimation for Cech operator $\delta=\delta_{\underline{X}}$).

There is a d.p.c. estimation map $E_{\underset{\sim}{H}}\in\underline{E}_{d.p}(q>0)$, with which we have:

$(2.6)_1$ $s^*Z_1^q(\underline{A}_\sigma(\underline{\tilde{X}}_r(P)),\underline{f}^m\underset{\sim}{H})_\partial\subset\delta C_1^{q-1}(\underline{A}_{\sigma'}(\underline{\tilde{X}}_{r'}(P)),\underline{f}^{m'}\underset{\sim}{H})_{\partial'}$, where

$(2.6)_1'$ $(r';\sigma';m';\partial')=E_{\underset{\sim}{H}}(r;\sigma;m;\partial)$(cf.Def.2.4).

Theorem 2.2_1. (D.p uniform estimation for resolution of $\underset{\sim}{H}$).

There is a d.p.c.map $E_{\underset{\sim}{H}}\in\underline{E}_{d.p}(q\geqq 0)$, with which we have:

$(2.6)_2$ $s^*Z_1^q(\underline{A}_\sigma(\underline{\tilde{X}}_r(P)),\underline{f}^m\underset{\sim}{H})_\partial\subset\omega_{\underset{\sim}{H}}Z_1^q(\underline{A}_{\sigma'}(\underline{\tilde{X}}_{r'}(P)),\underline{f}^{m'}\underset{\sim}{O}_{\underline{X}}^k)_{\partial'}$, where

$(2.6)_2'$ $(r';\sigma';m';\partial')=E_{\underset{\sim}{H}}(r;\sigma;m;\partial)$

---

*) $\underline{R}_{\tilde{\sigma}}^{+2}:=\{\sigma\in\underline{R}^{+2};\ \sigma\geqq\tilde{\sigma}\}$ and $\underline{Z}_{\tilde{m}}^+=\{m\in\underline{Z}^+;m\geqq\tilde{m}\}$(cf. the end of the introduction of Chap. I). In Th. 2.2_2 we are concerned with the structure sheaf O_X, and we should understand that $\tau_{\underset{\sim}{H}}=\tau_{\underset{\sim}{O}_{\underline{X}}},\ldots$

and $\omega_{\underset{\sim}{H}}:\underset{\sim}{\underline{O}}{}^{k}_{\underline{X}} \to \underset{\sim}{H}$ is the first resolution of $\underset{\sim}{H}$(Def.1.5).*)

Note that the sheaves $\underline{f}^{m}\underset{\sim}{\underline{O}}{}^{k}_{\underline{X}}$ in $(2.6)_2$ are, in general, not free sheaves. In order to complete the resolution in Th.2.2_1 we give:

Theorem 2.2_2.(D.p.uniform estimation for $\{\underline{f}^{m}\underset{\sim}{O}_{\underline{X}}\}_{m=1}^{\infty}$).

There is an element $E_{\underline{X}} \in \underline{E}_{d.p}(q \geqq 0)$, with which we have:

$(2.6)_3$ $s^{*}Z^{q}_{\ell}(\underline{A}_{\sigma}(\underline{X}_{r}(P)),\underline{f}^{m}\underset{\sim}{O}_{\underline{X}})_{\partial} \subset F^{m'}Z^{q}_{\ell}(\underline{A}_{\sigma'}(\underline{X}_{r'}(P)),\underset{\sim}{O}{}^{s}_{\underline{X}})_{\partial'}$, where

$(2.6)_3'$ $(r',\sigma';m';\partial')=E_{\underline{X}}(r,\sigma;m;\partial)$,

and $F^{m}:\underset{\sim}{O}{}^{s}_{\underline{X}} \longrightarrow \underset{\sim}{O}_{\underline{X}}$ is the m-th standard homomorphism for $\underline{f}$(n.2, § 2.1).

In Th.2.1 ~ Th.2.2_2, the parameter $(P;r,\sigma;m;\partial)$ is in the restricted parameter space $\mathcal{T}'_{\underset{\sim}{H}}(=D_{r\underline{X}}\times(0,\tilde{r})\times\underset{\sim}{\underline{R}}{}^{+2}_{\sigma}\times\overset{(\tau=)}{\underline{Z}}{}^{+}_{\tilde{m}}\times\underline{R}^{+2}_{1})$. We will rewrite Th.2.1 in the following diagram(cf.also Fig.III,n.4,§ 1.2).

Fig.I.(D.p.vanishing properties for Cech operator)**)

$$\begin{tikzcd}
\mathcal{T}_{\underline{X}} \ni \tau \arrow[rr, "Z^{q}_{\underset{\sim}{H}}"] \arrow[dd] \arrow[dr, "E_{\underset{\sim}{H}}"] & & Z^{q}(\underline{X},\underset{\sim}{H})^{g}_{p.g} \supset Z^{q}_{\underset{\sim}{H}}(\tau) \arrow[dd, dashed] \arrow[dr, dashed] \arrow[drrr, "s^{*}"] \arrow[dr, phantom, bend left, "\circlearrowright"] & & & \\
& \mathcal{T}_{\underline{X}} \ni E_{\underset{\sim}{H}}(\tau) \arrow[rr, "C^{q-1}_{\underset{\sim}{H}}"] \arrow[dd, "E'_{\underset{\sim}{H}}"] & & C^{q-1}(X,\underset{\sim}{H})^{d}_{p.g} \arrow[r, "\delta"] \arrow[dd, dashed] & \delta C^{q-1}(E_{\underset{\sim}{H}}(\tau)) \arrow[r, phantom, "\supset"] & s^{*}Z^{q}_{\underset{\sim}{H}}(\tau) \\
\mu_{\underline{X}} \arrow[rr, "u_{\underline{X}}"] \arrow[dr] & & \mathrm{Cov}_{0}(\underline{X})_{p.g} \arrow[dr] \arrow[d, phantom, "\circlearrowleft"] & & & \\
& \mu_{\underline{X}} \arrow[rr, "u_{\underline{X}}"] & & \mathrm{Cov}_{0}(\underline{X})_{p.g} & &
\end{tikzcd}$$

For the proof of Th.2.1 ~ Th.2.2_2, see § 2.3. Also applications of these results will be given in n.3, n.4, § 2.2 and in § 3. As we will see in § 2.3, Th.2.2_2 concerns an open map property of KoSzul complexes, which relates to a cohomological generalization of Hilbert zero point theorem (cf Lemma 2.5;see also Introduction). Th.2.2_2 will also fill the gap between the d.p. and p.g. estimations in §2 and §1, and may be most basic among Th.2.1 ~ Th.2.2_2.

---

*) Also, in Th.2.1 ~ Th.2.2_2, we drop the term 'g' from the sets of the cochains(cf. also $(2.5)_2$).

**) The map $E'_{\underset{\sim}{H}}$ in Fig.I is the first part of $E_{\underset{\sim}{H}}$(cf. n.4, §2.1).

2. An affine analogue. Letting the affine variety $\underline{X}' \in \underline{C}^n(z)$ be as in Th.1.3, we take a finite set $\underline{f}'=(f'_j)_{j=1}^{s} \subset \Gamma(\underline{X}',\underset{\sim}{\mathcal{O}}_{\underline{X}'};g_{\underline{X}'})_{p.g}$, where the p.g. function $g_{\underline{X}'}=|z|+1$ is as in n.5, §1.2. Similarly to n.1, we use the symbol X' also for $(\underline{X}',g_{\underline{X}'})$, $(\underline{X}',\underline{f}')$ and $(\underline{X}',g_{\underline{X}'},\underline{f}')$. We generalize here Th.1.3 and Th.1.4. The set of the p.g. coverings: $Cov_0(X')_{p.g} := \{\underline{A}_\sigma(X');\sigma\in \underline{R}_1^{+2}\}$, where $\underline{A}_\sigma(\underline{X}')$ is the p.g. covering of $\underline{X}'$ of size σ, is as in Th.1.3, Th.1.4. Next we set $\mathcal{T}_{\underline{X}'} := \mu_{\underline{X}'} \times (\underline{Z}^+ \cup 0)$, with $\mu_{\underline{X}'} := \underline{R}_1^{+2}$. Then taking a p.g. sheaf $\underset{\sim}{H}' \in Coh(\underline{X}')_{p.g}$(cf. Def. 1.$4_1$) , our d.p. cochains in n.2 will be parametrized as follows:

$$(2.7)_1 \quad C^q_{\underset{\sim}{H}}: \mathcal{T}_{\underline{X}'} = \underline{R}_1^{+2} \times (\underline{Z}^+ \cup 0) \ni (\sigma;m) \longrightarrow C^q_1(A_\sigma(\underline{X}'),f^k\underset{\sim}{H};g_{X'})_{p.g}(\text{cf. } (2.3)_3)$$

Taking an element $(\tilde{\sigma};\tilde{m}) \in R_1^{+2} \times Z^+$, we set$^{*)}$ $\mathcal{T}' := R^{+2}_{\tilde{\sigma}} \times Z^{+}_{\tilde{m}}m$ Our d.p. estimation soon below will work for elements in $\mathcal{T}'$. Thirdly, our estimation maps here will be of the following form:

$$(2.7)_1' \quad E' : \underline{R}_1^{+2} \times (\underline{Z}^+ \cup 0) \ni (\sigma;m) \longrightarrow \underline{R}_1^{+2} \times (\underline{Z}^+ \cup 0) \ni (\underline{L}(\sigma), [L(m)]),$$ with an el-map $\underline{L}$ and a linear map $L=ct;c>0$.

We write the collection of all such maps as $E'_{d,p}$.

Theorem 2.3. (D.p. uniform estimation for Cech operator $\delta=\delta_{\underline{X}'}$).

For a suitable $E_{\underset{\sim}{H}} \in \underline{E}'_{d.p.}$ we have:

$$(2.7)_2 \quad s^* Z^q(\underline{A}_\sigma(\underline{X}'),\underline{f}'^{\tilde{m}}\underset{\sim}{H}')_{p.g} \subset \delta C^{q-1}(\underline{A}_{\sigma'}(\underline{X}'),\underline{f}'^{m'}\underset{\sim}{H}')_{p.g},$$ with $(\sigma';m')=\underline{E}_{\underset{\sim}{H}}(\sigma;m$

Theorem 2.4_1. (D.p. uniform estimation for resolution).

For a suitable $E_{\underset{\sim}{H}} \in \underline{E}'_{d.p}$ we have:

$$(2.7)_3 \quad s^* Z^q(\underline{A}_\sigma(X'),\underline{f}'^{\tilde{m}}\underset{\sim}{H}')_{p.g} \subset \omega_{\underset{\sim}{H}'} Z^q(\underline{A}_{\sigma'}(X'),\underline{f}'^{m'}\cdot\underset{\sim}{H}')_{p.g},$$ with $(\sigma';m')=E_{\underset{\sim}{H}}(\sigma;m$

where $E_{\underset{\sim}{H}}$ is the first resolution of $\underset{\sim}{H}$(Def.1.5)

---

*) According as we are concerned with Th.2.3, Th.2.4_1 or Th.2.4_2, $(\tilde{\sigma};\bar{m})$ depends on $(\underline{X},\underset{\sim}{H})$ or $\underline{X}$. Thus we should understand that $\mathcal{T}'=\mathcal{T}'_{\underset{\sim}{H}}$ or $=\mathcal{T}'_{\underline{X}}$, according to the thorems just above.

Theorem 2.4_2. (D.p.uniform estimation for $\{\underline{f}'^m \underset{\sim}{O}_{\underline{X}'}\}_{m=0}^{\infty}$). For a suitable $E_{\underline{X}'} \in \underline{E}'_{d.p}$ we have:

$$(2.7)_4 \quad s^* Z^q(\underline{A}_{\sigma}(\underline{X}'), \underline{f}'^m \underset{\sim}{O}_{\underline{X}'})_{p.g} \subset F'^{m'} Z^q(\underline{A}_{\sigma'}(\underline{X}'), \underset{\sim}{O}^s_{\underline{X}'})_{p.g}, \text{ with } (\sigma';m') = E'_{\underline{X}}(\sigma;m)$$

In the above the parameter $(\sigma;m)$ is in $\mathcal{T} = \underline{R}^{+2}_{\underset{\sim}{\sigma}} \times \underline{Z}^{+}_{m}$

For the proof of Th. 2.3, Th. $2.4_{1,2}$, see § 2.3.

3. Open map properties. Here we show the properties in the title for some p.g.filtered cpmplexes. For this we first set:

$$(2.8)_0 \quad \begin{cases} (\underline{X}^*, \underset{\sim}{H}^*, \underline{f}^*, g^*) := (X, \underset{\sim}{H}, \underline{f}, g_X) \text{ or } (\underline{X}', \underset{\sim}{H}', \underline{f}', g_{\underline{X}'}) \\ \mathcal{U}_{\underline{X}^*} := \mathcal{U}_{\underline{X}} := (0, \tilde{r}) \times \underline{R}^{+2}_{\sigma} \text{ or} = \mathcal{U}_{\underline{X}'} := \underline{R}^{+2}_{\sigma} . \end{cases} \quad \text{(cf. n.1, n.2, § 2.2)}$$

We regard $\mathcal{U}_{\underline{X}^*}$ as the direct set in the manner as in n.6, §1.2. For an element $\mu = (r;\sigma) \in \mathcal{U}_{\underline{X}} = (0, \tilde{r}) \times \underline{R}^{+2}_{\sigma}$ or $= \sigma \in \mathcal{U}_{\underline{X}'}$, we denote by A_{μ} the p.g. covering $\underline{A}_{\sigma}(\underline{X}_r(P))$ or $\underline{A}_{\sigma}(\underline{X}')$ (cf. Th. 2.1 and Th. 2.3). We generalize Def. 1.9 to the present d.p.cohomology theory;

Definition 2.5_1. By left $(g^*, \underline{f}^*)$-p.g.filtered Cech complex for $(\underline{A}_{\mu}, \underset{\sim}{H}^*)$, we mean the following filtered complex (cf. n.1, §2.1):

$$(2.8)_1 \quad 0 \rightarrow Z^0(\underline{A}_{\mu}, \underset{\sim}{H}^*)_{p.g} \xrightarrow{i} \underline{C}^0_l(\underline{A}_{\mu}, \underset{\sim}{H}^*)^d_{p.g} \xrightarrow{\delta} \cdots \underline{C}^q_l(\underline{A}_{\mu}, \underset{\sim}{H}^*)^d_{p.g} \longrightarrow ,$$

where we set:

$$(2.8)'_1 \quad \underline{C}^q_l(\underline{A}_{\mu}, \underset{\sim}{H}^*)^d_{p.g} := \{C^q_l(\underline{A}_{\mu}, \underline{f}^{*m} \underset{\sim}{H}^*)_{p.g}\}_{m=0}^{\infty} \quad (\text{cf } (2.3)_3),$$

and $i :=$ inclusion: $Z^0(\underline{A}_{\mu}, \underset{\sim}{H}^*)_{p.g} \hookrightarrow C^0(\underline{A}_{\mu}, \underset{\sim}{H}^*)_{p.g}$

Next we write $\underset{\sim}{H}^*$ in the form of $(1.4)_1$: $0 \rightarrow \underset{\sim}{O}^{k_p}_{\underline{X}^*} \xrightarrow{K_{p-1}} \cdots \xrightarrow{K_1} \underset{\sim}{O}^{k_1}_{\underline{X}^*} \xrightarrow{\omega_{\underset{\sim}{H}^*}} \underset{\sim}{H}^* \rightarrow 0$,

and we set:

$$(2.8)_2 \quad \underline{Z}^q_l(\underline{A}_{\mu}, \underset{\sim}{H}^*)^d_{p.g} := \{Z^q_l(\underline{A}_{\mu}, \underline{f}^{*m} \underset{\sim}{H}^*)_{p.g}\}_{m=0}^{\infty} .$$

(We define $\underline{Z}^q_l(\underline{A}_{\mu}, \underset{\sim}{O}^{k_j}_{\underline{X}^*})^d_{p.g}$ in the similar manner to the above.)

Definition 2.5$_2$. By *q-th left* $(g^*, \underline{f}^*)$-*p.g.-filtered resolution comple*: *for* $(\underline{A}_\mu, \underset{\sim}{H}^*)(q \geqq 0)$, we mean:

$$(2.8)_3 \quad 0 \longrightarrow \underline{Z}_1^q(\underline{A}_\mu, \underset{\sim}{O}_{X^*}^{p})^d_{p.g} \xrightarrow{K_{p-1}} \cdots \xrightarrow{K_1} \underline{Z}_1^q(\underline{A}_\mu, \underset{\sim}{O}_{X^*}^{1})^d_{p.g} \xrightarrow{\omega_{\underset{\sim}{H}^*}} \underline{Z}_1^q(\underline{A}_\mu, \underset{\sim}{H}^*)^d_{p.g} \longrightarrow 0$$

where the augumentation map is understood to be the zero map(n.1,§2.1). We write the filtered complexes in Def.2.5$_1$,$_2$ as $\underline{C}_1^*(\underline{A}_\mu, \underset{\sim}{H}^*)^d_{p.g}$ and $\underline{Z}_1^q(\underline{A}_\mu, \underset{\sim}{H}^*)^d_{p.g}$. The *right* $(\underline{f}^*, g^*)$-*p.g. filtered complexes* $\underline{C}^*(\underline{A}_\mu, \overset{*}{\underset{\sim}{H}})^d_{p.g}$ *and* $\underline{Z}_r^q(\underline{A}_\mu, \overset{*}{\underset{\sim}{H}})^d_{p.g}$ will be defined similarly. Then we have:

Lemma 2.1. (*Open map properties of the left p.g. filtered Cech and resolution complexes*). *The direct systems of the left p.g. filtered complexes* $\{\underline{C}_1^*(\underline{A}_\mu, \underset{\sim}{H}^*)_{p.g}\}_\mu$, $\{\underline{Z}^q(\underline{A}_\mu, \underset{\sim}{H}^*)^d_{p.g}\}_\mu$ *satisfy the open map property and are* μ*-exact(Def. 2.1)* ,*where* μ *runs through* $\widetilde{\mu}_{\underline{X}}$(cf.$(2.8)_0$).

*Proof.**) Let $E'_{\underset{\sim}{H}}: (\underline{R}^+ \times \underline{R}^{+2}) \longrightarrow (\underline{R}^+ \times \underline{R}^{+2})$ and $L_{\underset{\sim}{H}}: \underline{Z}^+ \to \underline{Z}^+ \cup 0$ be the first and a.d. parts of the d.p.c map $E_{\underset{\sim}{H}}$ in Th.2.1 (cf. also n.4,§ 2.1). Then, letting the element $\mu = (r;\sigma) \in \widetilde{\mu}_{\underline{X}} (\subset \underline{R}^+ \times \underline{R}_1^{+2})$ be as in Def.2.5$_1$, we have directly the following from $(2.6)_1$ and $(2.4)'_1$:

$(2.8)_4 \quad s^* Z_1^q(\underline{A}_\mu, \underline{f}^m \underset{\sim}{H})_{p.g} \subset \delta\, C_1^{q-1}(\underline{A}_{\mu'}, \underline{f}^{m'} \underset{\sim}{H})_{p.g}$, with $\mu' := (r'; \sigma') = E'_{\underset{\sim}{H}}(r;\sigma)$ and $m' = [L_{\underset{\sim}{H}}(m)] \quad (m \gg 0)$.

Comparing this with the numerical criterion $(2.1)'_2$ for the open map property, we have that condition for the Cech complex defined for $\underline{X} \in \underline{An}_{1a}$. The open map property for the Cech complex for $\underline{X}' \in \underline{Aff}$ follows from Th.2.3 Similarly to the above. Also the open map properties for the resolution complexes defined for $\underline{X} \in \underline{An}_{1a}$ and $\underline{X}' \in \underline{Aff}$ follow*) from Th.2.2$_1$, Th.2.4$_1$. Finally, the μ-exactness condition for the Cech and the resolution complexes follow from the 'p.g. exactnesses', Cor.1.2, Th.1.3,Th.1.4 , and we finish the proof of Lemma 2.1. q.e.d.

---

*) cf. also (2.14)' , n.4,§ 2.3, which is used in the proof of Th.2.2$_1$ and Th.2.4$_1$.

61

For the right d.p. filtration we have the similar fact to Lemma 2.1:

Lemma 2.2. (Open map properties of the right d.p. filtrations).

The direct systems of the right p.g. filtered complexes $C_r^*(\underset{\sim}{A}_\mu, \underset{\sim}{H}^*)^d_{p.g}$ and $\{Z_r^q(\underset{\sim}{A}_\mu, \underset{\sim}{H}^*)^d_{p.g}, \cancel{*}\}$ satisfy the open map property and are μ-exact.

By Lemma 2.1 and Prop 2.1_2, the following lemma insures Lemma 2.2.

Lemma 2.3. (Equivalence of the left and right d.p. filtrations)

The direct systems $\{C_l^*(\underset{\sim}{A}_\mu, \underset{\sim}{H}^*)^d_{p.g}\}_\mu$ and $\{C_r^*(\underset{\sim}{A}_\mu, \underset{\sim}{H}^*)^d_{p.g}\}_\mu$ as welll as $\{Z_l^*(\underset{\sim}{A}_\mu, \underset{\sim}{H}^*)^d_{p.g}\, \cancel{*}\}$ and $\{Z_r^q(\underset{\sim}{A}_\mu, \underset{\sim}{H}^*)^d_{p.g}\, \cancel{*}\}$ are equivalent(Def. 2.1_2).

Recalling the definitions of 'equivalence' and Prop 2.2_2, we see easily that the proof of the following leads to Lemma 2 3:

Lemma 2.3'. We have the inclusion:

$(2\ 9)_1$ $\quad s^* C_r^q(\underset{\sim}{A}_\mu, \underset{\sim}{f}^m \underset{\sim}{H}^*)_{p.g} \subset C_l^q(\underset{\sim}{A}_{\mu'}, \underset{\sim}{f}^{m'} \underset{\sim}{H}^*)_{p.g}$, with $m' = [L_{X^*}(m)]$, and a suitable parameter $\mu' \in \mu_{\underset{\sim}{X}^*}$, where $m \gg 0$ and $L_{\underset{\sim}{X}^*}$ is chosen*) in an independent manner from $(\mu; \overline{m})$.

The proof of Lemma 2.3' is given in § 4.2.

4. P.g complexes. First, letting the pair $(\underset{\sim}{A}_\mu, \underset{\sim}{H}^*)$ and $\underset{\sim}{f}^* \subset \Gamma(\underset{\sim}{X}^*, \underset{\sim}{O}_{X^*})$ be as in n.3, we define:

$(2\ 9)_1$ $\quad C^*(\underset{\sim}{A}_\mu, \hat{H}^*)_{p.g} := \varprojlim_m C^*(\underset{\sim}{A}_\mu, \underset{\sim}{H}^*)_{p.g} / C_r^*(\underset{\sim}{A}_\mu, f^m \underset{\sim}{H}^*)_{p.g}$, where $\hat{\underset{\sim}{H}}^* = \varprojlim_m \underset{\sim}{H}^* / \underset{\sim}{f}^{*m} \underset{\sim}{H}^*$ (Def. 2.3).

---

*) L_{X^*} is, as in Lemma 2.1, a linear map. Also the pair $(\underset{\sim}{A}_\mu, \underset{\sim}{H}^*)$ has the similar meaning to Lemma 2.1.

Denoting by $\underline{X}_P$ the germ of $\underline{X}$ at P, we generalize Def.1.9 to the completion theory as follows:

Definition 2.6. (1) By p.g.Cech complexes for $(\underline{X}_P,\widehat{\underset{\sim}{H}})$ and $(\underline{X}',\widehat{\underset{\sim}{H}}')$ we mean the ones:

$$(2.9)_3 \quad C^*(\underline{X}_P,\widehat{\underset{\sim}{H}})_{p.g}:=\varinjlim_{\tilde{\mu}} C^*(\underline{A}_{\tilde{\mu}},\widehat{\underset{\sim}{H}})_{p.g}\ ,\ C^*(\underline{X}',\widehat{\underset{\sim}{H}}')_{p.g}:=\varinjlim_{\tilde{\mu}'} C^*(\underline{A}_{\tilde{\mu}'},\widehat{\underset{\sim}{H}}')_{p.g},$$

where $\tilde{\mu},\tilde{\mu}'$ run through $\tilde{\mathcal{U}}_{\underline{X}}\ ,\tilde{\mathcal{U}}_{\underline{X}'}$.

(2) By q-th p.g.resolution complexes for $(\underline{X}_P,\widehat{\underset{\sim}{H}})$ and $(\underline{X}',\widehat{\underset{\sim}{H}}')(q\geqq 0)$, we mean the complexes(cf.Def.2.5_2):

$$(2.9)_4\begin{cases} 0\rightarrow Z^q(\underline{X}_P,\underset{\sim}{O}_{\underline{X}}^{k_p})_{p.g}\xrightarrow{\hat{K}_{p-1}}--\xrightarrow{\hat{K}_1}Z^q(\underline{X}_P,\underset{\sim}{O}_{\underline{X}}^{k_1})_{p.g}\rightarrow Z^q(\underline{X}_P,\underset{\sim}{H})_{p.g}\xrightarrow{\frown}0\\ 0\rightarrow Z^q(\underline{X}',\underset{\sim}{O}_{X'}^{k_p})_{p.g}\xrightarrow{\hat{K}'_{p-1}}--\xrightarrow{\hat{K}'_1}Z^q(\underline{X}',\underset{\sim}{O}_{\underline{X}'}^{k_1})_{p.g}\rightarrow Z^q(\underline{X}',\underset{\sim}{H}')_{p.g}\rightarrow 0\end{cases}$$

(cf.$(2.8)_3$,Def.2.5_2), where $\hat{K}_{p-1},\ldots,\hat{\omega}_{\underset{\sim}{H}},\ldots$ are the completions of $K_{p-1},\ldots,\omega_{\underset{\sim}{H}}$.

Now, using the above p.g.complexes, we generalize Th:1.5,Th.1.6 to the completion theory as follows: first denoting by $H^q(\underline{X}_P,\hat{H})_{p.g},\ldots$ the q-th cohomology group of $C^*(\underline{X}_P,\hat{H})_{p.g},\ldots$, we have the following generalization of Th.1.5 from Lemma 2.3 and Prop.2.2_1:

Theorem 1.5. $H^q(\underline{X}_P,\widehat{\underset{\sim}{H}})_{p.g}\cong 0$ and $H^q(\underline{X}',\widehat{\underset{\sim}{H}}')_{p.g}\cong 0(q\geqq 1)$.

Also, applying Lemma 2.3 and Prop.2.1_1 to the p.g.complexes in $(2.9)_4$, we have the following generalization of Lemma 1.1:

Lemma 2.4. The p.g.complexes in $(2.9)_4$ are exact.

Finally we will determine the structure of $H^0(\underline{X}_P,\widehat{\underset{\sim}{H}})_{p.g}$, $H^0(\underline{X}',\widehat{\underset{\sim}{O}}_{\underline{X}'})_{p.g}$.

For this, letting the sheaves $\underset{\sim}{\mathcal{O}}_{X_0}(*D)$ of meromorphic functions over X_0 and $\underset{\sim}{\mathcal{O}}_{X',\mathrm{alg}}$ of (algebraic) regular functions over $\underline{X}'_{\mathrm{alg}}$ be as in Th.1.6_1 (cf. also $(1.13)'_7$) . Then we have:

Theorem 2.6_1. There are natural isomorphisms from the meromorphic and algebraic completions to the p.g.completions:

$$(2.9)_5 \quad \hat{\theta}_P:\hat{\underset{\sim}{\mathcal{O}}}_{X_0}(*D)_P \longrightarrow H^0(\underline{X}_P,\hat{\underset{\sim}{\mathcal{O}}}_{\underline{X}})_{p.g} \;,\; \hat{\theta}_{\underline{X}'}:\Gamma(\underline{X}'_{\mathrm{alg}},\hat{\underset{\sim}{\mathcal{O}}}_{\underline{X}',\mathrm{alg}})\longrightarrow H^0(\underline{X}',\hat{\underset{\sim}{\mathcal{O}}}_{\underline{X}'})_{p.g}$$

where the left sides are as follows:*)

$$(2.9)'_5 \quad \hat{\underset{\sim}{\mathcal{O}}}_{X_0}(*D):=\varprojlim_m \underset{\sim}{\mathcal{O}}_{X_0}(*D)/F^m\underset{\sim}{\mathcal{O}}^s_{X_0}(*D),\; \hat{\underset{\sim}{\mathcal{O}}}_{X',\mathrm{alg}}:=\varprojlim_m \underset{\sim}{\mathcal{O}}_{\underline{X}',\mathrm{alg}}/F'^m\underset{\sim}{\mathcal{O}}_{\underline{X}',\mathrm{alg}}.$$

Proof . We prove the first isomorphism in $(2,9)_5$ .The second is proven similarly. First, from the isomorphism $\theta_P:\underset{\sim}{\mathcal{O}}_{X_0}(*D)_P \xrightarrow{\sim} H^0(\underline{X}_P,\underset{\sim}{\mathcal{O}}_{\underline{X}})_{p.g}$, we see that the following natural homomorphism is an isomorphism:

(a) $\hat{\theta}_P:\hat{\mathcal{O}}_{X_0}(*D)_P\longrightarrow \varinjlim_{\mu}(\varprojlim_m Z^0(\underline{A}_\mu(P),\underset{\sim}{\mathcal{O}}_{\underline{X}})_{p.g}/F^mZ^0(\underline{A}_\mu(P),\underset{\sim}{\mathcal{O}}^s_{\underline{X}})_{p.g})$, where we write $\underline{A}_\mu:=\underline{A}_\sigma(\tilde{\underline{X}}_r(P))$ as $\underline{A}_\mu(P)$ (cf.also $(2.8)_0$) .

On the otherhand, $(2.9)'_4$, Th.2.2_2 and Lemma 2.3 imply that

(b) $H^0(\underline{X}_P,\underset{\sim}{\mathcal{O}}_{\underline{X}})_{p.g} \cong$ right side of (a),

and we have $(2,9)_5$. q.e.d.

Finally, applying Lemma 2.4 to Th.1.6_1, we generalize Th.1.6_2 to the completions $\hat{H}$,$\hat{H}'$ as follows:

Theorem 2.6_2. The following complexes are exact:

$$(2.9)_6 \quad \begin{tikzcd} \cdots \arrow[r] & \hat{\mathcal{O}}_{\underline{X}}(*D)_P^{k_2} \arrow[r, "\hat{K}_1"] & \hat{\mathcal{O}}_{\underline{X}}(*D)_P^{k_1} \arrow[r, "\hat{\omega}"] & \hat{H}^0(\underline{X}_P,\hat{\mathfrak{f}})_{p.g} \arrow[r] & 0 \\ \cdots \arrow[r] & \Gamma(\underline{X}'_{\mathrm{alg}},\hat{\mathcal{O}}^{k_2}_{\underline{X}',\mathrm{alg}}) \arrow[r, "\hat{K}'_1"] & \Gamma(\underline{X}'_{\mathrm{alg}},\hat{\mathcal{O}}^{k_1}_{\underline{X}',\mathrm{alg}}) \arrow[r, "\hat{\omega}'"] & \hat{H}^0(\underline{X}',\hat{\mathcal{O}}_{\underline{X}'})_{p.g} \arrow[r] & 0 \end{tikzcd}$$

Th.2.5 and Th.2.6 are a generalization of Th.1.5 and Th.1.6 as well as our anaologue of Th.A.B of H.Cartan in the p.g.cohomology theory in the completion. We give an application of Th.1.5,Th.2.6 to the analytic de Rham theory in §3.

§ 2.3. Key lemmas

Here we give key lemmas for Th.2.1 ~ Th.2.4, which concerns the open map properties of $\underset{\sim}{\mathcal{O}}_{\underline{X}^*}$, where $\underline{X}^*=\underline{X}$ or $\underline{X}'$ (cf.Lemma 2.5 ~ Lemma 2.7). Also, using those lemmas, we prove Th.2.1 ~ Th.2.4.

1. Koszul complexes. In our proof of Th.2.1 ~ Th.2.4, we will take Th.2.2$_2$ and Th.2.4$_2$, which concerns the a.d.properties of $\mathcal{O}_{X^*}$, as the starting point(cf.n.3, n.4). Recalling that the a.d.properties in Th.2.2,Th.2.4$_2$ are measured by the powers of $\underline{f}^*:=\underline{f}$ or $\underline{f}'$, we first attach to $\underline{f}^{*m}$ what we call m-th Koszul complex for $\underline{f}^*$:

$$(2.10)_0 \quad 0 \longrightarrow \underset{\sim}{\mathcal{O}}_{X^*} \xrightarrow{F_0^{*m}} \underset{\sim}{\mathcal{O}}_{X^*}^{s} \longrightarrow \cdots \underset{\sim}{\mathcal{O}}_{X^*}^{\binom{s}{p}} \xrightarrow{F_p^{*m}} \cdots \underset{\sim}{\mathcal{O}}_{X^*}^{s} \xrightarrow{F_{s-1}^{*m}} \underset{\sim}{\mathcal{O}}_{X^*} \longrightarrow 0.$$

Here the $\underset{\sim}{\mathcal{O}}_{X^*}$-homomorphism F_p^{*m} is given, as usual, in terms of the exterior product as follows*): for a point $Q \in X^*$, let Ω_Q^p denote $\underset{\sim}{\mathcal{O}}_Q$ $(:=\underset{\sim}{\mathcal{O}}_{X^*,Q})$-module consisting of differential forms of degree q with coefficients in $\underset{\sim}{\mathcal{O}}_Q$. Letting x be a (formal) indeterminate, we denote by i_Q the identification: $\underset{\sim}{\mathcal{O}}_Q^{\binom{s}{p}} \ni \varphi = (\varphi_J)_J \longrightarrow \Omega_Q^p \ni \sum_J \varphi_J dx_J$, where J exhausts all indices $J=(j_1<\cdots<j_p)$. Then F_p^m is defined by: $i_Q F_p^{*m} = \wedge\omega_m \cdot i_Q$, where

---

*) cf. J.P.Serre [ ].

where we set $\omega_m := \sum_{j=1}^{s} f_j^m dx_j$ and $\wedge$ denotes the symbols of the exterior product. Noting that $F_{s-1}^m = F^m$(cf. n.2, § 2.1)*), we use the Koszul complexes in $(2.10)_0$, F^m in sybbol, for analyays of of the sheaves $f^{*m}O_{X^*}(=F^{*m}O_{X^*}^s)$ (cf. n.2 soon below. The lemma in n. 2, Lemma 2.5, will be our key facts for the proof of Th. 2.2_2 ,Th.2.4_2, which concern the sheaves $f^m O_X$. In later arguments we use the symbols $F^m, F_p^m, \ldots$ or $F'^m, F_p'^m, \ldots$ for F^{*m}, F_p^{*m}, according as we are concerned with $X^* = X$ or X'.)

2. <u>Open map property for F^*</u>. Letting the parameter spaces $\mathcal{M}_X (\subset D_X \times R_1^{+2} \times R_1^{+2})$ and $\mathcal{M}_{X'} := R^{+2}$ be as in Th. 2.2_2 , Th.2.4_2(cf. also $(2.5)_1$, $(2.7)_1$), we form a product $\lambda_X := \mathcal{M}_X \times R_1^{+2} (\subset D_X \times R^+ \times R_1^{+2} \times R_1^{+2})$. Also we take a linear

---

*) Precisely the homomorphisms F_{s-1}^m and F^m are: $O_X^s \ni (\varphi_j)_{j=1}^s \longrightarrow O_X^s \ni \sum_{j=1}^s (-1)^j f_j^m \varphi_j$ and $\sum_{j=1}^s \varphi_j \cdot f_j^m$. This difference of the signatures does not cause differences for the applications of the results for F_{s-1}^{*m} .

have the similar meaning to Th.2.2_2, Th.2.4_2. Also we take a linear function $L_{0,\underline{X}^*}=c_{0,\underline{X}^*}t;c_{0,\underline{X}^*}>0$. Then we have:

Lemma 2.5. (Open map property for $\underline{F}^*$). Choose suitable d.p. estimation maps*) $E_{\underline{X}}\in \underline{E}_{d.p}$ and $E_{\underline{X}'}\in \underline{E}'_{d.p}$. Then, for each $(\tilde{m},m)\in Z^+\times Z^+$ satisfying $\tilde{m}>L_{0,\underline{X}^*}(m)$, we have the following inclusion ($1\leq s<p$):

$$(2.10)_1 \begin{cases} s^*(C^q(\underline{A}_\alpha(\underline{X}_r(P)),\underline{f}^{\tilde{m}}\underset{\sim}{O}_{\underline{X}}^{(\overset{s}{p})})_\delta\cap(F_p^m)^{-1}(0))\subset F_{p-1}^m C^q(\underline{A}_{\alpha'}(\underline{X}_{r'}(P)),\underline{f}^{\tilde{m}'}\underset{\sim}{O}_{\underline{X}}^{(p\overset{s}{-}1)})_{\delta'} \\ s^*(C^q(\underline{A}_\alpha(\underline{X}),\underline{f}^{\tilde{m}}\underset{\sim}{O}_{\underline{X}'}^{(\overset{s}{p})})_{p.g}\cap(F_p'^{m})^{-1}(0))\subset F'^m_{p-1}C^q(\underline{A}_{\alpha'}(\underline{X}),\underline{f}'^{\tilde{m}'}\underset{\sim}{O}_{\underline{X}'}^{(p\overset{s}{-}1)})_{p.g} \end{cases},$$

with $(r';\sigma';\tilde{m}';\delta')=E_{\underline{X}}(r;\sigma;\tilde{m};\delta)$ and $(\sigma';\tilde{m}')=E_{\underline{X}'}(\sigma;\tilde{m})$. Here the parameteres $(P;r;\sigma;\delta)$ are in $\Lambda_{\underline{X}}(\subset D_{1,\underline{X}}\times\underline{R}^+\times\underline{R}_1^{+2}\times\underline{R}_1^{+2})$ and σ is in $\Lambda_{\underline{X}'}(\subset\underline{R}_1^{+2})$.

If we fix an element $m\in\underline{Z}^+$, which defines the homomorphisms F_p^m, then Lemma 2.5 insures the open map property for F_p^m. As we will see soon below, Lemma 2.5 plays the most basic role in getting the d.p. uniform estimations in §2 from the p.g. estimations in §1. Also Lemma 2.5 will concern a cohomoloigcal genralization of Hilbert zero point theorem(cf. part B, §4.1). Lemma 2.5 will play the most important roles in the lemmas given in §2.3.

3. Here we will prove the following implication:

Lemma 2.6. (Reduction of d.p. uniform estimations to p.g. uniform estimations).

$$(2.10)_2 \quad \left\{\frac{\text{Th. 1.1}}{\text{Th. 1.3}}\right\}+\text{Lemma 2.5 for}\left\{\frac{\underline{F}}{\underline{F}'}\right\}\longrightarrow\left\{\frac{\text{Th. }2.2_2}{\text{Th. }2.4_2}\right\}.$$

---

*) cf. Def. 2.4 and $(2.7)_1'$

6 87

Precisely, in the above implication, we use Th.1.1 and Th.1.3, applied to the structure sheaves $\underset{\sim}{O}_{\underline{X}}$ and $\underset{\sim}{O}_{\underline{X}}'$. Note that Th.1.1 and Th.1.3 do not concern the a.d.properties of $\underset{\sim}{O}_{\underline{X}}$,.., while those properties are the basic factor in Th.2.2_2 and Th.2.4_2.; we rewrite Lemma 2.6 in the following symbolical form:

$(2.10)_2'$ $\underline{\text{p.g.uniform estimation for } \underset{\sim}{O}_{X^*}} \xrightarrow{\text{open map property for } \underline{F}^*} \underline{\text{d.p uniform estimation for } \underset{\sim}{O}_{X^*}}$.

We prove Lemma 2.6 in three steps. First we introduce a type of auxiliarly estimation maps, which is used to fill the gap between the estimations in Th.2.2_2,Th.2.4_2 and in Th.1.1,Th.1.3.

(i) Pre d.p.c.map. Denote by $\underline{D},\underline{D}'$ the parameter space $(\underline{R}^+\times\underline{R}^{+2})\times\underline{Z}^+\times\underline{R}^{+2}$ and $(\underline{R}^+\times\underline{R}^{+2})\times\underline{R}^{+2}$, on which the d.p.c. and p.g.c.maps operate. Then we make

Definition 2.5. By a pre d.p.c map we mean such a map:

$(2.11)_0$ $E:\underline{D}\to\underline{D}'$, where E is written as $E=\mathrm{pr}\cdot\tilde{E}$, with a d.p.c. map $\tilde{E}$(Def.2.4) and the projection $\mathrm{pr}:\underline{D}\to\underline{D}'$.

Fig. I

$$\begin{tikzcd} \underline{D} \arrow[r, "\tilde{E}"] \arrow[dr, "E"'] & \underline{D} \arrow[d, "\mathrm{pr}"] \\ & \underline{D}' \end{tikzcd}$$

Take p.g.c. and d.p.c. maps E_1,E_2. Then, for each $\tau=(r;\sigma;m;\alpha)\in(0,1)\times\underline{R}^{+2}\times\underline{Z}^+\times\underline{R}_1^{+2}$ and the pre d.p.c.map E as in $(2.11)_0$, we have:

$(2.11)_0'$ $\overline{E}_1(\tau)\succ E_1\cdot E(\tau)$, and $\overline{E}_2(\tau)\succ E\circ E_2(\tau)$, with suitable pre d.p.c maps $\overline{E}_1,\overline{E}_2$.(For the order $\succ$, see $(1.6)_3'$.)

(ii) Letting the parameter space $\Lambda_{\underline{X}^*}'$ and the linear function $L_{0,\underline{X}^*}$ be as in Lemma 2.5, we check that the symbol 'C^q' in Lemma 2.5 is changed by 'Z^q'(by using the pre d.p.c.map instead of the d.p.c.map).(In Lemma 2.6' soon below, the parameter $(P;r;\sigma;\alpha)$ or $\sigma\in\Lambda_{\underline{X}^*}'$ is as in Lemma 2.5. Also the elements $(\check{m},m)\in\underline{Z}^+\times\underline{Z}^+$ satisfies:$\tilde{m}>L_{0,\underline{X}^*}(m)$.)

Lemma 2.6'. (1) For a suitable pre d.p.c. map $\underline{E}'_{\underline{X}}$ we have $(1 \leqq p < s)$:

$(2.11)_1$ $s^*(Z^q(\underline{A}_\sigma(\underline{\tilde{X}}_r(P)), \underline{f}^{\tilde{m}}\underset{\sim}{O}_{\underline{X}}^{\binom{s}{p}})_\delta \cap (F^m_p)^{-1}(0)) \subset F^m_{p-1}Z^q(\underline{A}_{\sigma'}(\underline{\tilde{X}}_{r'}(P)), \underset{\sim}{O}_{\underline{X}}^{(p^s-1)})_{\delta'}$,

with $(r';\sigma';\delta') = E'_{\underline{X}}(r;\sigma;\tilde{m};\delta)$.

(2) For a suitable el-map $\underline{L}_{\underline{X}'}$ we have:

$(2.11)_2$ $s^*(Z^q(\underline{A}_\sigma(\underline{X}'), \underline{f}'^{\tilde{m}}\underset{\sim}{O}_{\underline{X}'}^{\binom{s}{p}})_{p.g} \cap (F'^m_p)^{-1}(0)) \subset F'^m_{p-1}Z^q(\underline{A}_{\sigma'}(\underline{X}'), \underset{\sim}{O}_{\underline{X}'}^{(p^s-1)})_{p.g}$,

with $\sigma' = \underline{L}_{\underline{X}'}(\sigma)$.

Proof. The proof of (1),(2) is parallel. We prove only (1). For this we first remark that, by applying Th.1.1 to the right side of $(2.11)_1$, we have:

$(2.11)'_1$ s^*(left side of $(2.11)_1$) $\subset \delta F^m_{p-1}C^{q-1}(\underline{A}_{\sigma'}(\underline{\tilde{X}}_{r'}(P)), \underset{\sim}{O}_{\underline{X}}^{(p^s-1)})_{\delta'}$ $(q \geqq 1)$,

where $(r';\sigma';\delta') = E''_{\underline{X}}(r;\sigma;\tilde{m};\delta)$, with a pre d.p.c. map $E''_{\underline{X}}$, which is determined by $\underline{E}'_{\underline{X}}$.

(Remark that the estimation in Th.1.1 is given by a p.g.c. map; and from $(2.11)_0$, we have $(2.11)'_1$.) Now, using $(2.11)'_1$, the proof of Lemma 2.6' is given inductively on p: if p=1, then, remarking that $F^m_0 : \underset{\sim}{O}_{\underline{X}} \to \mathcal{O}^s_{\underline{X}}$ is injective, we have $(2.11)_1$ directly from Lemma 2.5. Assume that $p \geqq 2$ and that $(2.11)_1$ (and so $(2.11)'_1$) holds for $\tilde{p} < p$. Take an elemenet φ from the left side of $(2.11)_1$, and we write φ as $\varphi = F^m_{p-1}\varphi'$, with $\varphi' \in C^q(\underline{A}_{\sigma'}(\underline{\tilde{X}}_{r'}(P)), \underline{f}^{\tilde{m}'}\underset{\sim}{O}_{\underline{X}}^{(p^s-1)})_{\delta'}$. Here $(r';\sigma';\tilde{m}';\delta') = \tilde{E}_{\underline{X}}(r;\sigma;m;\delta)$, with a d.p.c. map $\tilde{E}_X$ as in Lemma 2.5. Applying $(2.11)'_1$ for $\tilde{p}=p-1$ to $\delta\varphi'$, we have:

(a) $s^*\varphi' \in F^m_{p-2}C^q(\underline{A}_{\sigma''}(\underline{\tilde{X}}_{r''}(P)), \underset{\sim}{O}_{\underline{X}}^{(p^s-2)})_{\delta''} + Z^q(\underline{A}_{\sigma''}(\underline{\tilde{X}}_{r''}(P)), \underset{\sim}{O}_{\underline{X}}^{(p^s-1)})_{\delta''}$, with $(r'';\sigma'';\delta'') = \bar{E}_{\underline{X}}(r';\sigma';\tilde{m}';\delta')$, where the pre d.p.c. map $\bar{E}_{\underline{X}}$ is determined by the maps $\tilde{E}_{\underline{X}}$ just above and $E'_{\underline{X}}$ in $(2.11)_1$ (cf also $(2.11)_0$).

Finally, operating F^m_{p-1} to the both sides of (a), we have $(2.11)_1$. q.e.d.

(iii) Proof of Lemma 2.6. We derive Th.2.2_2 from $(2.11)_1$, Lemma 2.6. The proof of Th.2.4_2 is given by using Lemma 2.6' in the similar manner. Letting the linear function $L_0(t)=L_{0,\underline{X}}(t)$ be as in Lemma 2.5, Lemma 2.6, we take a suitable linear map $L(t)$. Then we have the following for each integer $m \gg 0$:

(a) $m-m' > L_0(m')$, with $m' = [L(m)]$.

Now take an element $\varphi \in Z^q(\underline{A}_\mu(P), \underline{f}^m \mathcal{O}_{\underline{X}})_\partial$, where $\underline{A}_\mu(P) := \underline{A}_{\sigma}(\underline{\tilde{X}}_r(P))$(cf. Th.2.$2_2$) Then setting $m' := [L(m)]$ and $\tilde{m} := m-m'$, one can write $\varphi = F^m_{s-1}\varphi'$, with $\varphi' \in C^q(\underline{A}_\mu(P), \underline{f}^{\tilde{m}} \mathcal{O}^s_{\underline{X}})_\partial$. By (a) one can apply Lemma 2.6' to $\delta\varphi'$, and we have:

(b) $s^*\varphi' \in F^m_{s-2} C^q(\underline{A}_{\sigma'}(\underline{\tilde{X}}_{r'}(P)), \mathcal{O}^{(p^s 2)}_{\underline{X}})_{\partial'} + Z^q(\underline{A}_{\sigma'}(\underline{\tilde{X}}_{r'}(P)), \mathcal{O}^s_{\underline{X}})_{\partial'}$, with $(r';\sigma';\partial') = E_{\underline{X}}(r;\sigma;\tilde{m};\partial)$. (Here $E_{\underline{X}}$ is the pre d.p.c.map as in Lemma 2.6'.)

Operating F^m_{s-1}, to the both sides, we have:

(c) $s^*\varphi \in F^m_{s-1} Z^q(\underline{A}_{\sigma'}(\underline{\tilde{X}}_{r'}(P)), \mathcal{O}^s_{\underline{X}})_{\partial'}$.

On the otherhand we see easily that the correspondence:

(d) $(r;\sigma;m;\partial) \longrightarrow (r';\sigma';m';\partial')$

defines a d.p.c.map, which is determined by E_X and $L(t)$. It is clear that (c) and (d) insure Th.2.2_2. q.e.d.

By Lemma 2.6 we see that the open map property for $\underline{F}^*$ in Lemma 2.5 suffices to get Th.2.2_2 and Th.2.4_2 from the p.g. uniform estimations in §1. In n.4 we give a lemma, which is used to get Th.2.2_1, Th.2.1 from Th.2.2_2(resp. Th 2.3, Th.2.4_1 from Th.2.4_2).

4. Letting the p.g sheaves $\underset{\sim}{H}, \underset{\sim}{H}'$ be as in Th. 2.1, Th. 2.3, we assume that $\underset{\sim}{H}, \underset{\sim}{H}'$ are in $\mathrm{Coh}'(\underline{X})_{p.g}$, $\mathrm{Coh}'(\underline{X}')_{p.g}$ (cf. $(1.4)_9$). Thus $\underset{\sim}{H} \subset \underset{\sim}{O}^k_{\underline{X}}$, $\underset{\sim}{H}' \subset \underset{\sim}{O}^{k'}_{\underline{X}'}$ with a suitable $k \in \underline{Z}^+$. Then letting the parameter spaces $\tau_{\underset{\sim}{H}}$, $\tau_{\underset{\sim}{H}'}$ and the estimation maps $E_{\underset{\sim}{H}} \in \underline{E}_{d.p}$ and $E'_{\underset{\sim}{H}'} \in \underline{E}'_{d.p}$ have the similar meanings to Th. 2.1, Th. 2.3, we have:

<u>Lemma 2.7.</u> <u>We have the following inclusions:</u>

$$(2.12)\begin{cases} s^*(C^q(\underline{A}_\sigma(\underline{\tilde{X}}_r(P)), \underline{f}^m \underset{\sim}{O}^k_{\underline{X}})_\partial \cap C^q(\underline{A}_\sigma(\underline{\tilde{X}}_r(P)), \underset{\sim}{H})) \subset \omega_{\underset{\sim}{H}} C^q(\underline{A}_{\sigma'}(\underline{\tilde{X}}_{r'}(P)), \underline{f}^{m'} \underset{\sim}{O}^{k'}_{\underline{X}})_{\partial'} \\ s^*(C^q(\underline{A}_\sigma(\underline{X}'), \underline{f}^m \underset{\sim}{O}^k_{\underline{X}})_{p.g} \cap C^q(\underline{A}_\sigma(\underline{X}'), \underset{\sim}{H})) \subset \omega_{\underset{\sim}{H}'} C^q(\underline{A}_{\sigma'}(\underline{X}'), \underline{f}^{m'} \underset{\sim}{O}^{k'}_{\underline{X}'})_{p.g} \end{cases},$$

<u>where</u> $(r';\sigma';m';\partial') = E_{\underset{\sim}{H}}(r;\sigma;m;\partial)$ <u>and</u> $(\sigma';m') = E'_{\underset{\sim}{H}'}(\sigma;m)$, <u>and the parameters</u> $(P;r;\sigma;m;\partial)$ <u>and</u> $(\sigma;m)$ <u>are in</u> $\tau_{\underset{\sim}{H}}(\subset \underline{D} \times \underline{R}^+ \times \underline{R}_1^{+2} \times \underline{Z}^+ \times \underline{R}_1^{+2})$ <u>and in</u> $\tau_{\underset{\sim}{H}'}(\subset \underline{R}_1^{+2} \times \underline{Z}^+)$. <u>Moreover,</u> $\omega_{\underset{\sim}{H}} : \underset{\sim}{O}^{k'}_{\underline{X}} \longrightarrow \underset{\sim}{O}^k_{\underline{X}}, \ldots$ <u>are the first resolution of</u> $\underset{\sim}{H}, \ldots$ (Remark 1.1)

We prove Lemma 2.7 in §4.2. Note that Lemma 2.7 concerns the exact complexes, and is of Artin-Rees theorem type. The role of Lemma 2.7 in our d.p. estimations in § 2 is similar to that of the above theorem in the completions of rings (cf. [12]). Here we check the implication:

(2.13) Th. 2.2_2 + Lemma 2.7 $\longrightarrow$ Th. $2.2_1 \longrightarrow$ Th. 2.1 (and Th. 2.4_2 + Lemma 2.7 $\longrightarrow$ Th. $2.4_1 \longrightarrow$ Th. 2.3).

(From a simple observation,*) we see that Th. $2.2_{1,2}$, together with Th. 1.1, imply Th. 2.1. Here we check the first implication in (2.13).) The key fact for (2.13) is the following inclusion, which is similar to $(2.11)'_1$:

$$(2.14)\quad s^*(Z^q(\underline{A}_\sigma(\underline{\tilde{X}}_r(P)), \underline{f}^m \underset{\sim}{O}^k_{\underline{X}})_\partial \cap Z^q(\underline{A}_\sigma(\underline{\tilde{X}}_r(P)), \underset{\sim}{H})) \subset \omega_{\underset{\sim}{H}} Z^q(\underline{A}_{\sigma'}(\underline{\tilde{X}}_{r'}(P)), \underline{f}^m \underset{\sim}{O}^{k'}_{\underline{X}})_{\partial'}.$$

(This follows using the similar inductive arguments to n. 3 on the length of $\underset{\sim}{H}$.) Actually, remarking that $Z^q(\underline{A}_\sigma(\underline{X}_r(P)), \underline{f}^m \underset{\sim}{H})_\partial \subset$ (left side of (2.14)), we easily have Th. 2.2_1 from (2.14), and we also have the first implication in (2.13).

---

*) As in n.3, we consider only the case of the local variety $\underline{X} \in \underline{An}_{1a}$.

We will conclude § 2.3 by the following proposition.

Proposition 2.3. For the proof of Th.2.1 ~ Th.2.4, it suffices to prove Lemma 2.5, Lemma 2.7 and Lemma 2.3.

For the proof of these lemmas, see §4.2.

Remark 2.2. Here we make some remarks, which are used in the later arguments. First, we saw the following implication in $(2.10)_2$: we recall that

$(2.15)_1$ Lemma 2.5 ⟶ $\text{Th}.2.2_2$.

Next we remark that the open map property in Lemma 2.5 is given in terms of the symbol 'C^q'. Using the similar (syzygy) arguments to Lemma 2.6', we see easily that Lemma 2.5 and $\text{Th}.2.2_2$ enable us to change the symbol 'C^q' in (2.12) in Lemma 2.7 to the one 'Z^q'. (Namely we have the following inclusion):

$(2.15)_2$ $s^*(Z^q(\underline{A_\sigma(\tilde{X}_r}(P)), \underline{f^m O_X^{(p^s)}})_{P.g} \cap (F_p^m)^{-1}(0)) \subset F_{p-1}^m Z^q(\underline{A_{\alpha'}(\tilde{X}_{r'}}(P)), \underline{f^{m'}} O_X^{(p^s-1)})_{P.g}$.

By $(2.15)_1$, we have the following implication:

$(2.15)_2$ Lemma 2.5 ⟶ $(2.15)_2$.

Thirdly, as we checked in (2.14), the symbol 'C^q' in Lemma 2.7 is changed to 'Z^q' (by using Lemma 2.7 and Th. 2.2_2). This fact, together with $(2.15)_1$, insures the implication:

$(2.15)_3$ Lemma 2.5 + Lemma 2.7 ⟶ (2.14).

We use $(2.15)_{1\sim 3}$ in the proof of Prop.4.2 (in n.4, § 4.2).

71- bis

§3. Application to Analytic de Rham theory

Here we summarize our applications of the cohomology theories in §1, §2 to the analytic de Rham theory. As was mentioned*), we use here our results on C^∞-de Rham theory for certain stratified spaces and real analytic varieties, which were announced in [15]$_{2\sim 4}$ and [17] (cf. Lemma 3.2 and Lemma 3.3). The details of those results will be published elsewhere in a near future (cf. [18]).

1. Letting the smooth local variety $\underline{X}=X_0-D$ and the smooth affine variety $\underline{X}'$ be as in §1, §2, we set*):

$$(3.1)_0 \begin{Bmatrix} \Omega_{\underline{X}',\mathrm{alg}} \\ \Omega_{\underline{X}}(*D) \end{Bmatrix} := \text{sheaf over} \begin{Bmatrix} \underline{X}'_{\mathrm{alg}} \\ X_0 \end{Bmatrix} \text{of} \begin{cases} \text{rational differential forms,} \\ \text{meromorphic differential forms with} \\ \text{the pole } D \text{ .} \end{cases}$$

Moreover, let the subvarieties V, V' of X_0, $\underline{X}'$ in §2, we set:

$$(3.1)_0' \ \hat{\Omega}_{\underline{X}}(*D) := \varprojlim_m \Omega_{\underline{X}}(*D)/\underset{\sim}{I}_V^m \cdot \Omega_{\underline{X}}(*D) \ , \ \hat{\Omega}_{\underline{X}',\mathrm{alg}} := \varprojlim_m \Omega_{\underline{X}',\mathrm{alg}}/\underset{\sim}{I}_{V',\mathrm{alg}}^m \cdot \Omega_{\underline{X}',\mathrm{alg}} \ ,$$

where $\underset{\sim}{I}_V$ and $\underset{\sim}{I}_{V',\mathrm{alg}}$ denote the ideal sheaves of V and V'_{alg}.
We write V-D ss $\tilde{V}$, and, taking a point $P \in D \cap V$, we denote by $\underline{X}_P$, $\tilde{V}_P$ the germs of $\underline{X}, \tilde{V}$ at P. We then set:

$$(3.1)_0'' \quad H^*(\underline{X}_P, \underline{C}) := \varinjlim_{U \to P} H^*(\underline{X} \cap U, \underline{C}) \ , \text{ where } U \text{ exhaust all neighborhoods of } P \text{ in}$$

X_0 (and we define $H^*(V_P, \underline{C})$ similarly).
Then we have*):

<u>Theorem 3.1</u> $H^*(\underline{X}_P, \underline{C}) \cong H^*(\Omega_{\underline{X}}(*D)_P)$, and $H^*(\underline{X}', \underline{C}) \cong H^*(\Gamma(\underline{X}'_{\mathrm{alg}}, \hat{\Omega}_{\underline{X}',\mathrm{alg}}))$.

<u>Theorem 3.2</u> $H^*(\tilde{V}_P, \underline{C}) \cong H^*(\hat{\Omega}_{\underline{X}}(*D)_P)$, and $H^*(V', \underline{C}) \cong H^*(\Gamma(V'_{\mathrm{alg}}, \hat{\Omega}_{\underline{X}',\mathrm{alg}}))$

---

*) Similarly to §1, §2, $\underline{X}'_{\mathrm{alg}}$ and V'_{alg} denote the algebraic varieties whose underlying analytic varieties are $\underline{X}'$, V'.

(Th.3.1 is given to the smooth varieties $\underline{X},\underline{X}_p$, while Th.3.2 is a generaliz--tion of Th.3.1 to the varieties $V,\widetilde{V}_p$, which have in general singularities.) Except the concrete style of the formulation, Th.3.1 is the well known theorems of A.Grothendirck in [5]. The second algebraic isomorphism in Th.3.2 is due to P.Deligne(cf.[7]). The first analytic isomorphism in Th.3.2 is seems to have been not known. In the both theorems, the analytic isomorphisms are stronger than the algebraic ones(The analytic isomorphisms, together with standard Gaga arguments, lead to the algebraic ones. The proof of the former is harder than the latter(cf.[5]).) We note that the arguments in [5], [7] use the resolution theorem of H.Hironaka. We also note that, in the proof of the first isomorphism in Th.3.1,[5] uses a comparison theorem of H.Grauert-R.Remmert(on the behaviors of coherent sheaves under proper maps). As we will see soon later, our proof of Th.3.1 and Th.3.2 is largely different from the arguments in [5], [7]. Our proof is more close to that of the holomorphic de Rham theorem for Stein manifolds(cf. H.Cartan [1]):

$(3.1)_1$ $H^*(Y,\underline{C})\cong H^*(\Gamma(Y,\Omega_Y))$, where Y is a Stein manifold and Ω_Y is the sheaf of holomorphic differential forms over Y.

As is well known, $(3.1)_1$ is a formal consequence of the following facts:

$(3.1)_2$ $H^*(Y,\Omega_Y)\cong 0(q\geqq 1)$, $H^0(Y,\Omega_Y)\cong\Gamma(Y,\Omega_Y)$ (Th.A,B of H.Cartan for Ω_Y).

$(3.1)_3$ Exactness of $\Omega_Y: 0\to\underline{C}\to\Omega_Y^0\to\cdots\to\Omega_Y^p\xrightarrow{d}\cdots$ (Poincare lemma).

Th.A,B is a main result in the theory of Stein varieties, while $(3.1)_3$ is based on the analytic contractiblity of the analytic manifolds. Our proof of Th.3.1 and Th.3.2 is patterned on the proof of $(3.1)_3$ indicated above We also use a topological fact(Lemma 3.5), whose correspondence does not

13

appear in the proof of $(3.1)_1$.

2. First letting $\Omega_X, \Omega_{X'}$ be the sheaves of holomorphic differential forms over $\underline{X}, \underline{X}'$, we set: $\widehat{\Omega}_{\underline{X}} := \varprojlim_m \Omega_{\underline{X}}/\underset{\sim}{I}_V^m \Omega_{\underline{X}}$ and $\widehat{\Omega}_{\underline{X}'} := \varprojlim_m \Omega_{X'}/I_{V'}^m \Omega_{X'}$, where $\underset{\sim}{I}_V$ and $\underset{\sim}{I}_{V'}$ denote the ideals of V, V'. Then, from our main results, Th.1.5, Th.1.6 and Th.2.5, Th.2.6, in §1, § 2, we have:

Lemma 3 1. (1) $H^q(X_P, \Omega_X)_{p.g} \cong 0$, $H^q(X', \Omega_{X'})_{p.g} \cong 0$, and $H^0(X_P, \Omega_X)_{p.g} \cong \Omega_X(*D)_P$, $H^0(X', \Omega_{X'})_{p.g} \cong \Gamma(X'_{alg}, \Omega_{X',alg})$.

(2) $H^q(X_P, \widehat{\Omega}_X)_{p.g} \cong 0$, $H^q(X', \widehat{\Omega}_{X'})_{p.g} \cong 0 (q \geqq 1)$, and $H^0(X_P, \widehat{\Omega}_X)_{p.g} = \widehat{\Omega}_X(*D)_P$, $H^0(X', \Omega_{X'})_{p.g} \cong \Gamma(X'_{alg}, \Omega_{X',alg})$.

Lemma 3.1 corresponds to $(3.1)_2$ in our p.g.cohomology theory and in 'p.g.cohomology theory in the completions'. As in the proof of $(3.1)_1$, Lemma 3 1 will play the most basic roles in our proof of Th.3.1 and Th.3.2. Also we note that Lemma 3.1 concerns the Stein and the algebraic properties of X, X', which may be the most important properties of these varieties(cf.also Introduction).

3. Next let j_X be the injection: $C \hookrightarrow \Omega^0_{X'}$, and we define:

$(3.1)_4$ $C^q(X', C)_{p.g} := \varinjlim_{\sigma} C^q(A_\sigma(X'), C)_{p.g}$, where $C^q(A_\sigma(X'), C)_{p.g} := j^{-1} C^q(A_\sigma(X'), \Omega^0_X)_{p.g}$, and the p.g.covering $A_\sigma(X')$ is as in Th.1.3. We define $C^q(X_P, C)_{p.g}$ similarly to the above. Then we have:

Proposition 3.1. (P.g.Poincare lemma). The following complexes are exact:

$$(3.1)_5 \quad \left\{ \begin{array}{l} 0 \longrightarrow C^q(X_P, C)_{p.g} \longrightarrow C^q(X_P, \Omega^0_X)_{p.g} \xrightarrow{d} C^q(X_P, \Omega^p_X)_{p.g} \longrightarrow \\ 0 \longrightarrow C^q(X', C)_{p.g} \longrightarrow C^q(X', \Omega^0_X)_{p.g} \xrightarrow{d} C^q(X', \Omega^p_{X'})_{p.g} \longrightarrow \end{array} \right.$$

where d denotes the exterior differential operator.

Prop.4.1 will correspond to $(3.1)_3$ in our proof of Th.3.1. The proof of Prop.3.1 is essentially very elementary(cf. the end of § 5.2). Next letting the finite sets $\underline{f}=(f_j)_{j=1}^{s} \subset \Gamma(X_0, \underset{\sim}{O}_{X_0})$ and $\underline{f}'=(f'_j)_{j=1}^{s} \subset \Gamma(\underline{X}', \underset{\sim}{O}_{\underline{X}'})_{p.g}$ be as in §2, we set:

$$(3.2)_3' \quad \Omega^{*}_{\underline{X},m} := \underline{f}^{m}\cdot\Omega^{*}_{\underline{X}} + d\underline{f}^{m+1}\wedge\Omega^{*-1}_{\underline{X}} \quad , \Omega^{*}_{\underline{X}',m} := \underline{f}'^{m}\cdot\Omega^{*}_{\underline{X}'} + d\underline{f}'^{m+1}\wedge\Omega^{*-1}_{\underline{X}'} \text{ , and}$$

$$(3.2)_3 \quad C^q(\underline{A}_{\sigma}(X'), \Omega_{X',m})_{p.g} := F'^{m}C^q(\underline{A}_{\sigma}(X'), (\Omega_{\underline{X}'})^{s})_{p.g} + d\underline{f}'^{m+1}\wedge C^q(\underline{A}_{\sigma}(\underline{X}'), \Omega^{*-1}_{\underline{X}'})_{p.g}$$

Also letting the p.g.covering $\underline{A}_{\mu}(P)$, attached to $\underline{X}_P$, be as in Cor.1.2, we define the p.g.complex $C^q(\underline{A}_{\mu}(P), \Omega_{\underline{X},m})_{p.g}$ similarly to the above.

Lemma 3.2.(P.g open map property for de Rham complex). We have the inclusions:

$$(3.2)_4 \begin{cases} (C^q(\underline{A}_{\mu}(P), \Omega^{p}_{\underline{X},m})_{p.g} \cap d^{-1}(0)) \subset dC^q(\underline{A}_{\mu'}(P), \Omega^{p-1}_{\underline{X},m'})_{p.g} \\ (C^q(\underline{A}_{\sigma}(X'), \Omega^{p}_{\underline{X}',m})_{p.g} \cap d^{-1}(0)) \subset dC^q(\underline{A}_{\sigma'}(\underline{X}'), \Omega^{p-1}_{\underline{X}',\tilde{m}'})_{p.g} \end{cases} (p \geqq 1,\ q \geqq 0),$$

where the parameters μ', σ' are chosen suitably in the mannner as in Lemma 2.3 Moreover, $m'=[L_{\underline{X}}(m)]$ and $\tilde{m}'=[L_{\underline{X}'}(m)]$, with linear maps $L_{\underline{X}}(t)=c_{\underline{X}}t$ and $L_{\underline{X}'}(t)=c_{\underline{X}'}\cdot t$; $c_{\underline{X}}, c_{\underline{X}'} > 0$.

For the proof of Lemm 3.2, see Lemma 4.7(cf.part B, §4.1) and the end of §5.2. Our proof of Lemma 3.2 uses certain open map properties for Koszul complexes and a.d.properties of (topological) contractiblity of analytic varieties(cf. § 5.2). Lemma 3.2 is, no longer, of obvious nature. Now, applying Prop. 2.1_1 to the open map property in Lemma3.2, we have:

Lemma 3.3.(P.g Poincare lemma in the completion theory). The following complexes are exact:

$$(3.2)_5 \begin{cases} 0 \to C^q(\tilde{V}_P, \underline{C})_{p.g} \longrightarrow C^q(\tilde{V}_P, \hat{\Omega}^{0}_{\underline{X}})_{p.g} \to \cdots \xrightarrow{d} C^q(\tilde{V}_P, \hat{\Omega}^{p}_{\underline{X}})_{p.g} \longrightarrow \\ 0 \to C^q(V', \underline{C})_{p.g} \longrightarrow C^q(V', \Omega^{0}_{\underline{X}'})_{p.g} \to \cdots \xrightarrow{d} C^q(V', \hat{\Omega}^{p}_{\underline{X}'})_{p.g} \longrightarrow \end{cases}$$

(In Lemma 3.3, the varieties in question are $\tilde{V}, V'$ (instead of $\underline{X}, \underline{X}'$ in Lemma 3.2 : by a simple observation of p.g.properties of the imbedded varieties $\tilde{V}, V'$, we have: $C^q(\tilde{V}_P, \hat{\Omega}^{*}_{\underline{X}})_{p.g} \cong C^q(\underline{X}_P, \hat{\Omega}^{*}_{\underline{X}})_{p.g}, \cdots$ (cf.[18]). Then, also from a simple observation, we easily check that $d^{-1}(0)$ at the first steps in $(3.2)_2$ are: $C^q(\underline{X}_P,\underline{C})_{p.g}, \ldots$ (cf. also [18]).

Now, from Lemma 3.1 ~ Lemma 3.3, we easily have:

Lemma 3.4. We have the following isomorphisms:

$$(3.2)_6 \left\{\begin{array}{l} H^*(\underline{X}_P,\underline{C})_{p.g} \cong H^*(\Omega_{\underline{X}}(*D)_P), \text{ and } H^*(\underline{X}',\underline{C})_{p.g} \cong H^*(\Gamma(\underline{X}'_{alg}, \Omega_{\underline{X}'},{}_{alg})) \\ H^*(\tilde{V}_P,\underline{C})_{p.g} \cong H^*(\hat{\Omega}_{\underline{X}}(*D)_P), \text{ and } H^*(V',\underline{C})_{p.g} \cong H^*(\Gamma(V'_{alg}, \hat{\Omega}_{\underline{X}'},{}_{alg})) \end{array}\right\}.$$

These isomorphisms summarize our applications of the p.g. cohomology theories as in Lemma 3.1 ~ Lemma 3.3. In order to get Th.3.1 and Th.3.2, we should drop the term 'p.g' from the cohomology groups $H^*(\underline{X}_P,\underline{C})_{p.g}, \ldots$ in the left sides in $(3.2)_6$. In this step we will use our main results on p.g. C^∞-de Rham theory for certain stratified spaces in $[15]_{2\sim4}$, [17].

4. Let $\underline{C}^n$ be the ambient (euclid) space of the local variety*) $\underline{X}$ and the affine variety $\underline{X}'$. We identify $\underline{C}^n$ with the real euclid space $\underline{R}^{2n}$ in a natural manner, and we fix coordinates $x=(x_j)_{j=1}^{2n}$ of $\underline{R}^{2n}$. The symbol $\mathcal{E}$ will denote the sheaf of C^∞(differentiable) differential forms over $\underline{R}^{2n}$. Taking an open set Y of $\underline{R}^{2n}$ and a p.g. function $g:Y \longrightarrow \underline{R}_1^+$, we set:

$(3.2)_0$ $\quad \mathcal{E}(Y;g)_{p.g} := \{\varphi \in \mathcal{E}(Y);\ \varphi = \sum_K \varphi_K dx^K$ satisfies the following for each suffix K and each element $J \in (\underline{Z}^+ \cup 0)^{2n}$:

$(3.2)_0'$ $\quad |D_J \varphi_K(P)| < a_J g(P)$ in Y, with a suitable $a_J \in \underline{R}_1^{+2}$, where $D_J := \partial^J/\partial x^J\}$

Next taking subsets Z, Z' of X, X', we define:

$$(3.2)_0'' \quad \left\{\begin{array}{l} \underline{B}_\sigma(Z) \\ \underline{B}_\sigma(Z') \end{array}\right. := g\text{-p.g. covering of } \left\{\begin{array}{l} Z \\ Z' \end{array}\right\} \text{ of size } \sigma \text{ in } \underline{C}^n (\cong \underline{R}^{2n}),$$

where g is the p.g. function $|h_{\underline{X}}^{-1}|$ or $|z|+1$ of $\underline{X}$ or $\underline{X}'$ (cf. Th.1.1 ~ Th.1.4). We use the symbols $\mathcal{N}_\sigma(Z)$, $\mathcal{N}_\sigma(Z')$ for supp $\underline{B}_\sigma(Z)$ and supp $\underline{B}_\sigma(Z')$. We may

---

*) Recall that $\underline{X}$ is of the form $\underline{X}=X_0-D$, with a variety X_0 in U_0. We are assuming that $U_0 \subset \underline{C}^n$.

may call $\underline{N}_\sigma(\underline{Z}), \underline{N}_\sigma(\underline{Z}')$ the p.g.neighborhoods of $\underline{Z}, \underline{Z}'$ in $\underline{C}^n$ of size σ. Such p.g.neighborhoods are suitable for investigations of the p.g.properties of imbedded varieties(cf. [19]. See also Prop.4.6 , §4.2 of the present paper, where we discuss p.g.properties in connection with extensions of cochains from imbedded varieties to their ambient spaces.) Now we set:

$$(3.2)_1' \left\{\begin{array}{ll} \mathcal{E}(\underline{X}_P)_{p.g} := \varinjlim\limits_{\sigma,r} \mathcal{E}(\underline{N}_\sigma(\underline{X}_r(P)),g)_{p.g}, & \mathcal{E}(\tilde{V}_P)_{p.g} := \varinjlim\limits_{\sigma,r} \mathcal{E}(\underline{N}_\sigma(\tilde{V}_r(P)),g)_{p.g} \\ \mathcal{E}(\underline{X}')_{p.g} := \varinjlim\limits_{\sigma} \mathcal{E}(\underline{N}_\sigma(\underline{X}'),g)_{p.g} & , \mathcal{E}(V')_{p.g} := \varinjlim\limits_{\sigma} \mathcal{E}(\underline{N}_\sigma(V'),g)_{p.g} \end{array}\right\}.$$

where $g=|h_{\underline{X}}^{**1}|$ or $|z|+1$. Also the manifold $\underline{X}_r(P):=\underline{X}\cap U_r(P)$, where $U_r(P):=$ disc in $\underline{C}^n$ of center P and radius r, is as in Th.1.1. Moreover, we set $\tilde{V}_r(P):=\tilde{V}\cap U_r($

Then our main result in [17] insures:*)

Lemma 3.5. We have the following isomorphisms:

$$(3.2)_1 \left\{\begin{array}{ll} H^*(\underline{X}_P,\underline{C}) \cong \underline{H}^*(\mathcal{E}(\underline{X}_P)_{p.g}), & H^*(\tilde{V}_P,\underline{C}) \cong \underline{H}^*(\mathcal{E}(\tilde{V}_P)_{p.g}) \\ H^*(\underline{X}',\underline{C}) \cong \underline{H}^*(\mathcal{E}(X')_{p.g}), & H^*(V',\underline{C}) \cong \underline{H}^*(\mathcal{E}(V')_{p.g}) \end{array}\right\}.$$

Note that the right sides in $(3.3)_1$ may be regarded as C^∞-analogues of the analytic de Rham cohomology groups as in $(3.1)_6$, Lemma3.3. Also note that the left sides in $(3.2)_1$ are the topological cohomology groups $H^*(X_P,C)$,- - -, while the left sides in $(3.1)_6$, $H^*(\underline{X}_P,\underline{C})_{p.g}$, ... contain the suffix 'p.g'.(This difference occurs from the following situation: first, in the definition of $H^*(\underline{X}_P,\underline{C})_{p.g}$, we used the p.g.coverings $\underline{N}_\sigma(\underline{X}_r(P))$, which are attached to $\underline{X}_r(P)$ (cf.Def.1.6_2) , and our use of such p.g.coverings is a main source for the suffix 'p.g'. mentioned just above. On the otherhand, our proof of Lemm 3.5 is based on a type of stratified spaces attached to real analytic varieties , which we call normalized series of stratified space

---

*) Lemm 3.5 is given in [17] for local analytic varieties, and is applied to the variety $\underline{X}_P$. On the otherhand, Remarking that, the affine variety X' is compactified (in $P^n(\underline{C})\supset\underline{C}^n$) and, applying the local results just above to each point of the completion of $\underline{X}'$, we get Lemma 3.5 for $\underline{X}'$.

(cf.[17]). Such stratified spaces admitt what we call p.g.simple coverings, where the word 'simple' is used in the similar sense to the 'simple covering' in the C^∞-de Rham theorem in [21]. The simpleness as above insures that the above coverings satisfy the strndard Leray condition for the constant sheaf $\mathbb{Z}$(and so for $\mathbb{R}$ and C), and they are used to determi the topological cohomology groups $H^*(X_P,C)$,... Such coverings are also suitable for treatments of the p.g.properties of C^∞-differential forms over analytic varieties. Using the above stratified spaces and the p.g. simple coverings of them, the proof of Lemma 3.5 is formal(cf.[17] and [18]. See also Remark 3.1 at the end of § 3.)

5. Finally we will see that Th.3.1 and Th.3.2 are derived from Lemma 3.3 and Lemma 3.5 in a formal fashion. For this we first let $\mathcal{E}_X, \mathcal{E}_{X'}$ denote the sheaves of C^∞-differential forms over X, X'. Letting the subsets Z, Z' of X, X' be as in $(3.2)_0$, we define 'p.g.complexes of C^∞-differentia forms':

$(3.2)'_2$ $\mathcal{E}_X(Z)_{p.g}$ and $\mathcal{E}_{X'}(Z')_{p.g}$ (in the similar manner to $(3.2)_0$, by using the coordinates of X, X' instead of those of C^n as in $(3.2)_0$). Moreover, we use the symbols $B'_\sigma(Z)$, $B'_\sigma(Z')$ for the p.g.coverings of Z, Z' in X, X' of size σ (cf.also $(3.2)''_0$). We also use the symbold $N'_\sigma(Z), N'_\sigma($ for supp $B'_\sigma(Z)$, supp $B'_\sigma(Z')$. (Thus $N'_\sigma(Z)$, $N'_\sigma(Z')$ are the p.g.neighborhoods of Z, Z' in X, X'.) Then, corresponding to $(3.2)'_1$, we define:

$$(3.2)''_2 \begin{cases} \mathcal{E}_X(X_P)_{p.g} := \lim\limits_{\sigma,r\to} \mathcal{E}_X(N'_\sigma(X_r(P)))_{p.g}, & \hat{\mathcal{E}}_X(\tilde{V}_P)_{p.g} := \lim\limits_{\sigma,r\to} \mathcal{E}_X(N'_\sigma(\tilde{V}_r))_{p.g} \\ \mathcal{E}_{X'}(X')_{p.g} := \lim\limits_{\sigma\to} \mathcal{E}_X(N'_\sigma(X'))_{p.g}, & \hat{\mathcal{E}}_{X'}(V')_{p.g} := \lim\limits_{\sigma\to} \mathcal{E}_X(N'_\sigma(V'_r))_{p.g} \end{cases}.$$

Then it is not difficult*) to check: $H^*(\mathcal{E}_X(X_P)_{p.g}) \cong H^*(\mathcal{E}(X_P)_{p.g})$, ..., and Lemma 3.5 is rewritten in the following manner:

---

*) From that X, X' are smooth, this isomorphism is rather easily checke (cf.[18]. See also Prop.4.6 for tretmants of the p.g.neighborgoods.)

$$(3.2)_2 \left\{\begin{array}{ll} H^*(\underline{X}_P,\underline{C}) \cong \underline{H}^*(\mathcal{E}_{\underline{X}}(\underline{X}_P)_{p.g}) , & H^*(\tilde{V}_P,\underline{C}) \cong \underline{H}^*(\hat{\mathcal{E}}_{\underline{X}}(\tilde{V}_P)_{p.g}) \\ H^*(\underline{X}',\underline{C}) \cong \underline{H}^*(\mathcal{E}_{\underline{X}'}(\underline{X}')_{p.g}) , & H^*(V',\underline{C}) \cong \underline{H}^*(\hat{\mathcal{E}}_{\underline{X}}(V')_{p.g}) \end{array}\right\}.$$

Now we denote by τ , $\hat{\tau}$ the natural homomorphisms from the analytic de Rham groups to the p. g. C^∞-de Rham groups:

$$(3.2)_3 \quad \tau: \Omega_{\underline{X}}(*D)_P \longrightarrow \mathcal{E}_X(*D)_{P.g}, \ \hat{\tau}: \hat{\Omega}_{\underline{X}}(*D)_P \longrightarrow \mathcal{E}_{\underline{X}}(\tilde{V}_P)_{p.g} .$$

Moreover, from a simple observation, we have natural homomorphisms:*)

$$(3.2)_4 \ \mu: \underline{H}^*(\mathcal{E}_{\underline{X}}(\underline{X}_P)_{p.g}) \longrightarrow H^*(\underline{X}_P,\underline{C})_{p.g} , \ \hat{\mu}: \underline{H}^*(\underline{X}_P', (\tilde{V}_P)_{p.g}) \longrightarrow H^*(\tilde{V}_P,\underline{C})_{p.g} .$$

Then we easily have the following diagrams

Fig.I.

$$\begin{tikzcd} \underline{H}^*(\Omega_{\underline{X}}(*D)_P) \arrow[r, "\sim"] \arrow[d, "\tau"] & H^*(\underline{X}_P,\underline{C})_{p.g} \arrow[d] \\ \underline{H}^*(\mathcal{E}_{\underline{X}}(\underline{X}_P)_{p.g}) \arrow[r, "\sim"] \arrow[ur, "\mu"] & H^*(\underline{X}_P,\underline{C}) \end{tikzcd} \qquad \begin{tikzcd} \underline{H}^*(\hat{\Omega}_{\underline{X}}(*D)_P) \arrow[r, "\sim"] \arrow[d, "\tau"] & \underline{H}^*(\tilde{V}_P,\underline{C})_{p.g} \arrow[d] \\ \underline{H}^*(\mathcal{E}_{\underline{X}}(\tilde{V}_P)_{p.g}) \arrow[r, "\sim"] \arrow[ur, "\mu"] & H^*(\tilde{V}_P,\underline{C}) \end{tikzcd}$$

(and the similar diagrams for $\underline{X}'$ and V'.) It is clear that the above diagram insures the implication:

$$(3.2)_5 \qquad \text{Lemma } 3.3 + \text{Lemma } 3.5 \longrightarrow \text{Th.}3.1 \text{ and Th.}3.2 .$$

Remark 3.1. As may be clear from the content of §3, the most important topological fact in getting Th.3.1 and Th.3.2 is the p. g. C^∞-de Rham isomorphism as in Lemma3.5. Also we use some a.d. properties of (topological) contractiblity of analytic varities in getting the open map property of the de Rham complex as in Lemma 3.4(cf.[15]$_2$. Also see the end of §5.2.) The underlying basic fact for the above topological facts is the existence of stratified spaces for analytic varieties(=normalized series of stratified spaces), which were mentioned previously. The details of the above topological facts are in [15]$_{2\sim4}$ and the author's forthcoming paper[18](The first three are outlines of what are mentioned soon above, while [18] will contain the details.)

---

*) Such homomorphisms are constructed, by using similar arguments to the one in[21], which attaches, to the closed differential forms, their cohomology classes. See also the arguments in §5.1 and n.5, n.6 in §5.2, where we give some cohomological arguments.(Such arguments have similar algebraic structures with the arguments in[21].

As may be from the context of §3, the above topological facts are indispensable in getting Th.3.1 and Th.3.2. However, as in the case of the holomorphic de Rham theorem $(3.1)_1$, and in the proof of Th.3.1 in [5], the most important facts in getting the analytic de Rham de Rham theorem(as in Th.3.1, Th.3.2) are the results on the coherent sheaves in Lemma 3.1.

Remark 3.1. At present, our results on the analytic de Rham theory are given separatedly, according as we are concerned with the analytic or topological aspects. The present paper covers the necessary analytic facts for the proof of Th.3.1,Th.3.2, while the necessary topological facts are summarized in $[15]_{2\sim 4}$,[17] and in [18]. The author plans to write a survey paper on the analytic de Rham theory, which will include (1) even treatments of the analytic and topological parts as above and (2) comparisons of our methods indicated as in § 3 and the methods taken in [5],[7].

75 –5

Chapter II. Uniform estimations on homomorphisms of coherent sheaves

§ 4. Uniform estimations with bound and algebraic division

In § 4.1, § 4.2, we give the first and second uniform estimations in the title, which are semi-global in nature(Lemma 4.1 ∼ 4.6). In § 4.3, § 4.4 we give global versions of the results in § 4.1, § 4.2. Also using such results, we prove the lemmas in § 1, § 2, whose proof is postponed until now. The proof of Lemma 4.1 ∼ 4.6 will be given in § 5.

§ 4.1. Uniform estimation with bound

1. First we take a datum:
$(4.1)_0$ $\underline{X}=(C^n(z),U_0,X_0,X_0',P_0)$ consisting of an analytic variety $X_0(\ni P_0)$ in an open set U_0 of a euclidean space $C^n(z)$ and a subvariety X_0' of X_0. The variety X_0' may be empty, but should satisfy:
$(4.1)_0'$ $X:=X_0-X_0'$ is smooth, and, when $X_0'\neq\phi$, X_0' contains P_0.
We fix the datum $\underline{X}$ in the remainder of § 4 and in § 5. The underlying variety of $\underline{X}$ will be: $X=X_0-X_0'$. Moreover, for convenience of the formulation of the estimations in § 4, we fix subvarieties X_1,X_2 of X_0 satisfying
$(4.1)_0''$ $X_0\supset X_1\supsetneqq X_2\supset X_0'$.

2. Underlyigng data. Setting $\tilde{X}_1:=X_1-X_2$ we first define a parametrization of manifolds in $X(=X_0-X_0')$:
$(4.1)_1$ $u_{\tilde{X}_1}:\mathcal{U}_{\tilde{X}_1}:=\tilde{X}_1\times R^+\ni\mathcal{U}=(P;r)\longrightarrow Ouv(X)\ni\tilde{U}_r(P):=\{Q\in X;d(P,Q)<r\}$,
where d=natural distance in $C^n(z)$(cf.n.1, § 1.2).

---

*) cf.n.1, § 1.2.

varieties with singularities. Finally, the procedure in § 4.2, which rewrites the non cohomological estimations in § 4.1 in cohomological forms, is essentially algebraic; large parts of § 4.2 is given in an abstract fashion in terms of the q-sheaves(Def.1.4_1). The content of §4.2 may be useful for general treatments of the p.g. and a.d. properties of q-sheaves.

§4. Uniform estimations with bound and algebraic divisions

§4.1. Non cohomological estimations

In part A,B we give non cohomological estimations of local forms, which concern the first and second properties in the title. In C we give a global version of the results in A,B.

A. Uniform estimations with bound

1. Geometric underlying data. In a similar manner to § 1.2, we start with giving the following geometric datum:

$(4.1)_0$ $\underline{X}:=(\underline{C}^n(z),U_0,X_0,X_0',P_0)$ consisting of an analytic variety $X_0(\ni P_0)$ in an open set U_0 of a euclidean space $\underline{C}^n(z)$ and a subvariety X_0' of X_0. The variety X_0' may be empty, but should satisfy:

$(4.1)_0'$ $X:=X_0-X_0'$ is smooth, and X_0' contains P_0, if $X_0' \neq \emptyset$.

(When X_0' is the divisor of an element $h \in \Gamma(X_0,\underset{\sim}{O}_{X_0})$, the datum $\underline{X}$ is of the form which was used in §1.2:$X \in An_{1a}$(cf.$(1.8)_0$). Note that, in this case, $X=X_0-X_0'$ is a Stein variety. In Chap.II we do not require this condition. The datum X is more general than geometric data in An_{1a} in §1.

We fix the geometric datum $\underline{X}$ in the remainder of Chap.II. The underlying variety of $\underline{X}$ will be $X=X_0-X_0'$. Moreover, for convenience of the formulatior of the estimations in§ 4, we fix subvarieties X_1,X_2 of X_0 satisfying

$(4.1)_0''$ $X_0 \supset X_1 \supsetneqq X_2 \supset X_0'$.

2. Parametrizations. Next we will define certain sets of cross sections to certain coherent sheaves, which are parametrized in an explici manner(cf.$(4.1)_2$ soon below). The parametrization here is of non cohomological form and is simpler than the one in § 1, § 2. However, the formulatic of the former has some similarities to the one in the latter: first setting $\tilde{X}_1:=X_1-X_2$ we define a parametrization of open manifolds in $X(=X_0-X_0')$:

$(4.1)_1$ $u_{\tilde{X}_1}: \mathcal{U}_{\tilde{X}_1}:=\tilde{X}_1 \times \underline{R}^+ \ni \mu=(P;r) \longrightarrow \mathrm{Ouv}(\tilde{X}_1) \ni \tilde{U}_r(P):=\{Q\in\tilde{X}_1; d(P,Q)<r\}$,

where d is the natural distance in $\underline{C}^n(z)$(cf.n.1, § 1.2).

Next taking a matrix $K:\underset{\sim}{O}_X^v \longrightarrow \underset{\sim}{O}_X^u (u,v>0)$, whose entries are in $\Gamma(X_0,\underset{\sim}{O}_{X_0})$, we write the image $K\underset{\sim}{O}_X^v (\subset \underset{\sim}{O}_X^u)$ as $\underset{\sim}{K}$.(Here $\underset{\sim}{O}_X,\underset{\sim}{O}_{X_0}$ are the structure sheaves of X,X_0.) We use the symbols $\theta_{\underset{\sim}{K}}$, $\theta'_{\underset{\sim}{K}}$ for the q-structures of $\underset{\sim}{K}$, which are induced from $K:\underset{\sim}{O}_X^v \rightarrow \underset{\sim}{K}$ and the injection:$\underset{\sim}{K} \hookrightarrow \underset{\sim}{O}_X^u$(Def.1.$4_2$). Setting $\lambda_{\tilde{X}_1}:=\mathcal{U}_{\tilde{X}_1} \times \underline{R}_1^+$, we take an element $(P;r;a)\in\lambda_{\tilde{X}_1}(\subset \tilde{X}_1 \times \underline{R}^+ \times \underline{R}_1^+)$. Then the sets of the cross sections, which are used in §4, §5, will be of the form:

$$(4.1)_2 \left\{\begin{matrix}\Gamma(\tilde{U}_r(P),\underset{\sim}{K};\theta_{\underset{\sim}{K}})_a \\ \Gamma(\tilde{U}_r(P),\underset{\sim}{K};\theta'_{\underset{\sim}{K}})_a\end{matrix}\right\} := \left\{\varphi\in\Gamma(\tilde{U}_r(P),\underset{\sim}{K});\ \left\{\begin{matrix}|\varphi(Q)|_{\underset{\sim}{K}} < a \\ |\varphi(Q)|'_{\underset{\sim}{K}} < a\end{matrix}\right\} \text{ in } \tilde{U}_r(P)\right\},$$

where $|\ |_{\underset{\sim}{K}}, |\ |'_{\underset{\sim}{K}}$ are the $\theta_{\underset{\sim}{K}}$- and $\theta'_{\underset{\sim}{K}}$-absolute values(Def.1.$4_1$).

Note that, by the definition of $\theta_{\underset{\sim}{K}},..$, the above sets are explicitly as follows:

$(4.1)_2'$ $\Gamma(\tilde{U}_r(P),\underset{\sim}{K};\theta_{\underset{\sim}{K}})_a = K\,\Gamma(\tilde{U}_r(P),\underset{\sim}{O}_X^v;\theta)_a$, $\Gamma(\tilde{U}_r(P),\underset{\sim}{K};\theta'_{\underset{\sim}{K}})_a = \Gamma(\tilde{U}_r(P),\underset{\sim}{O}_X^u;\theta')_a$ $\cap\ \Gamma(\tilde{U}_r(P),\underset{\sim}{K})$,where θ,θ' are the standard q-structures of $\underset{\sim}{O}_X^v,\underset{\sim}{O}_X^u$(Def.1.$4_3$).

Fig.I

$$\begin{tikzcd} \underline{R}^+ \times \underline{R}^+ \arrow[r] \arrow[d, "\widetilde{M}"] & \underline{R}^+ \arrow[d, "M_1"] \\ \underline{R}^+ \times \underline{R}^+ \arrow[r] & \underline{R}^+ \end{tikzcd}$$

3. Estimation maps. Letting $\underline{M}$ denote the collection of all positive monomials(n.5, §1.1), we set $\underline{\widetilde{M}}:=\underline{M}\times\underline{M}$, and we regard an element $\widetilde{M}=(M_1,M_2) \in \underline{\widetilde{M}}=\underline{M}\times\underline{M}$ as a map(cf.also Fig.I):

$(4.1)_3$ $\widetilde{M}:\underline{R}^+\times\underline{R}^+ \ni (r;a) \rightarrow \underline{R}^+\times\underline{R}^+ \ni (M_1(r),M_2(a/r))$.

We use such a map in the remainder of §4.1.

4. Bdd*⁾ uniform estimation---1. Letting the matrix K be as in n.2, we take an open subset $U_1=U_{1,K}(\ni P_0)$ of U_0 and an element $\sigma=\sigma_K\in\underline{R}_1^{+2}$. We then form the following parameter spaces:

$(4.1)_4$ $\mu_K:=\{(P;r)\in(U_1\cap\widetilde{X}_1)\times\underline{R}^+;\ r<\{\sigma g(P)\}^{-1}\}$, with**⁾ $g(P)=d(P,X_2)^{-1}$, and $\lambda_K:=\mu_K\times\underline{R}_1^+$.

Lemma 4.1'. (uniform estimation with bound--1). Take a suitable $\widetilde{M}_K\in\underline{\widetilde{M}}$. Then we have:

$(4.1)_5$ $i^*\Gamma(\widetilde{U}_r(P),\underset{\sim}{K};\theta'_{\underset{\sim}{K}})_a \subset \Gamma(\widetilde{U}_{r'}(P),\underset{\sim}{K};\theta_{\underset{\sim}{K}})_{a'}$, with $(r';a')=\widetilde{M}_K(r;a)$, where the parameter $(P;r;a)$ is in $\lambda_K(\subset\widetilde{X}_1\times\underline{R}^+\times\underline{R}_1^+)$ and i=inclusion:$\widetilde{U}_{r'}(P)\hookrightarrow\widetilde{U}_r(P)$.

We prove Lemma 4.1' in § 5.1.

2. Bdd uniform estimation--2. Here we give an another uniform estimation, which is derived from Lemma 4.1'(cf. § 5.1) and is sharper than Lemma 4.1' in some aspects(cf.Remark 4.1): first take a set $\underline{h}=\{h_u\}_{u=1}^{u_0}\subset\Gamma(X_0,\underset{\sim}{O}_{X_0})$ satisfying $\cap_u D_u = X'_0$, where D_u=divisor of h_u on X_0, and we set:

$(4.1)_6$ $\mathrm{Coh}(X_0;\underline{h})$:=collection of all coherent sheaves $\underset{\sim}{H}$ over X, which admits a resolution of the form:

$(4.1)'_6$ $0\rightarrow\underset{\sim}{O}_X^{k_p}\xrightarrow{K_{p-1}} - \ - \ - - \xrightarrow{K_1}\underset{\sim}{O}_X^{k_1}\xrightarrow{K_0}\underset{\sim}{H}(\subset\underset{\sim}{O}_X^k)\rightarrow 0$, where $K_j(0\leqq j\leqq p)$ are matrices with entries in $\Gamma(X,\underset{\sim}{O}_X)$ and satisfy:

$(4.1)''_6$ the entries of K_j are in $\Gamma(X_0,\underset{\sim}{O}_{X_0}(*D_u))$(for each j,u).

Here $\underset{\sim}{O}_{X_0}(*D_u)$ denotes the sheaf over X_0 of meromorphic functions with pole D_u.

---

*) Bdd := bounded

**) When $X_2=\phi$, we understand that $d(P,X_2)=1$.

Now taking an open subset $U_{1,\underline{h}}(\ni P_0)$ of U_0 and an element $\sigma_{\underline{h}} \in \underline{R}_1^{+2}$, we form parameter spaces $\mathcal{U}_{\underline{h}}(\subset \tilde{X}_1 \times \underline{R}^+)$ and $\lambda_{\underline{h}} := \mathcal{U}_{\underline{h}} \times \underline{R}_1^+$ in the similar manner to $(4.1)_4$(by using $(U_{1,\underline{h}},\sigma_{\underline{h}})$;). Then we have:

Lemma 4.1. (Uniform estimation with bound---2). There are maps $\mathcal{E}_{\underline{h}}:\mathrm{Coh}(X_0;h)\ni \underset{\sim}{H} \longrightarrow \underline{\tilde{M}} \ni \tilde{M}_{\underset{\sim}{H}}$ and $\mathcal{E}'_{\underline{h}}:\underline{Z}^+ \to \underline{M}$ which are factored as in Fig.II, and with which we have the following for each $\underset{\sim}{H} \in \mathrm{Coh}(X_0;\underline{h})$:

Fig.II.*)

$$\begin{tikzcd}
\mathrm{Coh}(X_0;h) \arrow[r, "\mathcal{E}_h"] \arrow[d, "\lg"] & \underline{\tilde{M}} \arrow[d] \\
\underline{Z}^+ \arrow[r, "\mathcal{E}'_h"] & \underline{M}
\end{tikzcd}$$

$(4.1)_7$ $\quad i^* \Gamma(\tilde{U}_r(P),\underset{\sim}{H};\theta_{\underset{\sim}{H}})_a \subset \Gamma(\tilde{U}_{r'}(P),\underset{\sim}{H},\theta_{\underset{\sim}{H}})_{a'}$, with $(r';a')=\tilde{M}_{\underset{\sim}{H}}(r;a)$, where the parameter $(P;r;a)$ is in $\lambda_{\underline{h}}(\subset \tilde{X}_1 \times \underline{R}^+ \times \underline{R}_1^+)$.

We derive Lemma 4.1 from Lemma 4.1'(cf. § 5.1).

Here we give a remark on the formulation in Lemma 4.1' and Lemma 4.1, which we use in getting Lemma 4.1 from Lemma 4.1'(cf. § 5.1).

Remark 4.1_1. (1) Take an element $h \in \Gamma(X_0,\mathcal{O}_{X_0})$, and let $\mathcal{O}_{X_0}(*D)$ denote the sheaf over X_0 of meromorphic functions with pole D(=diviosr of h). Then, replacing the condition: 'the entries of the matrix K are in $\Gamma(X_0,\mathcal{O}_{X_0})$' in Lemma 4.1' by 'those are in $\Gamma(X_0,\mathcal{O}_{X_0}(*D))$' , we get also the similar inclusion to $(4.1)_5$(in Lemma 4.1')for K (after the above change). Actually remark that $\tilde{K}=h^dK$ are in $\Gamma(X_0,\mathcal{O}_{X_0})$, with a suitable $d \in \underline{Z}^+$, and we apply Lemma 4.1' to $\tilde{K}$. Then, recalling the explicit form of the estimation: $(r;a) \to (r';a')$ as in Lemma 4.1', we get easily the inclusion mentioned soon above for K from the application of Lemma 4.1' to $\tilde{K}$.

(2) Lemma 4.1 is sharper than Lemma 4.1' in the point that (a) the parameter space $\mathcal{U}_{\underline{h}}$ is independent from the individual sheaf $\underset{\sim}{H} \in \mathrm{Coh}(X_0;\underline{h})$ and (b) Lemma 4.1 satisfies Fig.II(as in that lemma). The latter is used to get the similar diagrams in Cor.1.1, Cor.1.3 and in Lemma 1.2. Concerning the first, we remark that the open set $U_1=U_{1,K}$ in the parameter space $\mathcal{U}_K$ (cf. $(4.1)_4$) is taken independently from the individual matrix K as in Lemma 4.1. Actually, take an open set $U_1=U_{1,X}(\ni P_0)$ of U_0, and finite points P_{0r}

$P_{0,r}$

---

*) '$\lg$' in Fig. II is the length map(cf. $(1.4)_2$).

means for the measure of the a.d.properties. For this taking an open subset $U_{1,\tilde{X}_1}(\ni P_0)$ of U_0 and an element $\sigma_{\tilde{X}_1}\in R_1^{+2}$, we form a subset $\mathcal{M}_{\tilde{X}_1}$ of $\tilde{X}_1\times R^+$ in the manner in $(4.1)_4$, by using $(U_{1,\tilde{X}_1},\sigma_{\tilde{X}_1})$. Also taking an element $\bar{m}\in Z^+$ we set: $\mathcal{T}_{\tilde{X}_1}:=\mathcal{M}_{\tilde{X}_1}\times Z_{\bar{m}}^+\times R_1^+$. For an element $(P;r;m;a)\in \mathcal{M}_{\tilde{X}_1}(\subset \tilde{X}_1\times R^+\times Z^+\times R_1^+)$ we define:

$$(4.2)_2\begin{Bmatrix}\Gamma(\tilde{U}_r(P),f^mO_X)_a\\ \Gamma(\tilde{U}_r(P),O_X)_a^m\end{Bmatrix}:=\left\{\begin{array}{l}F^m\Gamma(\tilde{U}_r(P),O_X^s;\theta)_a(\text{cf.}(4.1)_2')\\ \{\varphi\in\Gamma(\tilde{U}_r(P),O_X);|\varphi(P)|\leqq a|f(P)|^m \text{ in } \tilde{U}_r(P)\}\end{array}\right\}$$

In the above θ is the standard q-structure of O_X^s(cf.$(4.1)_2$ and Def.1.4_1). We then have:

Lemma 4.2.(Algebraic and analytic comparison of a.d.properties).

For a suitable a.d.map $E_{\tilde{X}_1}\in E_{a.d}$ we have**)

$$(4.2)_3\quad i^*\Gamma(\tilde{U}_r(P),O_X)_a^m\subset\Gamma(\tilde{U}_{r'}(P),f^{m'}O_X)_{a'},\ \text{with } (r';m';a')=E_{\tilde{X}_1}(r;m;a)$$

(cf.$(4.2)_1$). Here $(P;r;m;a)$ is in $\mathcal{T}_{\tilde{X}_1}(\subset\tilde{X}_1\times R^+\times Z^+\times R^+)$.

Treatments of the left side of $(4.2)_3$ are sometimes easier than the right side;Lemma 4.2 is useful in treatments of the a.d.properties of O_X. Next we may regard Lemma 4.2 as an analogue of the comparison of 'p.g. and meromorphic'(as in Th.1.6) in our treatments of the a.d.properties. Moreover, as we will see in n.3, Lemma 4.2 implies Hilbert zero point theorem for f(Lemma 4.3'). Lemma 4.2 may be a basic fact in the a.d. properties of O_X.

3. Koszul complex-1. Taking a finite set $g=(g_j)_{j=1}^t\subset\Gamma(X_0,O_{X_0})$ satisfying the similar condition to $(4.2)_2'$, we denote by G the Koszul complex for $g:0\to O_X\xrightarrow{G_0}\cdots\to O_X^{\binom{t}{p}}\xrightarrow{G_p}\cdots\xrightarrow{G_{t-1}}O_X\to 0$. We assume:

$(4.2)_4'$ the locus W of $g\subset V$(=locus of f).

Now taking an open subset $U_{1,G}(\ni P_0)$ of U_0 and an element $\sigma_G\in R_1^{+2}$, we form a subset $\mathcal{M}_G(\subset\tilde{X}_1\times R^+)$ in the manner in $(4.1)_4$. Also taking an element $\bar{m}=\bar{m}_G\in Z^+$ we set: $\mathcal{T}_G:=\mathcal{M}_G\times Z_{\bar{m}}^+\times R_1^+$.

B'. Algebraic division unfiorm estimations

This part concerns mainly uniform estimations on open map properties of $\underset{\sim}{O}_X$-homomorphisms(Lemma 4.2 ~ 4.5). Such results are our main results on non cohomological uniform estimations in this paper.*) The proof of the results of § 4.2 will be given in §5.2.

1. A.d.estimation maps.**) We begin § 4.2 by the following

Definition 4.1. By an a.d.estimation map we mean the one:
$(4.2)_1$ $E:\underline{R}^+\times(\underline{Z}^+\times\underline{R}^+)\ni(r;m;a)\longrightarrow\underline{R}^+\times(\underline{Z}^+\times\underline{R}^+)\ni(r';m';a')$, where $r'=M_1(r)$, $m'=[L(m)]$ and $a'=M_2(a/r)\cdot\exp M_3(m)$. Here $M_i(1\leq i\leq 3)$ are positive monomials and L is a linear function:$L=ct;c>0$.

Fig.I

$$\begin{tikzcd}
\underline{R}^+\times(\underline{Z}^+\times\underline{R}^+) \arrow[r] \arrow[d, "E"] \arrow[dr, phantom, "G"] & \underline{R}^+\times\underline{Z}^+ \arrow[d, "M_1\times L"] \\
\underline{R}^+\times(\underline{Z}^+\times\underline{R}^+) \arrow[r] & \underline{R}^+\times\underline{Z}^+
\end{tikzcd}$$

We then set:
$(4.2)_1'$ $\underline{E}_{a.d}$:=collection of all a.d.maps.
Letting $E\in\underline{E}_{a.d}$ be as in $(4.2)_1$ we call M_1 and L the first and a.d.parts of E. The map E is factored as in Fig.I. (In Fig.I, the factor '$\underline{R}^+$' in the right side is the first factor of $\underline{R}^+\times(\underline{Z}^+\times\underline{R}^+)$.)

2. Algebraic and analytic a.d.properties. Take a finite set $\underline{f}=(f_j)_{j=1}^s\subset\Gamma(X_0,\underset{\sim}{O}_{X_0})$, which vanishes at $P_0\in X_0$(cf.$(4,1)_0$) and satisfies:
$(4.2)_2'$ $f_j\not\equiv 0(X_{0,i})$ for each $f_j(1\leq j\leq s)$ and each irreducible component $X_{0,i}$ of the germ of X_0 at P_0.
Then the set $\underline{f}^m:=(\underline{f}_j^m)_{j=1}^s$, the m-th homomorphism $F^m:\underset{\sim}{O}_X^s\to\underset{\sim}{O}_X$ and the sheaf $\underline{f}^m\underset{\sim}{O}_X:=F^m\underset{\sim}{O}_X^s(\subset\underset{\sim}{O}_X)$ will have the similar meaning to n.2, §2.1. As in §2 we use $\underline{f}^m$ to measure the a.d.properties of $\underset{\sim}{O}_X$-coherent sheaves. On the otherhand, letting V be the locus of $\underline{f}$, we also use the***) 'distance $|\underline{f}(P)|:=\Sigma_{j=1}^s|f_j(P)|$ to V' to measure such properties. We will compare the above two

---

*) cf.also the beginning of Chap.II.

**) 'A.d.' =algebrraic division(cf. § 2).

***) By Lojasiewicz inequality, we may regard $|\underline{f}(P)|$ also as the distance to V.

means for the measure of the a.d.properties. For this taking an open subset $U_{1,\tilde{X}_1}(\ni P_0)$ of U_0 and an element $\sigma_{\tilde{X}_1} \in \underline{R}_1^{+2}$, we form a subset $\mathcal{U}_{\tilde{X}_1}$ of $\tilde{X}_1 \times \underline{R}^+$ in the manner in $(4.1)_4$, by using $(U_{1,\tilde{X}_1}, \sigma_{\tilde{X}_1})$. Also taking an element $\bar{m} \in \underline{Z}^+$ we set: $\tau_{\tilde{X}_1} := \mathcal{U}_{\tilde{X}_1} \times \underline{Z}_{\bar{m}}^+ \times \underline{R}_1^+$. For an element $(P;r;m;a) \in \mathcal{U}_{\tilde{X}_1} (\subset \tilde{X}_1 \times \underline{R}^+ \times \underline{Z}^+ \times \underline{R}_1^+)$ we define:

$$(4.2)_2 \begin{Bmatrix} \Gamma(\tilde{U}_r(P), \underline{f}^m \underset{\sim}{O}_X)_a \\ \Gamma(\tilde{U}_r(P), \underset{\sim}{O}_X)_a^m \end{Bmatrix} := \begin{Bmatrix} F^m \Gamma(\tilde{U}_r(P), \underset{\sim}{O}_X^s; \theta)_a (\text{cf.}(4.1)_2') \\ \{\varphi \in \Gamma(\tilde{U}_r(P), \underset{\sim}{O}_X); |\varphi(P)| \leqq a|\underline{f}(P)|^m \text{ in } \tilde{U}_r(P)\} \end{Bmatrix}$$

In the above θ is the standard q-structure of $\underset{\sim}{O}_X^s$(cf.$(4.1)_2$ and Def.1.4_1). We then have:

Lemma 4.2.(Algebraic and analytic comparison of a.d.properties).

For a suitable a.d.map $E_{\tilde{X}_1} \in \underline{E}_{a.d}$ we have:*)

$$(4.2)_3 \quad i^* \Gamma(\tilde{U}_r(P), \underset{\sim}{O}_X)_a^m \subset \Gamma(\tilde{U}_{r'}(P), \underline{f}^{m'} \underset{\sim}{O}_X)_{a'}, \text{ with } (r';m';a') = E_{\tilde{X}_1}(r;m;a)$$

(cf.$(4.2)_1$). Here $(P;r;m;a)$ is in $\tau_{\tilde{X}_1} (\subset \tilde{X}_1 \times \underline{R}^+ \times \underline{Z}^+ \times \underline{R}^+)$.

Treatments of the left side of $(4.2)_3$ are sometimes easier than the right side;Lemma 4.2 is useful in treatments of the a.d.properties of $\underset{\sim}{O}_X$. Next we may regard Lemma 4.2 as an analogue of the comparison of 'p.g. and meromorphic'(as in Th.1.6) in our treatments of the a.d.properties. Moreover, as we will see in n.3, Lemma 4.2 implies Hilbert zero point theorem for f(Lemma 4.3'). Lemma 4.2 may be a basic fact in the a.d. properties of $\underset{\sim}{O}_X$.

3. Koszul complex--1. Taking a finite set $\underline{g} = (g_j)_{j=1}^t \subset \Gamma(X_0, \underset{\sim}{O}_{X_0})$ satisfying the similar condition to $(4.2)_2'$, we denote by $\underline{G}$ the Koszul complex**) for $\underline{g}: 0 \to \underset{\sim}{O}_X \xrightarrow{G_0} \cdots \to \underset{\sim}{O}_X^{\binom{t}{p}} \xrightarrow{G_p} \cdots \xrightarrow{G_{t-1}} \underset{\sim}{O}_X \to 0$. We assume:

$(4.2)_4'$ the locus W of $\underline{g} \subset$ V(=locus of $\underline{f}$).

Now taking an open subset $U_{1,\underline{G}}(\ni P_0)$ of U_0 and an element $\sigma_{\underline{G}} \in \underline{R}_1^{+2}$, we form a subset $\mathcal{U}_{\underline{G}}(\subset \tilde{X}_1 \times \underline{R}^+)$ in the manner in $(4.1)_4$. Also taking an element $\bar{m} = \bar{m}_{\underline{G}} \in \underline{Z}^+$ we set: $\tau_{\underline{G}} := \mathcal{U}_{\underline{G}} \times \underline{Z}_{\bar{m}}^+ \times \underline{R}_1^+$.

---

*) The symbol 'i' means the injection:$\tilde{U}_{r'}(P) \hookrightarrow \tilde{U}_r(P)$. In later arguments we use this symbol for the injections in question, without mentioning it (when no fear of confusions occurs).

**) In the terminology of § 2 3, $\underline{G}$=m(=1)-th Koszul homomorphism for $\underline{g}$.

Lemma 4.3.(Open map property for Koszul complex).

For a suitable a.d.map $E_{\underline{G}} \in \underline{E}_{a.d}$ we have$(1 \leqq p \leqq t-1)$:

$$(4.2)_4 \quad i^*(\Gamma(\tilde{U}_r(P),\underline{f}^m\underset{\sim}{O}_X^{\binom{t}{p}})_a \cap G_p^{-1}(0)) \subset G_{p-1}\Gamma(\tilde{U}_{r'}(P),\underline{f}^{m'}\cdot\underset{\sim}{O}_X^{\binom{t}{p-1}})_{a'}$$,with

$(r';m';a')=E_{\underline{G}}(r;m;a)$, where $(P;r;m;a)$ is in $\mathcal{V}_{\underline{G}}(\subset \tilde{X}_1 \times \underline{R}^+ \times \underline{Z}^+ \times \underline{R}_1^+)$.

We check that Lemma 4.2, 4.3 give a cohomological generalization of Hilbert zero point theorem. For this, taking a point $P \in \tilde{X}_1 \cap W$, we form a filtered complex $\underline{C}_P^*: 0 \to \underline{C}_P^0 \to \cdots \to \underline{C}_P^p \xrightarrow{G_p} \cdots \underline{C}_P^{t-1} \longrightarrow \underline{C}_P^t \to 0$, where $\underline{C}_P^p := \{\underline{f}^m \underset{\sim}{O}_{X,P}^{\binom{t}{p}}\}_{m}^{\infty}$ and the degree one map is $G_p(0 \leqq p \leqq t-1)$(cf.n.1, § 2.1).

Lemma 4.3'.(1) The complex $\underline{C}_P^*$ satisfies the open map property(Def.2.1).
(2) The open map property for $\underline{C}_P^*$ at the final step:$^{*)}$ $\underset{\sim}{O}_X^t \xrightarrow{G_{t-1}} \underset{\sim}{O}_X \to 0$ is equivalent to Hilbert zero point theorem for $(\underline{f},\underline{g})$:$\underline{g}\underset{\sim}{O}_{X,P} \supset \underline{f}^{\tilde{m}}\underset{\sim}{O}_{X,P}$, with a suitable $\tilde{m} \in \underline{Z}^+$.

Proof. The check of (2) is easy. To see (1) take an element $\lambda \in \underline{R}_1^{+2}$. Then '$f_j \equiv 0$ on W' implies: $|f_j(Q)| \leqq \lambda \cdot d(Q,W)$ in a small neighborhood U_P of P in X. By Lojasiewicz inequality we have: $|f_j(Q)^m| \leqq a \cdot |\underline{g}(Q)|^{m'}$, with suitable $m,m' \in \underline{Z}^+$ and $a \in \underline{R}^+$. Applying Lemma 4,2 to $\underline{g}$, we have: $f_j^m \cdot \underset{\sim}{O}_{X,P} \subset \underline{g}\underset{\sim}{O}_{X,P}$. By (2) this implies the open map property for $G_{t-1}: \underset{\sim}{O}_X^t \to \underset{\sim}{O}_X$. Finally, Lemma 4,3 insures the open map property for $G_p(0 \leqq p < t-1)$, and we have (1). q.e.d.

Hilbert zero point theorem may be the most basic fact on the a.d. properties of analytic varieties.Its cohomological generalization, Lemma 4.2 and Lemma 4.3,may be also basic in treatments of

---

*) This means that $G_{t-1}(\underline{f}^{m'}\underset{\sim}{O}_{X,P}^t) \supset \underline{f}^m\underset{\sim}{O}_X$ for $m \gg 0$.Here $m'=c(m)$, with a map $c:\underline{Z}^+ \to \underline{Z}^+$ satisfying $\lim_{m\to} c(m)=\infty$(cf.Def.2.1).

the a.d.properties. Also note that Koszul complex is intimately related to the structure of the cohomology groups $H^*(X_P-W_P,\underset{\sim}{O}_{X},_P),\ldots,$ where X_P,W_P are the germs of X,W at P. It looks like interesting to try applicablities of Lemma 4.2,4.3 to investigations of the structures of the above cohomology groups.(See also Lemma 4.7,n.6, where we use Lemma 4.1,4.2 in our analytic de Rham theory, by applying those lemmas to determine the structure of $H^*(X_P-W_P,\underset{\sim}{O}_X,_P),\ldots$) (We gave Lemma 4.3' in terms of the germ $\underset{\sim}{O}_X,_P$. Formulations of semi-global and global versions of Lemma 4.3 will be left to interested readers .)

<u>3. Koszul complex--2.</u> Lemma 4.3 concerns the single Koszul complex $\underline{G}$. Here we will be concerned with the family $\underline{F}=\{\underline{F}^m\}_{m=1}^{\infty}$ of the m-th Koszul complexes $\underline{F}_m$: $0\to\underset{\sim}{O}_X\xrightarrow{F_0^m}\cdots\to O_X^{\binom{s}{p}}\xrightarrow{F_p^m}\cdots\to O_X^s\xrightarrow{F_{s-1}^m}O_X\to 0$(cf.n.1,§ 2.3): letting the restricted parameter space $\mathcal{U}'_{\tilde{X}_1}(\subset\tilde{X}_1\times\underline{R}^+)$ be as in Lemma 4.2, we take a suitable linear function $L_0(t)=c_0t;c_0>0$. Then we have:

<u>Lemma 4.4.</u>(Open map property for the family $\underline{F}=\{\underline{F}^m\}_{m=1}^{\infty}$).

Choose a suitable a.d.map $E_{\tilde{X}_1}\in\underline{E}_{a.d}$. Then we have the following for each $m\in\underline{Z}^+(1\leq s\leq p-1)$:

$$(4.2)_5\quad i^*\{\Gamma(\tilde{U}_r(P),\underline{f}^{\tilde{m}}\underset{\sim}{O}_X^{\binom{s}{p}})_a\cap(\text{kernel of }F_p^m)\}\subset F_{p-1}^m\Gamma(\tilde{U}_{r'}(P),\underline{f}^{\tilde{m}'}\underset{\sim}{O}_X^{\binom{s}{p-1}})_{a'},$$

where $(r';\tilde{m}';a')=E_{\tilde{X}_1}(r;\tilde{m};a)$. Moreover, $(P;r)$ is in $\tilde{\mathcal{U}}'_{\tilde{X}_1}(\subset\tilde{X}_1\times\underline{R}^+)$, and the a.d.exponent $\tilde{m}\in\underline{Z}^+$ of $\underline{f}^{\tilde{m}}$ satisfies:$\tilde{m}>L_0(m)$(cf.also Lemma 2.5).

If we fix an element $m\in\underline{Z}^+$, which characterize the complex $\underline{F}^m$, then $(4.2)_5$ follows from Lemma 4.3. The independence of the map $E_{\tilde{X}_1}$ from such $m\in\underline{Z}^+$ is the key fact in Lemma 4.4. We use Lemma 4.4 to get the corresponding cohomological version, Lemma 2.5(cf.§ 4.4).

5. Exact complex. Letting the set $\underline{h}=\{h_u\}_{u=1}^{u_0}\subset\Gamma(X_0,\underset{\sim}{O}_{X_0})$ be as in Lemma 4.1, take a coherent sheaf $\underset{\sim}{H}\in\mathrm{Coh}(X_0;\underline{h})$ of the form in $(4.1)'_6$: $0\longrightarrow\underset{\sim}{O}_X^{k_p}\xrightarrow{K_{p-1}}---\xrightarrow{K_1}\underset{\sim}{O}_X^{k_1}\xrightarrow{K_0}\underset{\sim}{H}(\subset\underset{\sim}{O}_X^k)\rightarrow 0$. Moreover, taking a suitable open subset $U_1=U_{1,\underset{\sim}{H}}(\ni P_0)$ of $\mho_0$ and elements $\sigma=\sigma_{\underset{\sim}{H}}\in\underline{R}_1^{+2}$, $\bar{m}=\bar{m}_{\underset{\sim}{H}}\in\underline{Z}^+$, we form parameter spaces $\mathcal{U}_{\underset{\sim}{H}}$ and $\mathcal{T}_{\underset{\sim}{H}}$ by *)

$(4.2)'_6$ $\mathcal{U}_{\underset{\sim}{H}}:=\{(P;r)\in(\tilde{X}_1\cap U_1)\times\underline{R}^+;r<\{\sigma g(P)\}^{-1}\}$, $\mathcal{T}_{\underset{\sim}{H}}=\mathcal{U}_{\underset{\sim}{H}}\times\underline{Z}_{\bar{m}}^+\times\underline{R}_1^+$.

Lemma 4.5. (Open map property for exact complex).

For a suitable a.d.map $E_{\underset{\sim}{H}}\in\underline{E}_{a.d.}$ we have:

$(4.2)_6$ $i^*(\Gamma(\tilde{U}_r(P),\underline{f}^{\tilde{m}}\underset{\sim}{O}_X^k)_a\cap\Gamma(\tilde{U}_r(P),\underset{\sim}{H}))\subset K_0\Gamma(\tilde{U}_{r'}(P),\underline{f}^{\tilde{m}'}\underset{\sim}{O}_X^{k_1})_{a'}$, with $(r';\tilde{m}';a')=E_{\underset{\sim}{H}}(r;\tilde{m};a)$. Here $(P;r;\tilde{m};a)$ is in $\mathcal{T}_{\underset{\sim}{H}}(\subset\tilde{X}_1\times\underline{R}^+\times\underline{Z}_{\bar{m}}^+\times\underline{R}_1^+)$.

Note that Lemma 4.5 concerns an inclusion of Artin-Rees theorem type (cf. §2.1), and is used in the proof of the corresponding cohomological fact(cf.Lemma 2.7, §2.3. Also see §4.4.) As we will see in §5.2, the proof of Lemma 4.5 is easier than that of Lemma 4.3,Lemma 4.4, which concern the open map property of Koszul complexes.(Note that the Koszul complexes are not, in general, exact.)

6. Comparison of filtrations. Here we add a lemma, which is used in the proof of the comparison of the filtrations in Lemma 2.3. For this letting the sheaf $\underset{\sim}{H}\in\mathrm{Coh}(X_0;\underline{h})$ be as in Lemma 2.5, we define $\underset{\sim}{O}_X$-homomorphisms:

$(4.2)_7$ $K_{0,m}:=K_0+F^m:\underset{\sim}{O}_X^{k_1}+\underset{\sim}{O}_X^{sk}\ni\vartheta_1+\vartheta_2\longrightarrow\underset{\sim}{O}_X^k\ni K_0\vartheta_1+F^m\vartheta_2$, where we use the symbol F^m of the homomorphism: $\underset{\sim}{O}_X^s\rightarrow\underset{\sim}{O}_X$ also for its k-times direct sum: $\underset{\sim}{O}_X^{sk}:=\underset{\sim}{O}_X^s+---+\underset{\sim}{O}_X^s\longrightarrow\underset{\sim}{O}_X^k:=\underset{\sim}{O}_X+--+\underset{\sim}{O}_X$(cf.n.2, §2.1).

Also we use the symbol $\underset{\sim}{H}_m$ for the image of $K_{0,m}$: $\underset{\sim}{H}_m:=K_{0,m}(\underset{\sim}{O}_X^{k_1+sk})\subset\underset{\sim}{O}_X^k$.

---

*) cf. also $(4.1)_4$.

Next, in Lemma 2.6 soon below, we use an estimation map, which is slicely different from the a.d.maps in Lemma 4.2 ~4.5. For this we set:

$(4.2)_7''$ $\underline{E}'_{a.d}$:= collection of all maps $E':\underline{R}^+\times\underline{Z}^+\times\underline{R}^+\ni(r;m;a)\longrightarrow\underline{R}^+\times\underline{Z}^+\times\underline{R}^+\ni(r';m';a')$, where $(r';a')=\widetilde{M}(r;a)$, $m'=[L(m)]$, with a map $\widetilde{M}\in\underline{\widetilde{M}}$(cf.n.1,§4.1) and a linear map $L=ct;c>0$. In a similar manner to n.1, §4.2, we call the maps $M:\underline{R}^+\ni r\longrightarrow\underline{R}^+\ni r'$ and $L:\underline{Z}^+\ni m\longrightarrow\underline{Z}^+\ni m'$ the first and a.d.parts of E'. Then, as in n.1, §4.2, we have the factorization in Fig.II.

Fig.II

$$\begin{tikzcd}
\underline{R}^+\times\underline{Z}^+\times\underline{R}^+ \arrow[r] \arrow[d, "E"] \arrow[dr, phantom, "G"] & \underline{R}^+\times\underline{Z}^+ \arrow[d, "M\times L"] \\
\underline{R}^+\times\underline{Z}^+\times\underline{R}^+ \arrow[r] & \underline{R}^+\times\underline{Z}^+
\end{tikzcd}$$

Lemma 4.6. (Comparison of filtrations). For a suitable map $\mathcal{E}_{\underset{\sim}{H}}:\underline{Z}^+\ni m\longrightarrow\underline{E}'_{a.d}\ni E_{\underset{\sim}{H},m}$ satisfying Fig.III, we have:

$(4.2)_7$ $i^*(\Gamma(\widetilde{U}_r(P),\underset{\sim}{O}_X^k)_a\cap\Gamma(\widetilde{U}_r(P),\underset{\sim}{H}_m))\subset K_{0,m'}\Gamma(\widetilde{U}_{r'}(P),\underset{\sim}{O}_X^{k_1+sk})_{a'}$,

with $(r';m';a')=E_{\underset{\sim}{H},m}(r;m;a)$, where $(P;r;m;a)$ is as in Lemma 4.5. The proof of Lemma 4.6 is also given in §5.

Fig.III *)

$$\begin{tikzcd}
\underline{Z}^+ \arrow[r, "\mathcal{E}_{\underset{\sim}{H}}"] \arrow[d] \arrow[dr, phantom, "G"] & \underline{E}_a \arrow[d] \\
\{0\} \arrow[r] & \underline{M\times L}
\end{tikzcd}$$

Remark 4.2. We make here a remark on the explicit estimations in Lemma 4.2~4.5 and in Lemma 4.6. For this take an a.d.map $E\in\underline{E}_{a.d}$ of the form in $(4.2)_1$. We then define a series $\{E'_{E,m}\}_{m=1}^{\infty}$ of maps $E'_{E,m}\in\underline{E}'_{a.d}$, which satisfies the factorization in Fig.III, in the following manner:

$(4.2)_8$ $E'_{E,m}:\underline{R}^+\times\underline{Z}^+\times\underline{R}^+\ni(r;\tilde{m};a)\longrightarrow\underline{R}^+\times\underline{Z}^+\times\underline{R}^+\ni(r';\tilde{m}';a')$, where $(r';a')=(M_1(r),M_2(a/r)\cdot\exp M_3(m))$ and $\tilde{m}'[=L(\tilde{m})]$. (For the first and a.d.parts M_1,L of $E\in\underline{E}_{a.d}$ and the positive monomials M_2,M_3, see $(4.2)_1$.)

Also it is easy to see that the estimations in Lemma 4.2~4.5, for exampl that in Lemma 4.5, is given in the form:

$(4.2)_8'$ $(r';m';a')=E'_{E,m}(r;m;a)$, where $E\in\underline{E}_{a.d}$ is as in Lemma 4.5.

Remark that the dependence of $E'_{E,m}$ on $m\in\underline{Z}^+$ is quite explicit(cf.$(4.2)_8$), and we may say that the estimation $(4.2)_8$, derived from that in Lemma 4.5 ,.., is sharper than the one $(4.2)_7$ in Lemma 4.6.

---

*) The set $\{0\}$ consists of the single element $0\in\underline{Z}^+\cup 0$. Thus Fig.III claims that the first and a.d.parts of $E_{\underset{\sim}{H},m}(m\in\underline{Z}^+)$ are independent of $m\in\underline{Z}^+$. Moreover, we denote by Ln the set of all linear maps $L(t)=ct;c>0$.

7. De Rham complex. Here we assume that X_0 is irreducible at the origin P_0 of X_0. We also assume that the pair (X_1,X_2) is of the form: (X_0,X_0'), with a subvariety X_0' of X_0, and that the coordinares $z'=(z_1,..,z_k)$ provide a local parameter at each $P\in(X_0-X_0')$. We identify the sheaf Ω_X^p (cf.§3) with $O_X^{\binom{k}{p}}$ in the standard manner: $\Omega_X^p \ni \Sigma_I \varphi_I dx^I \xrightarrow{\mu} O_X^{\binom{k}{p}} \ni \{\varphi_I\}_I$, where I exhaust all indices of the form: $(i_1<.<..<i_p)$, with $1\leq i_1<..<i_p\leq k$. Letting the parameter space $\mathcal{T}_X$ be as in n 2, part B, § 4.1, we take a parameter $(P;r;a;m)\in\mathcal{T}_X(\subset X\times R^+\times R_1^+\times R_1^+\times Z^+)$, and we set:

$$(4.2)_9'' \quad \Gamma(\tilde{U}_r(P),f^m\cdot\Omega_X^p)_a := \mu^{-1}\Gamma(\tilde{U}_r(P),f^m O_X^{\binom{k}{p}})_a .$$

Then, letting d_X be the exterior differential operator on X, we have:

Lemma 4.7. (Open map property for de Rham complex). For a suitable a.d. map $E_X\in E_{a.d}$ we have $(p\leqq 1)$:

$$(4.2)_9 \quad i^*(\Gamma(\tilde{U}_r(P),f^m\Omega_X^p)_a \cap d_X^{-1}(0)) \subset d_X\Gamma(\tilde{U}_{r'}(P),f^{m'}\Omega_X^{p-1})_{a'} ,$$ with

$(r';m';a')=E_X(r;m;a)$, where $(P;r';m';a')$ is in $\mathcal{T}_X(\subset X\times R^+\times R^+\times R_1^+\times Z^+)$.

Lemma 4.7 is derived from Th.1.2_2, Lemma 4.3 and from our uniform estimation on the a.d.properties of (local) contractible properties of analytic varieties(cf.[$]_2$). The latter concerns some topological properties of the varieties, and the details of it will be given elsewhere in a near future. We summarize the key points of the proof of Lemma 4.7 at the end of § 5.2. Lemma 4.7 is used in the proof of our 'p. g. open map property for the de Rham complex' as in Lemma 3.2. The relation of Lemma 4.7 to Lemma 3.2 is also summarized at the end of § 5.2.

Now, letting the parameter (P;r) be as in Lemma 4.7, we form a filtered complex $C_r^*(P)$ by

$$(4.2)_9''' \quad 0\to\Gamma(\tilde{U}_r(P),C)\to\{\Gamma(\tilde{U}_r(P),\Omega_{M,m}^p)\}_{m=0}^{\infty}\to ---\xrightarrow{d_X}\{\Gamma(U_r(P),\Omega_{X,m}^p)\}_{m=0}^{\infty}\longrightarrow$$

where we set:

$(4.2)_9^{iv}$ $\Omega^p_{X,m} := f^m \cdot \Omega^p_X + (\Sigma_{j=1}^{s} f_j^{m+1} \cdot dx_j) \wedge \Omega^{p-1}_X$.

Corollary 4.1.(1) The direct system $\{C^*_r(P); r \in (0;1)\}$ satisfies the open map property.

(2) The following complex is exact(formal Poincare lemma):

$$0 \to C \to \varprojlim_m \Omega^0_X/f^m\Omega^0_X \to \quad \xrightarrow{d_X} \quad \to \varprojlim_m \Omega^p_X/f^m\Omega^p_X \to \quad .$$

The first follows easily from Lemma 4.7(by dropping the explicit estimation in it), and the second follows from the first by Prop 2.1_1 . It is in the form of (1), Cor 4.1 that S. Lubkin conjectured the open map property for the de Rham complex . The formal Poincare lemma $(4.9)_3^{'''}$ was proved by R Hartshorne and by A Fujiki, independently, by using the resolution theorem of H Hironaka. (Their methods are also independent.) The open map property in (1), Cor. 4.1 is also proven by A. Fujiki by using the resolution theorem. (His proof also uses some local contractible properties of analytic varieties.)

Remark. The content of part B, § 4.1 contains all examples of complexes , which we know, where the open map property hold. From the basic property of Artin-Rees theorem in the completion theory as well as from the scope of our examples of the open map properties as above, it looks like that the open map properties deserve to be studied for more general types of (geomtric complexes). The author hopes that the content of part B call attension of analytic geometeres, who are working with complexes of geometric nayute(on analytic varieties).

8. Some remarks. Here we summarize some remarks for Lemma 4.1~4.6, which will be used in the proof of those lemmas(cf. §5).

(i) Terminologies. We begin n.8, by arranging some terminologies for later convenience. First recall that the estimation in Lemma 4.1' was given to points of $\tilde{X}_1=X_1-X_2$ and that the underlying homomorphism was: $K:O_X^v\longrightarrow O_X^u$; we will use the terminology:

$(4.3)_1$ Lemma 4.1' holds for $(X_1,X_2;K)$

as a synonym for 'the estimation $(4.1)_5$(in Lemma 4.1') holds' for the parameter $(P;r;a)$ as in $(4.1)_5$. Here $(P;r;a)$ should be in the parameter space of the form λ_K as in $(4.1)_5$, and the estimation map should be of the form $M_K\in M$ as in $(4.1)_5$. Similarly to the above we use the terminology:

$(4.3)_2$ Lemma 4.2 holds for $(X_1,X_2;f)$(resp.Lemma 4.3 holds for $(X_1,X_2;f,G$ Lemma 4.4 holds for $(X_1,X_2;f)$, Lemma 4.5 holds for $(X_1,X_2;f,K)$ or Lemma 4.6 holds for $(X_1,X_2;f,K)$)

as a synonym for the **following:**

$(4.3)_2'$ the estimation $(4.2)_3$(resp.$(4.2)_4$,$(4.2)_5$,$(4.2)_6$ or $(4.2)_7$) holds for the parameter $(P;r;m;a)$ as in $(4.2)_3$(resp.$(4.2)_3$,....).

(Here note that $(4.2)_3\sim(4.2)_6$ are the explicit estimations in Lemma 4.2 ~ Lemma 4.6. Also remark that f,(f,G),.. are the underlying geometric data in Lemma 4.2 ~ Lemma 4.6.) Moreover, for the first terminology in $(4.3)_2$, the parameter $(P;r;m;a)$ should be in the parameter space $\mathcal{T}_{\tilde{X}_1}$ as in Lemma 4.2 and the estimation map should be of the form $E_{\tilde{X}_1}\in E_{a.d}$ as in Lemma 4.2. For the other terminologies in $(4.3)_2$, the parameter spaces and the estimation maps should be understood in the similar manner to the above.

(ii) Next taking subvarieties X_1', X_2' of X_0 satisfying: $X_1' \supset X_1, X_2' \subset X_2$, we have the following implication:

$(4.3)_3'$ Lemma $4.1'$ for $(X_1', X_2'; K) \longrightarrow$ Lemma $4.1'$ for $(X_1, X_2; K)$.

This is checked easily, by remarking that the estimations in the left and right sides are given to points in $(X_1' - X_2')$ and $(X_1 - X_2)$ and that the estimation in the left side is applied to the right side. (See also the explicit estimation in Lemma $4.1'$.) By $(4.3)_3'$ we have:

$(4.3)_3''$ Lemma $4.1'$ for $(X_0, X_{0,\text{sing}}) \longrightarrow$ Lemma $4.1'$ (=Lemma $4.1'$ for $(X_1, X_2; K)$).

We prove Lemma $4.1'$ in the form of the left side. The similar implications to the above hold for Lemma 4.2 $\sim$ Lemma 4.6.

(iii) Chain property. Thirdly take a subvariety X_3 of X_0 satisfying

$(4.3)_4$ $\quad X_0 \supset X_1 \supsetneqq X_2 \supsetneqq X_3 \supset X_0'$.

Then we have the following implication, which will play a role in the proof of Lemma $4.1'$ (cf. § 5.1):

Proposition 4.1. Lemma $4.1'$ for $(X_i, X_{i+1}; K)(i=1,2) \rightarrow$ that for $(X_1, X_3; K)$

The similar implication to the above also holds for Lemma 4.2 $\sim$ Lemma 4.6. The proof of Prop.4.2 is given in part C, App.I.

(iv) Here we add a technical remark for the proof of Lemma 4.5: rrcall that the sheaf $\underset{\sim}{H}$ in Lemma 4.5 is in the collection $\text{Coh}(X_0; \underline{h})_{p.g}$, where $\underline{h} = (h_u)_{u=1}^{u_0}$ is a subset of $\Gamma(X_0, \underset{\sim}{O}_{X_0})$. We then have:

$(4.3)_5$ Lemma 4.5 for $(X_0, X_0'; \underset{\sim}{H})$ for the case : $\# h = 1 \longrightarrow$ that for the genral case: $\# h \gtrsim 2$.

Actually, assuming that $\# h \gtrsim 2$, we easily have that $\underset{\sim}{H} \in \text{Coh}(X_0; h_u)$

It is easy to get "Lemma 4.1' for $(X_1,X_3;K)$" from the above two estimation: (The former is given to points $P\in(X_1-X_3)$, and the size r of $\tilde{U}_r(P)$ should satisfy the inequality of the form: $r<\{\sigma d(P,X_3)^{-1}\}^{-1}$. It is easy to fill the gap between what is mentioned just above and $(4.3)_{5,6}$, by using elementary distance properties of analytic varieties; see also the author's forthcoming paper [ ].)

(iv) Here we add a technical remark for the proof of Lemma 4.1 and Lemma 4.5, Lemma 4.6: recall that the sheaf $\underset{\sim}{H}$ in these lemmas is in $\mathrm{Coh}(X_0;\underline{h})_{p.g}$, where $\underline{h}=(h_u)_{u=1}^{u_0}$ is a subset of $\Gamma(X_0,\underset{\sim}{O}_{X_0})$. We then have:

$(4.3)_7$ Lemma 4.1 ~~holds~~ for $(X_0,X_0';K)$ for the case: $\#\underline{h}=1 \Rightarrow$ that for the general case: $\#\underline{h}\geq 2$, where X_0' =locus of $\underline{h}$ (and the similar fact for Lemma 4.5 and Lemma 4.6).

Actually, let $\underline{h}=\{h_u\}_{u=1}^{u_0}$ be as in Lemma 4.1, we apply the left side of $(4.3)_7$ to each $h_u(u=1,\dots,u_0)$. Then the inclusion of the form in $(4.1)_7$, Lemma 4.1 holds for each $P\in X_0-D_u$, and the size of the manifold $\tilde{U}_r(P)$ (cf.$(4.1)_7$) should satisfy: $r<\{\sigma d(P,D_u)^{-1}\}^{-1}$. (Here D_u is the locus of h_u.) But $X_0'=\bigcap_u D_u$ Also, by the Lojasiewicz inequality , we have:

$(4.3)_8$ $\underline{c}'d(P,X_0')<\sum_u d(P,D_u)<\underline{c}\,d(P,X_0')$, with suitable $\underline{c},\underline{c}'\in\underline{R}^{+2}$.

This implies:

$(4.3)_9$ $\underset{\sim}{\tilde{c}}d(P,X_0')<d(P,D_u)$, with a suitable index u, where the element $\underline{\tilde{c}}\in\underline{R}^{+2}$ is deteremined by the element $\underline{c}$ in $(4.3)_8$.

From this the inequality mentioned just above is replaced by $r<\{\sigma d(P,X_0')^{-1}\}^{-1}$, and we get $(4.3)_7$(cf also similar arguments in (iii)).

(v) A key proposition. Here we give a key proposition for Lemma 4.2 $\sim$ Lemma 4.5(cf. Prop. 4.2). In (v) we take an element $h \in \Gamma(X_0, \mathcal{O}_{X_0})$ satisfyin $D \supset X_{0,\mathrm{sing}}$, where D=divisor of h. Now we assume the following:

(4.4) Lemma 4.4 for $(X_0, D; \underline{f})$(resp. Lemma 4.3 for $(X_0, D; \underline{f}, \underline{G})$, Lemma 4.4 for $(X_0, D; \underline{f})$ or Lemma 4.5 for $(X_0, D; \underline{f}, \underset{\sim}{H})$)(cf. $(4.3)_1$), where the Koszul complex $\underline{G}$ and the sheaf $\underset{\sim}{H} \in \mathrm{Coh}(X_0; h)$ are as in Lemma 4.3 and Lemma 4.5. Then taking a suitable*) open set $U_1 = U_{1,h} (\ni P_0)$ of U_0 and an element $\sigma = \sigma_h \in \underline{R}_1^{+2}$, we define the following parameter space(cf. also $(4.1)_5$):

$(4.4)'$ $\mathcal{U}_h := \{(P;r) \in (D - X_{0,\mathrm{sing}}) \cap U_1) \times \underline{R}^+;\ r < \{\sigma d(P, X_{0,\mathrm{sing}})^{-1}\}^{-1}\}$.

Also we take a suitable $d = d_h \in \underline{Z}^+$, an a.d.map $E_h \in \underline{E}_{a.d}$, a linear function $L_{0,h} = L_{0,h} = c_{0,h} t; c_{0,h} > 0$ and an element $\bar{m} = \bar{m}_h \in \underline{Z}^+$. Then, from the four uniform estimations in (4.4), we get the following weakere version of Lemma 4.2 for $(D, X_{0,\mathrm{sing}}; \underline{f}), \ldots$, Lemma 4.5 for $(D, X_{0,\mathrm{sing}}; \underline{f}, \underset{\sim}{H})$.

Proposition 4.2. We have the following inclusions:

$(4.5)_1$ $i^* h^d \Gamma(\tilde{U}_r(P), \mathcal{O}_X)^{\tilde{m}}_a \subset \Gamma(\tilde{U}_{r'}(P), \underline{f}^{\tilde{m}'} \mathcal{O}_X)_{a'}$,

$(4.5)_2$ $i^* h^d \Gamma(\tilde{U}_r(P), \underline{f}^{\tilde{m}} \mathcal{O}_X^{\binom{t}{p}})_a \cap G_p^{-1}(0)) \subset G_{p-1} \Gamma(\tilde{U}_{r'}(P), \underline{f}^{\tilde{m}'} \cdot \mathcal{O}_X^{\binom{t}{p-1}})_{a'} (1 \leq p < t)$,

$(4.5)_3$ $i^* h^d (\Gamma(\tilde{U}_r(P), \underline{f}^{\tilde{m}} \mathcal{O}_X^{\binom{s}{p}})_a \cap (F_p^m)^{-1}(0)) \subset F_{p-1}^m \Gamma(\tilde{U}_{r'}(P), \underline{f}^{\tilde{m}'} \mathcal{O}_X^{\binom{s}{p-1}})_{a'} (1 \leq p < s)$

$(4.5)_4$ $i^* h^d (\Gamma(\tilde{U}_r(P), \underline{f}^{\tilde{m}} \mathcal{O}_X^k)_a \cap \Gamma(\tilde{U}_r(P), \underset{\sim}{H})) \subset K_0 \Gamma(\tilde{U}_{r'}(P), \underline{f}^{\tilde{m}'} \mathcal{O}_X^k)_{a'}$.

(In $(4.5)_4$, the homomorphism K_0 is as in Lemma 4.5) In the above, the estimation is: $(r'; \tilde{m}'; a') = E_h(r; \tilde{m}; a)$. Moreover, the parameter $(P;r)$ is in $\mathcal{U}_h$ $(\subset (D - X_{0,\mathrm{sing}}) \times \underline{R}^+)$ and a is in $\underline{R}^+$. The element $\tilde{m} \in \underline{Z}^+$ in $(4.5)_{1,2,4}$ satisfie $\tilde{m} > \bar{m}$, while the pair $(\tilde{m}, m) \in \underline{Z}^+ \times \underline{Z}^+$ in $(4.5)_3$ satisfies: $\tilde{m} > L_0(h)$.

We prove Prop. 4.2 in §4.2. Note that if, we drop the term 'h^d' from the left sides of $(4.5)_{1\sim4}$, we get 'Lemma 4.2 for $(D, X_{0,\mathrm{sing}}; \underline{f})$', ..., and Prop. 4.2 will play a key role in the proof of Lemma 4.2 $\sim$ Lemma 4.5(cf. §5.2).

---

*) Precisely, the data $(U_{1,h}, \sigma_h)$ and (d_h, E_h), .. depend also on $\underline{G}$ or $\underset{\sim}{H}$, according as we are concerned with $(4.5)_2$ or $(4.5)_4$. (The above data also depend on $\underline{f}$. As in Lemma 4.2$\sim$Lemma 4.6, we write E_h, ... instead of $E_{h,\underline{f}}$, ...)

92

C. Global version of the results in A,B

Taking an analytic variety*) $\underline{\tilde{X}} = (C^n(z) \times C^n(z'), \tilde{X} = C^n \times U_0', P_0') \in An_0$ (cf. $(1.7)_0$ and an affine variety*) $\underline{X}' \in C^n(z)$ (cf. $(1.11)_0$), we give here a global version**) of Lemma 4.1 to $\underline{X}^* := \underline{\tilde{X}}$ or $= \underline{X}'$ and that of Lemma 4.4~Lemma 4.6 to X' (In the above the euclidean space $C^n(z)$, ... and the open set $U_0'(\ni P_0')$ in $C^{n'}$ are as in $(1.7)_0$.)

1 Global version of Lemma 4.1. First taking an open set $U_1'(\ni P_0')$ of U_0' and an element $\sigma = \sigma_{\underline{X}^*} \in R_1^{+2}$, we attach to $\underline{X}^*$ the following parameter space (cf. also $(4.1)_1$):

$(4.6)_0$ $\mu_{\underline{X}^*} := \{(Q;r) \in \underline{X}_1^* \times \underline{R}^+;\ r < \{\sigma g_{\underline{X}^*}(Q)\}^{-1}\}$, where $\underline{X}_1^* := \left\{ \begin{matrix} \underline{C}^n \times U_1' \\ \underline{X}' \end{matrix} \right\}$ and $g_{\underline{X}^*} := \left\{ \begin{matrix} |\tilde{z}| + 1 \\ |z| + 1 \end{matrix} \right\}$, according as $\underline{X}^* = \left\{ \begin{matrix} \underline{\tilde{X}} \\ \underline{X}' \end{matrix} \right\}$ (Recall that $g_{\underline{X}^*}$ is the p.g. function of $\underline{X}^*$ and $\tilde{z} = (z, z')$ (cf. n.1 and n.5, § 1.2).)

$(4\ 6)_0'$ $\lambda_{\underline{X}^*} := \mu_{X^*} \times \underline{R}_1^+$.

Next we set:

$(4.6)_0''$ $\mathrm{Coh}^*(\underline{X}^*)_{p.g} := \mathrm{Coh}'(\underline{\tilde{X}})_{p.g}$ or $\mathrm{Coh}''(\underline{X}')_{p.g}$ (cf. $(1.4)_9$ and $(1.18)_1$). (Recall that such collections consist of the p.g. coherent sheaves over $\underline{X}^*$ satisfying certain algebraic conditions (as in $(1.4)_9$ and $(1.18)_1$).

---

*) As in Chap. I, we use the symbol $\underline{\tilde{X}}$ also for its underlying variety $\tilde{X}$. Also recall that the affine variety $\underline{X}'$ in $(1.11)_0$ consists of the data $(\underline{C}^n(z), X', H_{X'})$, where X' is the underlying variety of $\underline{X}'$ and $H_{X'}$ is the resolution of the structure sheaf of X' (cf. $(1.11)_0$). We also use the symbol $\underline{X}'$ for its underlying variety X'.

**) Similar global results for Lemma 4.2, Lemma 4.3 also hold for $\underline{X}^* = \underline{\tilde{X}}$ or $\underline{X}'$. Also, for $\underline{X}^* = \underline{\tilde{X}}$, similar global facts for Lemma 4.4~Lemma 4.6 hold. But we do not use such facts (cf § 2).

For an element $\underset{\sim}{H} \in \mathrm{Coh}^*(\underline{X}^*)_{p.g}$, the q-structures $\theta_{\underset{\sim}{H}}, \theta'_{\underset{\sim}{H}}$ will be as in $(4.1)_2$. Also for an element $(P;r;a) \in \lambda_{\underline{X}^*}(\subset \underline{X}_1^* \times \underline{R}^+ \times \underline{R}_1^+)$, the set of the cochains $\Gamma(\widetilde{U}_r(P), \underset{\sim}{H}; \widetilde{\theta}_{\underset{\sim}{H}})_a$, where $\widetilde{\theta}_{\underset{\sim}{H}} = \theta_{\underset{\sim}{H}}$ or $\theta'_{\underset{\sim}{H}}$, will have the similar meaning to $(4.1)_2$. (Here, as in $(4.1)_2$, we set $\widetilde{U}_r(P) := \{Q \in \underline{X}^*; d(P,Q) < r\}$, with the natural metric d in $\underline{C}^n(z) \times \underline{C}^n(z')$ or $\underline{C}^n(z)$.)

Lemma 4.8_1. (Uniform estimation with bound for $\underline{X}^* = \underline{X}$ or $\underline{X}'$).

There are maps*) $\mathcal{E}_{\underline{X}^*} : \mathrm{Coh}^*(\underline{X}^*)_{p.g} \ni \underset{\sim}{H} \to \widetilde{\underline{M}} \ni \widetilde{M}_{\underset{\sim}{H}}$ and $\mathcal{E}'_{\underline{X}^*} : \underline{Z}^+ \to \underline{M}$, which satisfy the factorization in Fig.I and with which we have the following**) for each $\underset{\sim}{H} \in \mathrm{Coh}^*(\underline{X}^*)_{p.g}$:

$(4.6)_1$ $i^* \Gamma(\widetilde{U}_r(P), \underset{\sim}{H}; \theta'_{\underset{\sim}{H}})_a \subset \Gamma(\widetilde{U}_{r'}(P), \underset{\sim}{H}; \theta_{\underset{\sim}{H}})_{a'}$, with $(r';a') = \widetilde{M}_{\underset{\sim}{H}}(r;a)$. Here $(P;r;a)$ is in $\lambda_{\underline{X}^*}(\subset \underline{X}_1^* \times \underline{R}^+ \times \underline{R}_1^+)$.

Fig.I*)

$$\begin{tikzcd}
\mathrm{Coh}^*(\underline{X}^*)_{p.g} \arrow[r, "\mathcal{E}_{\underline{X}^*}"] \arrow[d, "lg"'] \arrow[dr, phantom, "\circlearrowright"] & \widetilde{\underline{M}} \arrow[d] \\
\underline{Z}^+ \arrow[r, "\mathcal{E}'_{\underline{X}^*}"'] & \underline{M} \\
\underline{X}^* \arrow[u, phantom] &
\end{tikzcd}$$

We prove Lemma 4.8_1 in n.3, and we derive Lemma 1.2 from Lemma 4.8_1 in §4.4

2. Globalization of Lemma 4.4 ~ Lemma 4.6. to $\underline{X}'$. In n.2 we let the set $\underline{f}' = (f'_j)_{j=1}^s \subset \Gamma(\underline{X}', \underset{\sim}{O}_{X'})_{p.g}$, the m-th Koszul complex $\underline{F}'^m$ for $\underline{f}'$: $0 \to \underset{\sim}{O}_{X'} \xrightarrow{F_0'^m} \cdots \to \underset{\sim}{O}_{X'}^{\binom{s}{p}} \xrightarrow{F_p'^m} \cdots \to \underset{\sim}{O}^s_{X'} \xrightarrow{F'^m} \underset{\sim}{O}_{X'} \to 0$ and the sheaves $\underline{f}'^m \underset{\sim}{O}_{X'} (= \underline{F}'^m \underset{\sim}{O}^s_{X'})$ be as in Lemma 2.5, §.2.3. In Lemma $4.8_{2\sim4}$ we globalize Lemma 4.4 ~ Lemma 4.6 to $\underline{X}'$, by using the sheaves $\underline{f}'^m \underset{\sim}{O}_{X'}, \ldots$ The estimation maps in Lemma $4.8_{2\sim4}$ will be in $\underline{E}'_{a.d}$ (cf. $(4.2)''_7$). We use the symbols π_1, π_2 for

---

*) For the sets $\widetilde{\underline{M}}, \underline{M}$ of the estimation maps and the map 'lg', ..., see Lemma 4.1, § 4.1.

**) i=inclusion: $\widetilde{U}_{r'}(P) \hookrightarrow \widetilde{U}_r(P)$. When there is no fear of confusions, we use the symbol 'i' for the inclusion in question, without mentioning it (cf. also § 4.1, § 4.2).

the assignments: $\underline{E}'_{a.d} \ni E' \longrightarrow M, L$, where M, L are the first and a.d. parts of E'. Then, taking a suitable linear function $L_0 = L_{0,X}(t) = c_0 t; c_0 > 0$, we globalize Lemma 4.4 to $\underline{X}'$ in the following manner:

Lemma 4.8_2. (Open map property for the Koszul complexes $\{\underline{F}'^m\}_{m=1}^{\infty}$). There is a map $\mathcal{E}_{\underline{X}'} : \underline{Z}^+ \ni \tilde{m} \longrightarrow \underline{E}'_{a.d} \ni E_{\underline{X}', \tilde{m}}$, which satisfies the similar factorization to Fig. III, Lemma 4.6, and with which we have the following for each $(\tilde{m}, m) \in \underline{Z}^+ \times \underline{Z}^+$ satisfying $\tilde{m} > L_0(m) (1 \leqq p < s)$:

$$(4.6)_2 \quad i^*(\Gamma(\tilde{U}_r(P), \underline{f}'^{\tilde{m}} \Omega_{\underline{X}'}^{(\substack{s\\p})})_a \cap (F_p'^m)^{-1}(0)) \subset F_{p-1}'^m \Gamma(\tilde{U}_{r'}(P), \underline{f}'^{\tilde{m}'} \Omega_{\underline{X}'}^{(\substack{s\\p-1})})_{a'},$$

with $(r'; \tilde{m}'; a') = E_{\underline{X}', \tilde{m}}(r; \tilde{m}; a)$. Here $(P; r; a) \in (\underline{X} \times \underline{R}^+ \times \underline{R}_1^+)$ is as in Lemma 4.8_1.

Next letting the sheaf $H' (\subset O_{X'}^k)$ and the homomorphism $K'_{0,m} : O_{X'}^{k_1} \to H'$ be as in*) Lemma 2.7, we define a sheaf H'_m and a homomorphism $K'_{0,m} : O_{X'}^{k_1 + sk}$ $\to H'_m (m \in Z^+)$ in the similar manner to $(4\ 2)'_7$:

$$(4.6)'_3 \quad K'_{0,m} : O_{X'}^{k_1} + O_{X'}^{sk} \ni \varphi_1 + \varphi_2 \longrightarrow O_{X'}^k \ni K'_0 \varphi_1 + F^m \varphi_2, \text{ and } H'_m := K'_{0,m} O_{X'}^{k_1 + sk} (\quad O_X^k \}$$

Moreover, taking suitable $\sigma = \sigma_{H'} \in R_1^{+2}$ and $\bar{m} = \bar{m}_{H'} \in Z^+$, we form parameter spaces:

$$(4.6)''_3 \quad \mu_{H'} := \{(P; r) \in X' \times R^+; r < \{\sigma g_{X'}(P)\}^{-1}\}, \quad \tau_{H'} := \mu_{H'} \times Z_{\bar{m}}^+ \times R_1^+.$$

Lemma 4.8_3. There is a map $\mathcal{E}_{H'} : Z^+ \ni \tilde{m} \quad E'_{a.d} \ni E_{H', \tilde{m}}$, which satisfies the similar factorization to Fig. III, Lemma 4.6, with which we have the following:

(1) (Open map property for the sheaf H'):

$$(4.6)_3 \quad i^*(\Gamma(\tilde{U}_r(P), f'^m O_{X'}^k)_a \cap \Gamma(\tilde{U}_r(P), H')) \subset K'_0 \Gamma(\tilde{U}_{r'}(P), f'^m O_{X'}^{k_1})_{a'}$$

(2) (Comparison of filtrations):

$$(4.6)_4 \quad i^*(\Gamma(\tilde{U}_r(P), O_{X'}^k)_a \cap \Gamma(U_r(P), H'_m)) \subset K'_{0,m} \Gamma(\tilde{U}_{r'}(P), O_{X'}^{k_1 + sk})_{a'}$$

In the above $(r'; \tilde{m}'; a') = E_{H', \tilde{m}}(r; \tilde{m}; a)$, and $(P; r; m; a)$ is in $\tau_{H'} (\subset X' \times R^+ \times Z^+ \times R^+)$.

*) The map K'_0 is the first resolution, denoted by $\omega_{H'}$, in the sheaf H' in Lemma 2.7. The symbol 'K'_0' is concordant to the one in Lemma 4.5, Kemma 4.6, and is convenient for the arguments on Lemma 4.8_3.

We use Lemma $4.8_{2,3}$ in the proof of Lemma 2.3,2.5 and Lemma 2.7(cf§4.2).

3. Proof of Lemma $4.8_{1\sim 3}$. (i) For the proof of Lemma $4.8_{1\sim 3}$, we will give a natural compactification(=completion) of $\underline{X}^*=\tilde{\underline{X}}$ or $=\underline{X}'$. For this we first set:

$(4.7)_1$ $P^n(\underline{C}):=\bigcup_{j=0}^{n}\underline{C}_j^n$, with $\underline{C}_0^n:=\underline{C}^n$. (Namely, $P^n(\underline{C})$=projective space, which is the natural completion of $\underline{C}^n$. Moreover, $\underline{C}_j^n$ are euclidean spaces, which cover $P^n(\underline{C})$.)

$(4.7)_2$ $\bar{\underline{X}}^*:=P^n(\underline{C})\times U_1'$(cf.$(4.6)_0$) or the completion $\bar{\underline{X}}'$ of $\underline{X}'$ in $P^n(\underline{C})$, and

$(4.7)_3$ $D^*:=D\times U_1'$ or $D\cap\bar{\underline{X}}'$, with $D:=P^n(\underline{C})-\underline{C}^n$.

We then take a point $P\in D^*$ and a small neighborhood $\tilde{U}$ of P in $\bar{\underline{X}}^*$. Also taking an element $\sigma=\sigma_P\in\underline{R}_1^{+2}$, we form parameter spaces μ_P, λ_P similarly to $(4.6)_0$:

$(4.7)_4$ $\mu_P:=\{(Q;r)\in(\tilde{U}-D^*)\times\underline{R}^+;\ r<\{\sigma g_{\underline{X}^*}(Q)\}^{-1}\}$, $\lambda_P:=\mu_P\times\underline{R}_1^+$.

Then the following analogue of Lemma 4.1,Lemma 4.4 at the 'point at infinity' $P\in D^*$' will suffice to insure Lemma $4.8_{1,2}$:

Lemma 4.1''. Take a suitable map $\mathcal{E}_P:\mathrm{Coh}^*(\underline{X}^*)_{p.g}\ni\underset{\sim}{H}\to\tilde{\underline{M}}\ni\tilde{M}_{\underset{\sim}{H}}$. Then $\mathcal{E}_P$ satisfies the similar factorization to Fig.I,Lemma 4.8_1,and we have the following for each $\underset{\sim}{H}\in\mathrm{Coh}^*(\underline{X}^*)_{p.g}$:

$(4.8)_1$ $i^*\Gamma(\tilde{U}_r(Q),\underset{\sim}{H};\theta'_{\underset{\sim}{H}})_a\subset\Gamma(\tilde{U}_{r'}(Q),\underset{\sim}{H};\theta_{\underset{\sim}{H}})_{a'}$, with $(r';a')=\tilde{M}_{\underset{\sim}{H}}(r;a)$. Here $(P;r;a)$ is in $\lambda_P(\subset(\tilde{U}-D^*)\times\underline{R}^+\times\underline{R}_1^+)$.

Lemma 4.4'. Take a suitable map $\mathcal{E}_P:\underline{Z}^+\ni\tilde{m}\longrightarrow\underline{E}'_{a.d}\ni E_{P,\tilde{m}}$ and a linear function $L_0=L_{0,P}(t)=c_0t;c_0>0$. Then we have the similar factorization to Fig.II, Lemma 4.8_2 and we also have the following for each $(\tilde{m},m)\in\underline{Z}^+\times\underline{Z}^+$ satisfying $\tilde{m}>L_0(m)(1\leqq s<p)$:

$(4.8)_2$ $i^*(\Gamma(\tilde{U}_r(Q),\underline{f}'^{\tilde{m}}\underset{\sim}{O}_{\underline{X}'}^{(\overset{s}{p})})_a\cap(F_p'^{m})^{-1}(0))\subset F_{p-1}'^{m}\Gamma(\tilde{U}_{r'}(Q),\underline{f}'^{\tilde{m}'}\underset{\sim}{O}_{\underline{X}'}^{(\overset{s}{p}-1)})_{a'}$,

with $(r';\tilde{m}';a')=E_{P,\tilde{m}}(r;\tilde{m};a)$, where $(Q;r;a)$ is in λ_P.

We use Lemma $4.8_{2,3}$ in the proof of Lemma 2.3, 2.5 and Lemma 2.7(cf.§4.2).

3. Proof of Lemma $4.8_{1\sim 3}$. (i) For the proof of Lemma $4.8_{1\sim 3}$, we will give a natural compactification(=completion) of $\underline{X}^*=\underline{\tilde{X}}$ or $=\underline{X}'$. For this we first set:

$(4.7)_1$ $P^n(C):=\bigcup_{j=0}^{n} C_j^n$, with $C_0^n:=C^n$. (Namely, $P^n(C)$=projective space, which is the natural completion of C^n. Moreover, C_j^n are euclidean spaces, which cover $P^n(C)$.)

$(4.7)_2$ $\bar{\underline{X}}^*:=P^n(C)\times U_1'$(cf.$(4.6)_0$) or the completion $\bar{\underline{X}}'$ of $\underline{X}'$ in $P^n(C)$, and

$(4.7)_3$ $D^*:=D\times U_1'$ or $D\cap\bar{\underline{X}}'$, with $D:=P^n(C)-C^n$.

We then take a point $P\in D^*$ and a small neighborhood $\tilde{U}$ of P in $\bar{\underline{X}}^*$. Also taking an element $\sigma=\sigma_P\in R_1^{+2}$, we form parameter spaces μ_P, λ_P similarly to $(4.6)_0$:

$(4.7)_4$ $\mu_P:=\{(Q;r)\in(\tilde{U}-D^*)\times R^+;\ r<\{\sigma g_{\underline{X}^*}(Q)\}^{-1}\}$, $\lambda_P:=\mu_P\times R_1^+$.

Then the following analogue of Lemma 4.1, Lemma 4.4 at the 'point at infinity' $P\in D^*$ will suffice to insure Lemma $4.8_{1,2}$:

Lemma 4.1''. Take a suitable map $\mathcal{E}_P:\mathrm{Coh}(\underline{X}^*)_{p.g}\ni H\to\underline{\tilde{M}}\ni\tilde{M}_H$. Then $\mathcal{E}_P$ satisfies the similar factorization to Fig. I, Lemma 4.8_1, and we have the following for each $H\in\mathrm{Coh}^*(\underline{X}^*)_{p.g}$:

$(4.8)_1$ $i^*\Gamma(\tilde{U}_r(Q),H;\theta'_H)_a\subset\Gamma(\tilde{U}_{r'}(Q),H;\theta_H)_{a'}$, with $(r';a')=\tilde{M}_H(r;a)$. Here $(P;r;a)$ is in $\lambda_P(\subset(\tilde{U}-D^*)\times R^+\times R_1^+)$.

Lemma 4.4''. Take a suitable map $\mathcal{E}_P:Z^+\ni\tilde{m}\longrightarrow E'_{a.d}\ni E_{P,\tilde{m}}$ and a linear function $L_0=L_{0,P}(t)=c_0t;c_0>0$. Then we have the similar factorization to Fig. II, Lemma 4.8_2 and we also have the following for each $(\tilde{m},m)\in Z^+\times Z^+$ satisfying $\tilde{m}>L_0(m)(1\leqq s<p)$:

$(4.8)_2$ $i^*(\Gamma(\tilde{U}_r(Q),f'^{\tilde{m}}O_{X'}^{(\frac{s}{p})})_a\cap(F_p'^{m})^{-1}(0))\subset F_{p-1}'^{m}\Gamma(\tilde{U}_{r'}(Q),f'^{\tilde{m}'}O_{X'}^{(p^{s}-1)})_{a'}$,

with $(r';\tilde{m}';a')=E_{P,\tilde{m}}(r;\tilde{m};a)$, where $(Q;r;a)$ is in λ_P.

$(4.8)_5'$ $\tilde{U}_r^j(Q) := \{R \in \underline{X}^* \cap \underline{C}_j^{*n};\ d_j^*(R,Q) < r\}$.

Then the following distance comparison between d^* and d_j^* is checked easily, by using $(4.8)_4'$:

Proposition 4.3. For a suitable positive monomial $M_P \in \underline{M}$, we have the following for each $(Q;r) \in (\tilde{U}-D^*) \times \underline{R}^+$ satisfying $r < \{\tilde{\sigma} g_{\underline{X}^*}(Q)\}^{-1}$, with a suitable $\tilde{\sigma} \in \underline{R}_1^{+2}$.

$(4.8)_5$ $T^{*j}(\tilde{U}_{r'}(Q)) \subset \tilde{U}_r^j(Q)$, and $T^{*j}(\tilde{U}_r(Q)) \supset \tilde{U}_{r'}^j(Q)$, with $r' = M_P(r)$, where the open set $\tilde{U}_r(Q)$ is as in Lemma 4.8_1.

Now, Lemma 4.1" is checked as follows: taking an open set $\hat{U}(\ni P)$ in the ambient space $\underline{C}_j^{*n}$ of $\tilde{U}(\subset \underline{X}^*)$ satisfying $\hat{U} \cap \underline{X}^* = \tilde{U}$, we define a local analytic variety $\underline{X}_P^* \in \underline{An}_{la}$ (cf. $(1.8)_0$) by

(a) $\underline{X}_P^* := (\underline{C}_j^{*n}(z^{*j}), \hat{U}, \tilde{U}, z_j^j, P_0)$,

and we attach the collection $\mathrm{Coh}''(\underline{X}_P^*)_{p,g}$ to $\underline{X}_P^*$ by $(1.18)_1$. Then, remarking that D^* is the divisor of z_j^j, we have:

(b) $\mathrm{Coh}^*(\underline{X}^*)_{p,g} \subset \mathrm{Coh}''(\underline{X}_P^*)_{p,g}$ (cf. $(4.6)_0''$).

By this we apply Lemma 4.1 to*) $H \in \mathrm{Coh}(\underline{X}^*)_{p,g}$. Then, we have the similar inclusion to $(4.8)_1$, Lemma 4.1", by changing the open set $\tilde{U}_r(P)$, which is required in Lemma 4.1", to $\tilde{U}_r^j(Q)$; using Prop. 4.3, we can replace $\tilde{U}_r^j(Q)$ by $\tilde{U}_{r'}(Q)$, and we have Lemma 4.1" from Lemma 4.1.

Next, the check of Lemma 4.4' is similar to the above, and is as follows: first we remark that

$(c)_1$ $\underline{\tilde{f}}' := (z_j^j)^d \underline{f}' \subset \Gamma(\tilde{U}, \underset{\sim}{O}_{\tilde{U}})$, with a suitable $d \in \underline{Z}^+$,

and that $\underline{\tilde{f}}'^m \underset{\sim}{O}_{\underline{X}'} = \underline{f}'^m \underset{\sim}{O}_{\underline{X}'}$ in $\tilde{U}-D^*$. Then we easily have:

$(c)_2$ $\Gamma(\tilde{U}_r, \underline{\tilde{f}}'^m \underset{\sim}{O}_{\underline{X}'})_a \subset \Gamma(\tilde{U}_r, \underline{f}'^m \underset{\sim}{O}_{\underline{X}'})_a$, $\Gamma(\tilde{U}_r, \underline{f}'^m \underset{\sim}{O}_{\underline{X}'})_a \subset \Gamma(\tilde{U}_r, \underline{\tilde{f}}'^m \underset{\sim}{O}_{\underline{X}'})_{a'}$,

where $a' = M_{P,m}(a/r)$, with a positive monomial $M_{P,m}$. which is independent from $Q \in \tilde{U}-D^*$. (Here we write $\tilde{U}_r(Q)$ as $\tilde{U}_r$.)

Using a similar argument to the proof of Lemma 4.1", we see easily that the comparisons of the distance and cochains as in Prop. 4.3 and $(c)_2$ lead

---

*) Recall that Lemma 4.1 is also applied to such a sheaf $\underset{\sim}{H}$ by (1), Remark 4.1 at the end of part A, § 4.1 ,

(d) Lemma 4.4 $\longrightarrow$ Lemma 4.4 .

(Here we apply Lemma 4.4 to the set $\tilde{f}'$ at the point at infinity $P \in D$.) Thus we checked Lemma 4.1" and Lemma 4.4', and we also have Lemma 4.8_1 and Lemma 4.8_2(cf. (1)).

4. Finally we add here an elementary uniform estimation on local parametrization of the variety $\underline{X}=(C^n(z),U_0,X_0,X_0',P_0)$(cf. $(4.1)_0$). We set $X=X_0-X_0'$, and we assume that $X_0'=X_{0,\text{sing}}$. Also taking a suitable open subset $U_{1,X}(\ni P_0)$ of U_0 and an element $\sigma_X \in R_1^{+2}$, we form a parameter space*) $\mathcal{U}_X := \{(Q;r) \in (U_{1,X} \cap X) \times R^+ ; r < \{\sigma_X \cdot g_X(P)\}^{-1}\}$, with $g_X(P):=d(P,X_0')^{-1}$.

Proposition 4.4. For a suitable positive monomial $M_X \in \underline{\tilde{M}}$ we have the following for each $(P;r) \in \mathcal{U}_X (\subset X \times R^+)$:

(4.9) there is an analytic map $\omega : U_{r'}(P) \longrightarrow \tilde{U}_r(P)$, which is the identity on $\tilde{U}_{r'}(P)$, with $r' := M_X(r)$. Here $U_r(P) := \{Q \in C^n ; d(Q,P) < r\}$, and $\tilde{U}_r(P) := U_r(P) \cap$

We use Prop. 4.4 for the proof of Lemma 1.4(for the local variety $X \in An_{1a}$ as in Lemma 1.4).The check of Prop. 4.4 is given in part A, App I.

Next letting the affine variety $X' \subset C^n(z)$ and the divisor $S \subset C^n \times U_0$ be as in Lemma 1.4 and Lemma 1.3*), we fake elements $\sigma = \sigma_{X'}$, $\tilde{\sigma} = \tilde{\sigma}_S \in R_1^{+2}$ and an open subset U' of U_0, and we form parameter spaces:

(a) $\mathcal{U}_{X'} := \{(P;r) \in X' \times R^+ ; \; r < \{\sigma g_{X'}(P)\}^{-1}$, and $\mathcal{U}_S := \{(P;r) \in (S \cap (C \times U_0)) \times R^+ ; r < \{\tilde{\sigma} g_S(P)\}^{-1}\}$, where the p.g. function $g_{X'}$ of X' is as in Lemma 1.4, and we set $g_S := |w|$, with the coordinate of C(cf. Lemma 1.3).

We then have the following analogur of Prop. 1.4 for X' and S.

Proposition 4.4'. For suitable positive monomials $M_{X'}$, M_S, we have the following for each $(P;r) \in \mathcal{U}_{X'}$ and $(P';r) \in \mathcal{U}_S$:

(4.9)' there are analytic maps $\omega : U_{r'}(P) \to (U_r(P) \cap X')$ and $\omega' : U_{r''}(P') \to (U_r(P') \cap S)$, which are the identities on $(U_{r'}(P) \cap X')$, $(U_{r''}(P') \cap S)$, where $r' = M_X(r)$ and $r'' = M_S(r)$. Also the discs $U_r(P), U_r(P')$ in $C^n(z)$, $C(w) \times C^n(z)$ have the similar meaning to the one $U_r(P)$ in Prop 4.4.

---

*)

We use Prop 4.4 for the proof of Lemma 1.4 for $\underline{X}'$ and of Lemma 1.3 .

Proof. Let $\overline{\underline{X}}'$ and $\overline{S}$ denote the completions of X', S in $P^n(C)$, $P(C)\times U_0$. Then, taking a points $P_0 \in \overline{X}'-X'$ and $\tilde{P}_0 \in (P(\underline{C})-\underline{C}) \times U_0$, we have the similar fact to Prop. 4.4 for (X', P_0) and $(S, \tilde{P}_0)$(using a similar arguments to Lemma 4.1"). Then, using the distance comparison, Prop.4.3, and the similar arguments to the ones in n.3, we have Prop.4.4' from the above analogues of Prop.4.4 at the points at infinity P_0, $\tilde{P}_0$ and from Prop 4.4 (applied to finite points $P\in\underline{X}'$ and $\tilde{P}\in S$). q.e.d .

§4.2. Proof of the lemmas in Chapter I

In n.1∼n.3 we give a cohomological version of Lemma 4.1∼4.8, and, using such a result, we prove the lemmas in Chap.I, Lemma 1.2 and Lemma 2.3 Lemma 2.5 as well as Lemma 2.7, which concern the uniform estimations on the sheaf homomorphisms. Also, using the results in n.1∼n.3, we prove Prop.4.2 in n.4. Moreover, we prove Lemma 1.3,1.4 in n.5,n.6, by using Prop.4.~~4.~~ 4

1. Comparison of cohomological and non cohomological estimations.

Here we give propositions, which play a key role in the proof of the first set of the lemmas just above(cf.Prop.$4.5_{1,2}$). Such propositions will be given in an abstract manner in terms of q-sheaves and is more general than the one used in the proof of the lemmas. In n.1 we fix q-sheaves $(\underset{\sim}{K},\theta_{\underset{\sim}{K}}),(\underset{\sim}{H},\theta_{\underset{\sim}{H}})$ and a homomorphism $\omega:\underset{\sim}{K}\longrightarrow\underset{\sim}{H}$(of abelian sheaves). Also we fix a p.g.function $g:X\longrightarrow\underline{R}_1^+$ and a distance function $d:X\times X\rightarrow[0,\infty)$ satisfying $d=0$ on the diagonal $\Delta_X(\subset X\times X)$(cf.Def.$1.4_4$ and Def.1.6_1). Moreover, we fix an element $\tilde{\sigma}\in\underline{R}_1^{+2}$, and we assume the following for each $P\in X$:

$(4.10)_0$ $g(Q)/2 < g(P) < 2g(Q)$ for each $Q\in\tilde{U}_{\tilde{\sigma}}(P;g):=\{Q\in X;d(P,Q)<\sigma g(P)\}^{-1}$

$(4.10)_0'$ the triangular inequality:$d(Q_1,Q_3)\leqq d(Q_1,Q_2)+d(Q_2,Q_3)$: hold for any $Q_i\in U_{\tilde{\sigma}}(P;g)(1\leqq i\leqq 3)$.

For a point $P\in X$ and an element $(r;a)\in\underline{R}^+\times\underline{R}_1^+$ we set:

$(4.10)_0''$ $\Gamma(\tilde{U}_r(P),\underset{\sim}{H};\theta_{\underset{\sim}{H}})_a:=\{\varphi\in\Gamma(\tilde{U}_r(P),\underset{\sim}{H});|\varphi(Q)|_{\underset{\sim}{H}}<a$ in $\check{U}_r(P)\}$, where $|\;|_H=\theta_H$-absolute value*) and $\check{U}_r(P):=\{Q\in X;d(P,Q)<r\}$.

(We use the simialr notation for $(\underset{\sim}{K},\theta_{\underset{\sim}{K}})$.) Also, for the formulation of Prop.$4.5_{1,2}$, we fix a subset Y of X, and we form the following parameter spaces(cf.also $(4.1)_4$):

---

*) cf.Def.1.4_1.

$(4.10)_0$ $\mathcal{U}_Y := \{(P;r) \in Y \times \underline{R}^+;\ r < \{\tilde{\sigma}g(P)\}^{-1}\}$, $\lambda_Y := \mathcal{U}_Y \times \underline{R}_1^+$.

(i) Bdd estimations$\rightarrow$p.g.estimations*). First taking an estimation map $\widetilde{M}=(M_1,M_2) \in \widetilde{\underline{M}}=\underline{M}\times\underline{M}$(cf.n.1, § 4.1), we assume the following uniform estimation for $\omega:\underset{\sim}{K}\longrightarrow\underset{\sim}{H}$ on Y:

$(4.10)_1$ $i^*\Gamma(\widetilde{U}_r(P),\underset{\sim}{H};\theta_{\underset{\sim}{H}})_a \subset \omega\Gamma(\widetilde{U}_{r'}(P),\underset{\sim}{K};\theta_{\underset{\sim}{K}})_{a'}$, with $(r';a')=\widetilde{M}(r;a)$, where $(P;r;a)$ is in $\lambda_Y(\subset Y\times\underline{R}^+\times\underline{R}_1^+)$ and i=inclusion:$\widetilde{U}_{r'}(P)\hookrightarrow\widetilde{U}_r(P)$.

Note that $(4.10)_1$ is of similar form to the estimation in Lemma 4.8_1. In Prop.4.5_1 soon below we give a cohomological version of $(4.10)_1$, which is of similar form to the estimations in Lemma 1.2. In Prop.4.5_1 we fix an estimation map $\widetilde{\underline{L}}\in\widetilde{\underline{L}}:\underline{R}^{+2}\times\underline{R}^{+2}\longrightarrow\underline{R}^{+2}\times\underline{R}^{+2}$, which is determined by $\widetilde{M}\in\widetilde{\underline{M}}$. (For the explicit dependence of $\widetilde{L}$ on $\widetilde{M}$, see $(4.10)_7$,(iii) in the proof of Prop.4.5_1.)

Proposition 4.5_1.(Bdd estimations$\rightarrow$p.g.estimations).. For each element $(Y';\sigma;\alpha)\in 2^Y\times\underline{R}_{\tilde{\alpha}}^{+2}\times\underline{R}_1^{+2}$ we have:

$(4.10)_2$ $s^*C^q(\underline{A}_\alpha(Y'),\underset{\sim}{H};\theta_{\underset{\sim}{H}})_\alpha \subset \omega C^q(\underline{A}_{\alpha'}(Y'),\underset{\sim}{K};\theta_{\underset{\sim}{K}})_{\alpha'}$, with $(\sigma';\alpha')=\widetilde{\underline{L}}(\sigma;\alpha)$, where

$(4.10)_2'$ $\left\{\begin{matrix}\underline{A}_\alpha(Y')\\ C^q(\underline{A}_\alpha(Y'),\underset{\sim}{H};\theta_{\underset{\sim}{H}})_\alpha\end{matrix}\right\} := \left\{\begin{matrix}\text{g-p.g.covering of } Y' \text{ in } Y \text{ of size } \sigma \text{(Def.1.6}\\ \text{set of } (g,\theta_{\underset{\sim}{H}})\text{-}\alpha\text{-growth cochains with value}\\ \text{in } \underset{\sim}{H}\text{(cf.}(1.3)_6\text{).}\end{matrix}\right.$

Moreover, s is the p.g.refining map:$\underline{A}_{\alpha'}(Y')\hookrightarrow\underline{A}_\alpha(Y')$(cf.Def.$1.6_2$).

We prove Prop.4.5_1 in (iii). In n.2 we use Prop.4.5_1 for the proof of Lemma 1.2.

(ii) A.d.estimations$\longrightarrow$D.p.estimations.

We give here a key proposition,Prop.4.5_2, for the implication:

Fig.I

$$\begin{tikzcd}
O^{sk} \arrow[r, "F^m"] \arrow[dr] & O^k \arrow[d, "\omega_K"'] & O^{sh} \arrow[r, "F^m"] \arrow[dr] & O^h \arrow[d, "\omega_H"'] & \\
 & K \arrow[rr, "\omega_m"] & & H \arrow[r, "\omega'_{H'}"] & H'
\end{tikzcd}$$

$(4.10)_3$ Lemma $4.8_{2,3}$ $\longrightarrow$Lemma 2.5,2.7 and Lemma 2.3.

---

*) The estimations of the left and right sides in the title of (i),(ii) concern respectively non cohomological and cohomological uniform estimation (cf.Lemma $4.8_{1\sim3}$ and Lemma 1.2,Lemma 2.5,2.7,....).

**) $\widetilde{\underline{L}}=\underline{L}\times\underline{L}$, with the collection $\underline{L}$ of all el-maps(cf.n.5, § 1.2).

For this we fix a sheaf O of ring over X, and we assume that (1) K, H are O-modules, and (2) K, H are homomorphic images of O^k, O^h $(h,k>0)$: $O^k \xrightarrow{\omega_K} K \to 0$, $O^h \xrightarrow{\omega_H} H \to 0$. Moreover, we fix an abelian sheaf H' and series of homomorphisms $\omega_m : K \to H$, $\omega'_m : H \to H'$ $(m=1,2,\ldots)$ satisfying $\omega'_m \circ \omega_m = 0$. Furthermore, we fix a subset $\underline{f}=(f_j)_{j=1}^{s} \subset \Gamma(X, O_X; g)_{p.g}$ (cf. $(1.3)'''_1$), and we use the symbol $\underline{F}^m$ (=m-th homomorphism for $\underline{f}$): $O^s \to O$ also for its k- and h-direct sums: $O^{sk} \to O^k$, $O^{sh} \to O^h$ (cf. n.2, § 2.1. See also Fig.I.)

For an element $(P;r;\tilde{m};a) \in X \times \underline{R}^+ \times \underline{Z}^+ \times \underline{R}^+$ we set:

$(4.10)''_3$ $\Gamma(\tilde{U}_r(P), \underline{f}^{\tilde{m}}K)_a := \omega_K F^{\tilde{m}} \Gamma(\tilde{U}_r(P), O^{sk})_a$ (cf. $(4.10)''_0$ and Fig.I).

(We use the similar notation for H.) Now taking a linear function $L_0(t)= c_0 t; c_0>0$ and an a.d.map $\tilde{E} \in \underline{E}_{a.d}$ (cf. n.1, § 4.2), we assume the following uniform estimations for the series $\{\omega_m\}$, $\{\omega'_m\}$ $(m \in \underline{Z}^+)$:

(4.10)$_3$ For each $(\tilde{m},m) \in \underline{Z}^+ \times \underline{Z}^+$ satisfying $\tilde{m} > L_0(m)$ we have:

$(4.10)'_3$ $i^*(\Gamma(\tilde{U}_r(P), \underline{f}^{\tilde{m}}H)_a \cap \omega'^{-1}_m(0)) \subset \omega_m \Gamma(\tilde{U}_{r'}(P), \underline{f}^{\tilde{m}'}K)_{a'}$, with $(r';\tilde{m}';a') = \tilde{E}(r;\tilde{m};a)$, where $(P;r;a)$ is in $\lambda_Y (\subset Y \times \underline{R}^+ \times \underline{R}_1^+)$ (cf. $(4.10)'''_0$).

Note that (4.10)$_3$ is a similar inclusion to Lemma 4.4. In Prop.4.5$_2$ soon below, taking a d.p.estimation map $E \in \underline{E}_{d.p}$ (cf. n.4, § 2.1)*), we give a cohomological version of (4.10)$_3$, which is similar to Lemma 4.8$_2$:

Proposition 4.5$_2$.(Add.estimation → d.p.estimation).

For each $(\tilde{m},m) \in \underline{Z}^+ \times \underline{Z}^+$ satisfying $\tilde{m} > L_0(m)$ we have:

$(4.10)_4$ $s^*(C^q_\ell(\underline{A}_\sigma(Y'), \underline{f}^{\tilde{m}}H)_\alpha \cap \omega'^{-1}_m(0)) \subset \omega_m C^q_\ell(\underline{A}_{\sigma'}(Y'), \underline{f}^{\tilde{m}'}K)_{\alpha'}$, with $(\sigma';\tilde{m}';\alpha') = E(\sigma;m;\alpha)$, where $(Y';\sigma;\alpha)$ is in $2^Y \times \underline{R}^{+2}_{\sigma} \times \underline{R}^{+2}_1$, and

$(4.10)'_4$ $C^q_\ell(\underline{A}_\sigma(Y'), \underline{f}^m H)_\alpha$:=left $(\underline{f},g)$-d.p.filtered set of cochains with value in H (cf. $(2.3)_{3,5}$).

---

*) As in Prop.4.5$_1$, the map $E \in \underline{E}_{d.p}$ is determined by $\tilde{E} \in \underline{E}_{a.d}$ in a simple fashion (cf. $(4.10)_7$).

(iii) Proof of Prop.$4.5_{1,2}$. (1) Letting the positive monomials M_1, M_1' be the first parts*) of the estimation maps M,E in $(4.10)_3$, we first define el-maps $\bar{L}_1, \bar{L}_1' : R^{+2} \ni \sigma \longrightarrow R^{+2} \ni \sigma', \sigma''$ by the equation:

$(4.10)_5'$ $(\sigma'.t)^{-1} = M_1((\sigma t)^{-1})$, $(\sigma'' t)^{-1} = M_1'((\sigma t)^{-1})$, where t is a variable.

Then taking an el-map $L_0 : R^{+2} \ni (\alpha_1, \alpha_2) \longmapsto R^{+2} \ni (4\alpha_1 2^{\alpha_2^2}, \alpha_2)$, we define:

$(4.10)_5$ $L_1 = L_0 \cdot L_1 \cdot L_0$, $L_1' = L_0 \cdot \bar{L}_1' \cdot L_0$.

Such el-maps will be the first components of the desired estimation maps L,E in Prop.$4.5_{1,2}$(cf.$(4.10)_7$). Then, letting $\sigma \in R_1^{+2}$ be as in Prop.$4.5_{1,2}$ we set:

$(4.10)_5''$ $\sigma'' = L_0(\sigma), \tilde{\sigma}'' = \tilde{\bar{L}}_1(\sigma'')$ and $\tilde{\sigma}' = L_0(\tilde{\sigma}'')(= \tilde{L}_1(\sigma))$, where $(\tilde{\bar{L}}_1, \tilde{L}_1) = (\bar{L}_1, L_1)$ or $(\bar{L}_1', L_1')$.

Next taking an element $A' \in N^{q+1}A_{\sigma'}(Y')$(cf.Prop.$4.5_{1,2}$) and a point $Q \in |A'|$, we set: $A = s(A') \in N^{q+1}A_{\sigma}(Y')$, where s=p.g.refining map: $A_{\sigma'}(Y') \hookrightarrow A_{\sigma}(Y')$ (Def.1.6_2). Then we have the following from $(4.10)_5''$ and $(4.10)_0, (4.10)_0'$:

$(4.10)_5'''$ $|A| \supset \tilde{U}_r(Q) \supset \tilde{U}_{r'}(Q) \supset |A'|$, where $r = \{\sigma'' g(Q)\}^{-1}, r' = \{\tilde{\sigma}'' g(Q)\}^{-1}$.

(Note that, by $(4.10)_5''$, we have: $r' = \tilde{M}_1(r)$, with $\tilde{M}_1 = M_1$ or M_1' (cf.$(4.10)_{1,3}$)

The relation $(4.10)_5'''$ will be a key fact for the proof of Prop.$4.5_{1,2}$.

(2) Now taking elements $\varphi \in C^q(A_\sigma(Y'), H; \theta_H)_\alpha$ and $\tilde{\Psi} \in C^q(A_\sigma(Y'), f^m H)_\alpha$ (cf.$(4.10)_{2,4}$), we write $\tilde{\Psi}$ explicitly as $\tilde{\Psi} = \omega_H F^m \Psi$, with $\Psi \in C^q(A_\sigma(Y'), O^{sk})_\alpha$ (cf.$(4.10)_{2,4}$). Then from $(4.10)_0$ we have:

$(4.10)_6''$ $|\varphi_A(R)|, |\Psi_A(R)| < a := \alpha' g(Q)$ in $\tilde{U}_r(Q)$, where $\alpha' = (\alpha_1 2^{\alpha_2^2}, \alpha_2)$ with $\alpha = (\alpha_1, \alpha_2)$.

We will apply $(4.10)_{1,3}$ to $\varphi_A, \tilde{\Psi}_A$ in $U_r(Q)$. Then there are elements $\varphi_1 \in \Gamma(\tilde{U}_{r'}(Q), K)_{a'}$ and $\tilde{\Psi}_1 \in \Gamma(\tilde{U}_{r''}(Q), f^{m'}K)_{a''}$ satisfying

$(4.10)_6'$ $\varphi_A = \omega\varphi_1$, $\tilde{\Psi}_A = \omega_m \tilde{\Psi}_1$, with $(r';a') = \tilde{M}(r;a)$ and $(r'';\tilde{m}';a'') = E'(r;\tilde{m};a)$, where the estimation maps $\tilde{M} \in \tilde{M}$ and $E' \in E_{a.d}$ are as in $(4.10)_{1\sim 3}$.

We write $\tilde{\Psi}_1$ as $\tilde{\Psi}_1 = \omega_K F^m \Psi_1$, with $\Psi_1 \in \Gamma(\tilde{U}_{r''}(Q), 0^{sk})_{a''}$ (cf. Fig. I). We note that $(4.10)_6'$ implies:

$(4.10)_6''$ $|\varphi_1| < a'$ and $|\Psi_1| < a''$.

Then from that $r = \{\tilde{\sigma}'' g(Q)\}^{-1}$ (cf. $(4.10)_5$) and $(4.10)_0$, together with the explicit forms of the estimation maps $\tilde{M}, E'$, we easily see that $(4.10)_6''$ is rewritten in the form:

$$(4.10)_6 \begin{Bmatrix} |\varphi_1(R)| < \partial' \cdot g(R) \\ |\Psi_1(R)| < \partial'' \cdot g(R) \end{Bmatrix}, \text{ with } \begin{Bmatrix} \partial' = L_2(\partial + \sigma) \\ \partial'' = \exp M(m) \cdot L_2'(\partial + \sigma) \end{Bmatrix} ,$$

where the el-maps L_2, L_2' and the positive monomial M are determined by the maps $\tilde{M}, E'$ in $(4.10)_3$.

Finally, letting $L: Z^+ \longrightarrow Z^+$ be the a.d. part of E' (cf. $(4.2)_1$), we define a map $\tilde{L} \in \underset{\sim}{\tilde{L}} = \underset{\sim}{L} \times \underset{\sim}{L}$ and a d.p.c. map $E \in E_{d.p}$ by the following.

$$(4.10)_7 \begin{cases} \tilde{L}: R^{+2} \times R^{+2} \ni (\sigma; \partial) \rightarrow R^{+2} \times R^{+2} \ni (L_1(\sigma), L_2(\partial + \sigma)) \\ E: R^{+2} \times R^{+2} \times Z^+ \ni (\sigma; \partial; m) \longrightarrow R^{+2} \times R^{+2} \times Z^+ \ni (L_1'(\sigma), \exp M(m) \cdot L_2'(\partial + \sigma), L(m) \end{cases}$$

We take the estimation maps $\tilde{L}$ and E to be the desired ones in Prop. $4.5_{1,2}$

Then remarking that the restrictions $\varphi', \tilde{\Psi}'$ of $\varphi_1, \tilde{\Psi}_1$ to A' satisfy:

$(4.10)_8$ $s^* \varphi = \omega\varphi'$ and $s^* \tilde{\Psi} = \omega_n \tilde{\Psi}'$,

we have $(4.10)_{2,4}$. q. e. d.

Prop. 4.5_2 will be used in the proof of Lemma 2.5, when the variety is the local one $X \in An_{1a}$ (cf. n.3, §4.2). Here we give a slice modification of Prop. 4.5_2, which is used in the proof of the other lemmas in § 2.

Remark 4.3.(i) First take a series $\{E_{\tilde{m}}\}_{\tilde{m}=1}^{\infty}$ of estimation maps $E_{\tilde{m}} \in E'_{a.d}$(cf. $(4.2)''_7$), which satisfies the similar factorization to Fig. III, Lemma 4.6, and we make the following change of the estimation in $(4.10)'_3$:

$(4.10)'_9$ $\quad (r;\tilde{m};a) \longrightarrow (r';\tilde{m}';a') = E_{\tilde{m}}(r;\tilde{m};a)$

Then, letting the el-map $L'_1 : R^{+2} \longrightarrow R^{+2}$ and the linear map L be as in $(4.10)_5$, we have the following inclusion, which is similar to $(4.10)_4$, from the arguments in the proof of Prop 4.5_2(cf., in particular, $(4.10)_{5\sim 8}$):

$(4.10)_9$ $\quad s^*(C_1^q(A_{\sigma}(Y'), f^{\tilde{m}}H)_{p.g} \cap \omega'^{-1}_{\tilde{m}}(0)) \subset \omega_{\tilde{m}}\, C_1^q(A_{\alpha}(Y'), f^{\tilde{m}'}K)_{p.g}$, with $(\sigma';\tilde{m}') = (L(\sigma), [L(\tilde{m})])$. (For the p.g. subgroup as above, see $(2,3)_6$.)

We use the above remark in the proof of Lemma 2.5, when the variety is $X' \in$ Aff.

(2) Next we assume that the homomorphisms ω_m, ω'_m in Prop. 4.5_2 are independent of $m \in Z^+$: $\omega = \omega_1 = \omega_2 = , \ldots$ and $\omega' = \omega'_1 = \omega'_2 =, \ldots$. Also take an element $\bar{m} \in Z^+$. Then, assuming the similar inclusion to $(4.10)'_3$ for each*) $\tilde{m} \geqq \bar{m}$, we obviously have the similar inclusion to $(4.10)_4$ for such $\tilde{m} \in Z^+$. We use this fact for the proof of Lemma 2.7(given to $X \in An_{1a}$). Finally, we assume that the similar inclusion to $(4.10)_3$ holds for each $\tilde{m} \geqq \bar{m}$, by changing the estimation in $(4.10)_3$ to $(4.10)_9$. Then we have the similar inclusion to $(4.10)_9$ for each $\tilde{m} \geqq \bar{m}$. We use this for the proof of Lemma 2.7(given to $X' \in$ Aff). (We also use a slice modification of Prop. 4.5_2 in the proof of Lemma 2.3. Such a modification is given in the proof of Lemma 2.3 in n. 3, § 4.2.)

---

*) Precisely, we replace the inequality: $\tilde{m} > L_0(m)$ in $(4.10)_3$ by $\tilde{m} \geqq \bar{m}$

2. Proof of Lemma 1.2. Here we derive Lemma 1.2 from Lemma 4.1, Lemma 4.8_1, by using Prop 4.5_1:

$(4.11)_1$ non cohomological estimation in Lemma 4.1, Lemma 4.8_1 $\xrightarrow{\text{Prop } 4.5_1}$ cohomologcal estimation in Lemma 1.2.

For this we set:

$$(a) \begin{cases} X^* := \tilde{X} \in An_0, \ X \in An_{1a} \text{ or } X' \in \mathrm{Aff}(\text{cf.Lemma 1.2}), \\ \mathrm{Coh}^*(\underline{X}^*)_{p.g} := \mathrm{Coh}'(\tilde{\underline{X}})_{p.g}, \ \mathrm{Coh}''(\underline{X})_{p.g} \text{ or } \mathrm{Coh}''(\underline{X}')_{p.g} (\text{cf.}(1.4)_8 \ (1.18)_1 \end{cases}$$

Also we denote by $\mathcal{E}_{X^*}$ the map: $\mathrm{Coh}^*(X^*)_{p.g} \ni H \rightarrow \underline{M} \ni M_H$ as in Lemma 4.1*) or Lemma 4.8_1 (according as $\underline{X}^* = \tilde{\underline{X}}$ or $\underline{X}, \underline{X}'$). Moreover, we set:

(b) $\tilde{\underline{Y}}^* := C^n \times U_1' (\subset \tilde{X})$, $U_1 \cap \underline{X} (\subset \underline{X})$ or $\underline{X}'$. (For the open set $U_1' \subset C^{n'}$, $U_1 \subset C^n$, see $(4.1)_4$ and $(4.7)_2$.)

Then one can apply, to each $H \in \mathrm{Coh}^*(X^*)_{p.g}$ and $Q \in \tilde{\underline{Y}}^*$, the estimation in the left side of $(4.11)_1$ (cf. also the explicit formulations in Lemma 4.1, Lemma 4.8_1). Now let the manifold**) $\underline{Y}^* := \tilde{\underline{X}}_r(P'), \underline{X}_r(P)$ or $\underline{X}'$, the p.g. covering $A_\sigma(\underline{Y}^*)$ and the sets of the cochains $C^q(A_\sigma(\underline{Y}^*), H; \Psi)_\partial$, $C^q(A_\sigma(\underline{Y}^*), H; \Psi')_\partial$ be as in Lemma 1.2. Then, by applying Prop 4.5_1 to the non cohomological estimation mentioned soon above, we get the desired inclusion of the cohomological form in Lemma 1.2 (cf. §1.3):

(c) $s^* C^q(A_\sigma(Y^*), H; \Psi')_\partial \subset C^q(A_{\sigma'}(Y^*), H; \Psi)_{\partial'}$, where $(\sigma'; \partial') = L_H(\sigma; \partial)$, with the element $L_H \in \tilde{\underline{L}} (= \underline{L} \times \underline{L})$ determined by M_H (cf. (a)) in the manner in $(4.10)_7$.

---

*) When we apply Lemma 4.1 to $X^* = X \in An_{1a}$, we understand that $X = X_0 - X_0'$ in Lemma 4.1 is of the form: $X = X_0 - D$, where the divisor D of X_0 is as in

**) By Remark 1.4 (cf. the end of §1.3), the proof of Lemma 1.2 for such a sheaf H suffices for that of Lemma 1.2 in its original form as in n.2, §1.3.

***) $\tilde{\underline{X}}_r(P') = \underline{C}^n \times U_r(P')$ and $\underline{X}_r(P) = \underline{X} \cap U_r(P)$, with the discs $U_r(P')$, $U_r(P)$ in $\underline{C}^{n'}, \underline{C}^n$ of center P', P and radius r (cf. Lemma 1.2). Also note that $\underline{Y}^* \subset \tilde{\underline{Y}}^*$ (cf. Lemma 1.2), and one can apply Lemma 1.2 to $\underline{Y}^*$.

Next, recall that, in Lemma 1.2, we imposed the factorization on the correspondence: $\mathrm{Coh}^*(X^*)_{p.g} \ni H \to \tilde{L} \ni \tilde{L}_H$ (cf. Fig. III, Lemma 1.2). This follows from the corresponding factorization in Lemma 4.1, 4.8_1, and we finish the proof of Lemma 1.2. q.e.d.

3. Proof of Lemma 2.5, 2.7. The proof of these lemmas is similar to Lemma 1.2. We summarize the key point of the proof: first letting the Koszul homomorphism $F_p^m : O_X^{\binom{s}{p}} \to O_X^{\binom{s}{p+1}}$ $(0 \leq p < s)$ be as in Lemma 2.5, Lemma 4.4, we note that (1) the non cohomological estimation (given to F_p^m) is of the form in $(4.10)_3$ and (2) Prop. 4.5_2 is given to $(4.10)_3$; applying Prop. 4.5_2 to the estimation in Lemma 4.4, we get the following implication:

$(4.11)_2$ Lemma 4.4 (which concerns the open map property of F_p^m) $\xrightarrow{\text{Prop.}4.5_2}$ Lemma 2.5 (given to F_p^m) = cohomological version of Lemma 4.4

Also recall that Lemma 4.8_1 concerns the non cohomological estimation on the Koszul homomorphisms $F_p'^m : O_X^{\binom{s}{p}} \to O_X^{\binom{s}{p+1}}$ and is of the form $(4.10)_8$. Thus the using the implication: $(4.10)_8 \to (4.10)_9$ (= cohomological version of $(4.10)_8$) (cf. (ii), Remark 4.3), we have the similar implication to $(4.11)_2$:

$(4.11)_2'$ Lemma $4.8_1 \longrightarrow$ Lemma 2.5 (given to $F_p'^m$) = cohomological version of Lemma 4.8_1.

Next, for the homomorphisms $K_0 : O_X^{k_1} \to H$, $K_0' : O_X^{k_1} \to H'$ as in Lemma 2.7, we have the following implications:

$(4.11)_3$ Lemma 4.5 $\longrightarrow$ Lemma 2.7 (given to K_0), Lemma $4.8_3 \to$ Lemma 2.7 (given to K_0').

Actually remark that the estimations in the left sides are of the form in $(4.10)_{3,8}$, while those in the right sides are of the form in $(4.10)_{4,9}$. Thus, using the implication:$(4.10)_3 \longrightarrow (4.10)_4$(which is insured by Prop.4.5 and the one:$(4.10)_8 \longrightarrow (4.10)_9$ (as in Remark 4.3), we have $(4.11)_3$ in the parallel manner to $(4.11)_2, (4.11)_2'$.

4. Proof of Lemma 2.3. Finally, letting the homomorphism $K_{0,m}$: $O_X^{k_1+sk} \longrightarrow O_X^k$ and its imahe $H_m = K_{0,m}(O_X^{k_1+sk})$ be as in Lemma 4.6, Lemma 2.3' we note that the estimations in Lemma 4.6, Lemma 2.3' are of the form in $(4.10)_{8,9}$. Thus, applying the implication:$(4.10)_8 \longrightarrow (4.10)_9$ to Lemma 4.6, we have the following implication parallely to $(4.11)_3$:

$(4.11)_4$ Lemma 4.6 $\longrightarrow$ Lemma 2.3'(given to $X \in An_{1a}$).

Recalling that Lemma 2.3' insures Lemma 2.3(cf. §2.2), we have Lemma 2.3 (given to $X \in An_{1a}$) from $(4.11)_4$. Next, for the sheaf $H_m' = K_{0,m}'(O_{X'}^{k_1+sk})$, whic was given to $X' \in$ <u>Aff</u> in Lemma 2.3,Lemma 4.8_1, we get the following implication in the parallel manner to $(4.11)_4$:

$(4.11)_4'$ Lemma $4.8_3 \longrightarrow$ Lemma 2.3'(goven to $X' \in$ <u>Aff</u>).

Thus we have Lemma 2.3, and we also finsh the proof of all the uniform estimations in §1, § 2, which concern $O_{\tilde{X}}$-,$O_{\bar{X}}$-and O_X-homomorphisms. The remaining uniform estimations in §1,2,2, Lemma 1.3 and Lemma 1.4, will be checked in n.5,n.6 soon below, by using Prop.4.4.

*) Note that the a.d.exponent $m \in Z^+$ of the sheaf $f^m O_X$,.. in Lemma 2.7 and the index $m \in Z^+$ of the sheaf H_m,.. in Lemma 2.3 satisfies the inequality of the form:$m > \bar{m}$ with a suitable $m \in Z^+$(instead of the inequality of the form in $(4.10)_{3,8}$). Thus, in applying Prop.4.5_2 and the implication:$(4.10)_8 \longrightarrow (4.10)_9$ to $(4.11)_{3,4}$ and $(4.11)_4'$, we should use the remarks in (i),(ii),Remark 4.3.

4. Proof of Prop.4.2. Here, using the implications $(4.11)_{2\sim 4}$, we prove Prop.4.2. First we prove $(4.5)_3$(cf.Prop.4.2), which is a key fact for the proof of Lemma 4.4(cf.§ 5.2). For this letting the element $h \in \Gamma(X_0, O_{X_0})$ and the divisor D of h be as in $(4.5)_3$, we have the following from $(4.11)_2$:

(a) 'Lemma 4.4 for $(X_0,D;f)$' $\longrightarrow$ 'Lemma 2,5 for $(X_0,D;f)$'.

Now, letting the parameters $(P;r) \in \mathcal{U}_h(\subset (D-X_{0,sing}) \times R^+)$, $a \in R_1^+$ and the elements $(\tilde{m},m) \in Z^+ \times Z^+$ be as in $(4.5)_3$, we define:

$(b)_1$ $\partial:=(a,1) \in R_1^{+2}$, $\tilde{r}:=r/2$ and $\sigma:=(2/r,1) \in R_1^{+2}$.

Then, for the set of the holomorphic functions as in $(4.5)_3$, we easily ha

$(b)_2$ $\Gamma(\tilde{U}_r(P), f^{\tilde{m}}O_X^{(\overset{s}{p})})_a \subset Z^0(A_\sigma(\tilde{X}_{\tilde{r}}(P)), f^{\tilde{m}}O_X^{(\overset{s}{p})})_\partial$, where the p.g.covering $A_\sigma(\tilde{X}_{\tilde{r}}($ and the set $Z^0(\cdot\cdot)$ in the right side are as in Lemma 2.5(cf.§ 2.3).

Applying Lemma 2.5 to the right side(cf.(a)), we have:

$(b)_3$ $s^*(Z^0(A_\sigma(\tilde{X}_{\tilde{r}}(P)), f^{\tilde{m}}O_X^{(\overset{s}{p})})_\partial \cap (F_p^m)^{-1}(0)) \subset F_{p-1}^m Z^0(A_{\alpha'}(X_{r'}(P)), f^{\tilde{m}'}O_X^{(\overset{s}{p-1})})_\partial$

with $(r';\sigma';\tilde{m}';\partial')=E(r;\sigma;\tilde{m};\partial)$, where the d.p.map E is as in Lemma 2.5.

On the otherhand, applying*) Th.2.2_2 to the right side, we have:**)

$(b)_4$ $s^*Z^0(A_{\alpha'}(\tilde{X}_{r'}(P)), f^{\tilde{m}'}O_X)_\partial \subset F^{\tilde{m}''}Z^0(A_{\alpha''}(\tilde{X}_{r''}(P)), O_X^s)_{\partial''}$, with $(r'';\sigma'';\tilde{m}'';\partial'')$ $=E'(r';\sigma';\tilde{m}';\partial')$, where the d.p.map $E' \in E_{d.p}$ is as in Th.$2.2_2'$.

From $(b)_{3,4}$, we have:

$(b)_5$ (left side of $(b)_3$) $\subset F_{p-1}^m(F^{\tilde{m}''}Z^0(A_{\tilde{\sigma}''}(X_{\tilde{r}''}(P)), O_X^{(\overset{s}{p-1})s})_{\tilde{\partial}''}$, with $(\tilde{r}'';\tilde{\sigma}'';\tilde{m}'';\tilde{\partial}'')=\tilde{E}(r;\sigma;\tilde{m};\partial)$, where the d.p.map $\tilde{E}$ is determined by E,E'.

---

*)**) Recall that Lemma 2.5 implies Th.2.2_2 and the inclusion of the form $(b)_3$(cf.Remark 2.2 at the end of §2).

***) For the homomorphism $F^m=F_{s-1}^m$, see n.2, § 2.1 and § 2.3

4. Proof of Lemma 1.4. (i) Take an open set U of a euclidean space $C^n(z)$ and a subset X of U. Also taking a p.g function $g:U \to R_1^+$ and an element $\tilde{\sigma} \in R_1^{+2}$ we assume(cf.also $(4.10)_0$):

$(4.12)_0$ $g(Q)/2 < g(P) < 2g(Q)$ for each $P \in U$ and $Q \in U_{\tilde{\sigma}}(P;g) \cap U$,

where $U_{\tilde{\sigma}}(P;g) := \{Q \in C^n; d(P,Q) < \{\tilde{\sigma}.g(P)\}^{-1}\}$.

For an open subset U' of U we say that U is a $(g,\tilde{\sigma})$-d-envelop of U', if

$(4.12)_0'$ $U_{\tilde{\sigma}}(Q;g) \subset U$ for each $Q \in U'$.

We fix such an open set U' in the remainder of n.4. Now take an el-map L_0 of the form $L_0 : R^{+2} \ni (a_1, a_2) \longrightarrow R^{+2} \ni (4a_1\overset{\sigma_2}{\alpha}{}^2, \sigma_2)$(cf.n.1).

Then setting

$(4.12)_1$ $U_\sigma(X;g) := \bigcup_{P \in X} U_\sigma(P;g)$,

we easily have:

$(4.12)_2$ $U_{\sigma'}(Q;g) \cap X = \phi$ for each $Q \in U' - U_\sigma(X;g)$, where $\sigma' = L_0(\sigma)$.

(This follows also if U is a (g,σ'')-d-envelop of U', where $\sigma'' \in R_1^{+2}$ satisfies: $\sigma'' > \sigma'$.

(ii) Extension of cochains. Here we give a key proposition for Lemma 1.
For this we assume that X is an analytic variety in U and that there are open sets $\tilde{U}, \tilde{U}_0$ in C^n and varieties $\tilde{X}, \tilde{X}_0$ in $\tilde{U}, \tilde{U}_0$ satisfying

(a) $\tilde{U}$, $\tilde{U}_0$ are $(g,\tilde{\sigma})$-d-envelop of $U, \tilde{U}$, and $\tilde{X} = \tilde{X}_0 \cap \tilde{U}$, $X = \tilde{X} \cap U$.

Also taking a positive monomial M(cf.n.5,§1.1), we assume the following uniform estimation(cf.Prop.4.4): for each $Q \in \tilde{X}$ and $r \in R^+$ satisfying $r < \{\tilde{\sigma} g(Q)\}^{-1}$, there is an analytic map

$(4.12)_3$ $\omega : U_{r'}(P) \hookrightarrow \tilde{U}_r(P)$, which is the identity on $\tilde{U}_{r'}(P)$. Here $r' = M(r)$, and we set $U_r(P) := \{Q \in C^n; d(P,Q) < r\}$, $\tilde{U}_r(P) = U_r(P) \cap \tilde{X}_0$.

///

For an element $\sigma \in R^{+2}_1$ we set:

$(4.12)_4$ $\left\{\begin{matrix} A_\sigma(X) \\ B_\sigma(U') \end{matrix}\right\}$:=g-p. g.-covering of $\left\{\begin{matrix} X \\ U' \end{matrix}\right\}$ in $\left\{\begin{matrix} \tilde{X} \\ C^n \end{matrix}\right\}$ of size σ (Def.1.6_1).

Next we define an el-map $L':R^{+2} \longrightarrow R^{+2}$ from M in the manner in $(4.10)_5$, and we set $\tilde{L}':=L_0 \circ L' \circ L_0$, where L_0 is as in $(4.12)_3$. Also denoting by L_0' the el-map:$R^{+2} \ni \sigma \longrightarrow R^{+2} \ni L_0(2\cdot\sigma)$, we set:

$(4.12)_5$ $L:= L_0' \circ \tilde{L}'$

Then, denoting by O_X,O and ω^*_X respectively the structure sheaf of X,C^n and the natural homomorphism:$O \to O_X$, we have:

Proposition 4 6.(Extension of cochains). For any $\sigma \in R^{+2}_{\tilde{\sigma}}$ and $\partial=(\partial_1,\partial_2)$ $\in R^{+2}_1$ we have a map:

$(4.12)_6$ $e^*:Z^q(A_\sigma(X).O_X)_\partial \hookrightarrow C^q(B_{\sigma'}(U'),O)_{\partial'}$, where $\partial'=(4\partial_1 2^{\partial_2},\partial_2)$, $\sigma'=L(\sigma)$, which satisfies $\omega^*_X \delta e^* = 0$ and

$(4.12)_6'$ $s^*=\omega^*_X e^*$, with the p.g.refining map $s:A_{\sigma'}(X) \hookrightarrow A_\partial(X)$.

(The similar facts to the above holds by changing (U',σ') to $(\tilde{U}',\tilde{\sigma}')$, with an open subset $\tilde{U}'$ of U' and an element $\tilde{\sigma}' \in R^{+2}_1$ satisfying*) $\tilde{\sigma}' > \sigma'$.)

Proof.(1) First we extend cochains on X to its small p.g.neighborhood. For this setting $B_\sigma(X)$:=g-p g -covering of X in C^n of size σ(Def.1.6_1), we sjow the existence of a map:

$(4.12)_7$ $e'^*:Z^q(A_\sigma(X),O_X)_\partial \hookrightarrow C^q(B_{\sigma''}(X),O)_{\partial'}$, satisfying $\omega_{X*} \delta e^*=0$ and $s^*=\omega^*_X e^*$, with the p.g.refining map $s:A_{\sigma''}(X) \hookrightarrow A_\sigma(X)$,

where the element $\sigma'' \in R^{+2}_1$ is defined as follows:

$(4.12)_7'$ $\tilde{\sigma}''=L_0(\sigma)$, $\bar{\sigma}''=L'(\tilde{\sigma}'')$, and $\sigma''=L_0(\bar{\sigma}'')(=\tilde{L}'(\sigma))$.

Take an element $B'=\{B_j'\}_{j=1}^{q+1} \in N^{q+1}B_{\sigma''}(X)$ satisfying $|B'| \cap X \neq \phi$, and

---

*) Note that this implies that U is a $(g,\tilde{\sigma}')$-d-envelop of $\tilde{U}'$.

we set $A' = \{A'_j\}_{j=1}^{q+1}$, with $A'_j = B'_j \cap \tilde{X}$, and $A = s(A') \in N^{q+1}A_{\sigma}(X)$ Then, taking a point $Q \in |A'|$, we have:

$(a)_1$ $|A| \supset \tilde{U}_r(Q)$, $U_{r'}(Q) \supset |B'|$, with $r = \{\tilde{\sigma}'' g(Q)\}^{-1}$, $r' = \{\bar{\sigma}'' g(Q)\}^{-1} (= M(r))$ (cf. $(4.12)_5$ and $(4.10)_5$).

By $(4.12)_3$ take an analytic map $\omega: U_{r'}(Q) \hookrightarrow \tilde{U}_r(Q)$, which is the identity on $\tilde{U}_{r'}(Q)$. Now, for an element $\varphi \in Z^q(A_{\sigma}(X), O_X)_{\alpha}$, we set $\varphi'_{B'} := \omega^* \varphi_{A'} \in \Gamma(|B'|, O)$.

Then we have:

$(a)_2$ $\omega_X^* \varphi'_{B'} = \varphi_{A'}$, and $|\varphi'_{B'}(Q)| < \partial' g(Q)$.

We then define an element $\varphi' := e'^* \varphi$ by

$(a)_3$ $\varphi'_{B'} := \omega_X^* \varphi'_{B'}$ or $=0$, according an $|B'| \cap \tilde{X} \neq \phi$ or $= \phi$.

Then it is easy to see that $(a)_2$ insures $(4.12)_7$.

(2) Next setting $\bar{\sigma}' = 2\alpha''$ and $\alpha' = L_0(\bar{\alpha}')(= L'_c(\alpha''))$, we set $B' := \{U_{\alpha'}(Q;g);\ Q \in U' \cap \text{supp } B_{\tilde{\alpha}'}(X)\}$ (cf. $(4.12)_7$) . Then we have a refining map

$(b)_1$ $t: B' \hookrightarrow B_{\alpha''}(X)$ so that $t(U_{\alpha'}(Q;g)) = U_{\alpha''}(Q;g)$, if $Q \in X \cap U'$.

Also we note that $(4.12)_2$ insures:*)

$(b)_2$ $U_{\alpha'}(Q;g) \cap X = \phi$, if $Q \in U'$-supp $B_{\tilde{\alpha}'}(X)$,

Then we set:

$(b)_3$ $N^{q+1}B_{\alpha'}(U') = N^{q+1}B' \cup B''$, where $B'' := \{B''_{\mu} \in N^{q+1}B_{\alpha'}(U');$ where one of elements $U_{\alpha'}(Q;g) \in B''_{\mu}$ satisfies: $Q \notin \text{supp } B_{\tilde{\alpha}}\}$.

Note that $(b)_2$ implies:

$(b)_4$ $|B''_{\mu}| \cap X = \phi$ if $B''_{\mu} \in B''$

Now letting $\varphi' = e'^* \varphi$ be as in $(a)_3$, we define an element $\psi = e^* \varphi \in C^q(B_{\alpha'}(U'), O)$ by the following:

(c) $\psi = t^* \varphi'$ on $N^{q+1}B'$, and $=0$ on B'' .

By $(b)_{1\sim 4}$ and $(a)_3$ we easily have $(4.12)_6$. q.e.d.

---

*) $(b)_2$ holds for the pair $(\tilde{U}', \tilde{\sigma}')$ as in the remark soon below Prop. 4.6, and also the remark itself holds .

we set $A' = \{A'_j\}_{j=1}^{q+1}$, with $A'_j = B'_j \cap \tilde{X}$, and $A = s(A') \in N^{q+1}A_\sigma(X)$ Then, taking

point $Q \in |A'|$, we have:

$(a)_1$ $|A| \supset \tilde{U}_r(Q)$, $U_{r'}(Q) \supset |B'|$, with $r = \{\tilde{\sigma}'' g(Q)\}^{-1}$, $r' = \{\sigma'' g(Q)\}^{-1} (= M(r))$ (cf. $(4.12)_5$ and $(4.10)_5$).

By $(4.12)_3$ take an analytic map $\overset{\omega}{\downarrow}$ $\omega : U_{r'}(Q) \hookrightarrow \tilde{U}_r(Q)$, which is the identity on $\tilde{U}_{r'}(Q)$. Now, for an element $\varphi \in Z^q(A_\sigma(X), O_X)_d$, we set $\varphi'_B := \omega^* \varphi_A \in \Gamma(|B|, O)$.

Then we have:

$(a)_2$ $\omega_X^* \varphi'_{B'} = \varphi_{A'}$, and $|\varphi'_{B'}(R)| < d' g(R)$ in $|\underline{B'}|$.

We then define an element $\varphi' := e'^* \varphi$ by

$(a)_3$ $\varphi'_{B'} := \omega_X^* \varphi'_{B'}$ or $=0$, according an $|B'| \cap \tilde{X} \neq \phi$ or $= \phi$.

Then it is easy to see that $(a)_2$ insures $(4.12)_7$.

(2) Next setting $\tilde{\sigma}' = 2\cdot\sigma''$ and $\sigma' = L_0(\tilde{\sigma}')(= L'_c(\sigma''))$, we set $B' := \{U_{\alpha'}(Q;g);\ Q \in U' \cap \operatorname{supp} B_{\tilde{\alpha}'}(X)\}$ (cf. $(4.12)_7$) . Then we have a refining map

$(b)_1$ $t: B' \hookrightarrow B_{\alpha''}(X)$ so that $t(U_{\alpha'}(Q;g)) = U_{\alpha''}(Q;g)$, if $Q \in X \cap U'$.

Also we note that $(4.12)_2$ insures:*)

$(b)_2$ $U_{\alpha'}(Q;g) \cap X = \phi$, if $Q \in U' - \operatorname{supp} B_{\tilde{\alpha}'}(X)$,

Then we set:

$(b)_3$ $N^{q+1}B_{\alpha'}(U') = N^{q+1}B' \cup B''$, where $B'' := \{B''_\mu \in N^{q+1}B_{\alpha'}(U')$; where one of elements $U_{\alpha'}(Q;g) \in B''_\mu$ satisfies: $Q \notin \operatorname{supp} B_{\tilde{\alpha}}\}$.

Note that $(b)_2$ implies:

$(b)_4$ $|B''_\mu| \cap X = \phi$ if $B''_\mu \in B''$

Now letting $\varphi' = e'^* \varphi$ be as in $(a)_3$, we define an element $\Psi = e^* \varphi \in C^q(B_{\alpha'}(U'), 0$

by the following:

(c) $\Psi = t^* \varphi'$ on $N^{q+1}B'$, and $=0$ on B'' .

By $(b)_{1\sim4}$ and $(a)_3$ we easily have $(4.12)_6$. q.e.d.

---

*) $(b)_2$ holds for the pair $(\tilde{U}', \tilde{\sigma}')$ as in the remark soon below Prop. 4.6, and also the remark itself holds .

(iii) Proof of Lemma 1.4. Now Lemma 1.4 follows from Prop.4.6 almost directly as follows. First let the local variety $X \in An_{1a}$ be as in Lemma 1.4. Then letting the parameter space $\mathcal{U}_X(\subset X \times R^+)$ and the positive monomial M_X be as in Prop.4.4, we assume the estimation (4.9) in Prop.4.4 for $(\mathcal{U}_X, M_X)$. The symbol L_X denotes the el-map:$R^{+2} \rightarrow R^{+2}$, which is formed from M_X in the manner in Prop.4.6. Next let the manifolds $Y_r(P)=U_r(P)-D_0$, $X_r(P)=Y_r(P)\cap X$ and their p.g.coverings $A_\sigma(Y_r(P)),A_\sigma(X_r(P))$ in C^n, X be as in Lemma 1.4. Then choosing suitable neighborhoods U',U'' of the origin P_0 of X(cf.$(1.8)_0$) in C^n, and an element $\tilde{\sigma} \in R_1^{+2}$, we have:

(a) $U'-D_0$(resp.$U''-D_0$) is$^{***)}$ a (g_X,α)-d-envelop of $U''-D_0$(resp.$Y_r(P)$)

Thirdly, let E_X' be the first part of the p.g.c.map E_X as in Lemma 1.4. Then setting $(r';\sigma')=E_X'(r;\sigma)$ we have the following from the explicit form of the map E_X'(cf.n.5,§ 1.1).

(b) $Y_r(P)$ is a (g_X,σ')-d-envelop of $Y_{r'}(P)$, and $\sigma' > L_X(\sigma)$.

By (a),(b), one can apply Prop.4.6 to $(A_{\alpha'}(Y_{r'}(P)),A_\sigma(X_r(P)))$ and the structure sheaves O_X,O of X,C^n: writing the p.g.resolution H_X of O_X over U_0-D_0 in the form: $\rightarrow O \xrightarrow{\omega_X^*} H_X \rightarrow 0$, where H_X coincides with O_X as the coherent sheaf over $U-D_0$, we have a map e^* from $(4.12)_6$ in Prop.4.6:

(c) $e^*:Z^q(A_\sigma(X_r(P)),O_X)_\partial \rightarrow Z^q(A_{\alpha'}(\overset{Y}{\cancel{X}}_{r'}(P)),H_X)_\partial$, which satisfies:$s^*=\omega_X^* e^*$.

(Here s=p.g.refining map:$A_{\alpha'}(X_{r'}(P)) \hookrightarrow A_\sigma(X_r(P))$, and we use the symbol

---

*) When we apply Prop.4.4 to X, we assume that the pair (X_0,X_0') in Prop.4.4 is of the form:$(X_0,X_0')=(X_0, D^{-})$, with the divisor D of X_0 as in Lemma 1.4.($=D_0\cap X_0$, with the divisor D_0 of U_0).

**) $U_r(P):=\{Q \in C^n;d(P,Q) < r\}$ (cf.$(1.17)_1''$).

***) From the explicit form of the paparmeter space $\mathcal{U}_X$, one can take U',U'' independently from the manifold $Y_r(P)$ in Lemma 1.4.

****) $g_X(=|h_X|^{-1}$, where $h_X \in \Gamma(U_0,O_{U_0}))$ is the p.g.function for X(cf. §1).

$\omega_X^*: O \to H_X$ also for the obvious homomorphism: $H_X \to O_X$. We note that the map ω_X^* in Lemma 1.4 was used in the latter sense.)
It is clear that (c) insures the desired inclusion in Lemma 1.4:

(d) $\quad s^*Z^q(A_\sigma(X_r(P)),O_X)_\partial \subset \omega_X^*Z^q(A_{\sigma'}(Y_{r'}(P)),H_X)_{\partial'}$,

and we have Lemma 1.4 for the local variety $X \in An_{1a}$. For the affine variety $X' \in$ Aff, we note that the ambient space C^n is (g_X,σ)-d-envelop of C^n itself for any $\sigma \in R_1^{+2}$. Using this remark and Prop. 4.4 for X', the proof of Lemma 1.4 for $X' \in$ Aff is given similarly (and more easily) to the case of $X \in An_{1a}$, and we finish the proof of Lemma 1.4.

6. Proof of Lemma 1.3. Let the divisor $S \subset \tilde{X} := C(w) \times U_0 (\subset C(w) \times C^n(z))$ be as in Lemma 1.3(cf. also $(1.16)_0$). Then for points $\tilde{Q} \in S, Q \in X = U_0 - D$ and $\sigma \in R_1^{+2}$, we set(cf. Lemma 1.3):

$$(4.13)_0 \begin{Bmatrix} \tilde{U}_\sigma(\tilde{Q};g_S) \\ \tilde{U}_\sigma(Q;g_X) \end{Bmatrix} := \begin{Bmatrix} R \in S;\ d(Q,R) < (\sigma g_S(Q))^{-1} \\ R \in X;\ d(Q,R) < (\sigma g_X(Q))^{-1} \end{Bmatrix}, \text{with} \begin{Bmatrix} g_S \\ g_X \end{Bmatrix} := \begin{matrix} |w| \\ |h_X^{-1}| \end{matrix} .$$

(Here $h_X \in \Gamma(U_0, O_{U_0})$ and its divisor D in U_0 are as in Lemma 1.3(cf. also $(1.15)_4$.) Next take a suitable open subset U_X of U_0, an element $\sigma = \sigma_X \in R_1^{+2}$ and an el-map $L_X: R^{+2} \longrightarrow R^{+2}$(cf. n.5, § 1.1). Then, from a simple observation, we have the following comparison of the p.g. properties of S and X:

Proposition 4.7_1. (1) $g_X(Q)/2 < g_S(\tilde{Q}) < 2g_X(Q)$.

(2) $\pi_X(\tilde{U}_{\sigma'}(\tilde{Q})) \subset \tilde{U}_\sigma(Q)$ and $\pi_X(\tilde{U}_\sigma(\tilde{Q})) \supset \tilde{U}_{\sigma'}(Q)$, with $\sigma' = L_X(\sigma)$.

Here $(Q;\sigma)$ is in $(U_X - D) \times R^{+2}$, and $Q = \pi_X^{-1}(Q)$. Moreover, π_X is the natural projection: $S \to X = U_0 - D$(cf. n.4, § 1.3). Also we write $U_\sigma(\tilde{Q};g_S)$ as $U_\sigma(\tilde{Q}), \ldots$

Letting the point $P \in D$ and the element $r \in R^+, \sigma \in R_1^{+2}$ be as in Lemma 1.3, we set:

(a)$_1$ $S_r(P) := S \cap \tilde{X}_r(P)$, with $\tilde{X}_r(P) := C \times U_r(P)$, and $\tilde{X}_r(P) := U_r(P) - D_0$ (cf.(1.16)$_2$),

$$(a)_2 \begin{Bmatrix} A_\sigma(S_r(P)) \\ A_\sigma(\tilde{X}_r(P)) \\ A_\sigma(X_r(P)) \end{Bmatrix} := \begin{Bmatrix} g_S \\ g_S \\ g_X \end{Bmatrix} \text{-p.g. covering of } \begin{Bmatrix} S_r(P) \\ \tilde{X}_r(P) \\ X_r(P) \end{Bmatrix} \text{of size } \sigma \text{ in} \begin{Bmatrix} S \\ \tilde{X} \\ C^n \end{Bmatrix}.$$

Then from Prop. 4.7$_1$ we easily have:

Proposition 4.7$_2$. There are (natural) refining maps s_S and s_X:

$$(4.13)_2 \begin{cases} s_S : A_{\sigma'}(S_{r'}(P)) \ni \tilde{U}_{\sigma'}(\tilde{Q}) \to \pi_X^{-1} A_\sigma(X_r(P)) \ni \pi_X^{-1}(U_\sigma(\pi_X(\tilde{Q}))) \\ s_X : A_{\sigma'}(X_{r'}(P)) \ni \tilde{U}_{\sigma'}(Q) \longrightarrow \pi_X \ A_\sigma(S_r(P)) \ni \pi_X(U_\sigma(\pi_X^{-1}(\tilde{Q}))) \end{cases}, \text{with } \sigma' = L_X(\sigma).$$

Thirdly let the estimation: $(r;\sigma;\partial) \longrightarrow (r';\sigma';\partial') = E_X(r;\sigma;\partial)$, where $(r;\sigma;\partial) \in R^+ \times R_1^{+2} \times R_1^{+2}$ and $E_X \in E_{p.g}$, be as in Lemma 1.3. Then from the explicit form of E_X (cf. Def. 1.5), we easily have:

(b) $X_r(P)$ is a (g_X, σ)-d-envelop of $X_{r'}(P)$ (cf.(i), n.5, § 4.2).

By this we apply Prop. 4.6 to $(A_\sigma(S_r(P)), A_\sigma(\tilde{X}_r(P)))$ (cf. also Prop. 4.4), and we have:

Proposition 4.7$_2$. (Extension of cochains). There is a map:

$(4\ 13)_3$ $e^* : Z^q(A_\sigma(S_r(P)), O_S)_\partial \longrightarrow Z^q(A_{\sigma'}(X_{r'}(P)), H_S)_{\partial'}$, with $(r';\sigma';\partial') = E_X(r;\sigma;\partial)$, which satisfies: $s^* = \omega_S^* e^*$

Here the homomorphism $\omega_S : H_S \to O_S$ is as in Lemma 1.3 and s:=p.g. refining map: $A_{\sigma'}(S_{r'}(P)) \hookrightarrow A_\sigma(S_r(P))$.

(Note that Prop. 4.7$_3$ insures, in the similar manner to n.5, the following inclusion:

$(4.13)_4$ $\omega_S^* Z^q(A_{\sigma'}(\tilde{X}_{r'}(P)), H_S)_{\partial'} \supset s^* Z^q(A_\sigma(X_r(P)), O_S)_\partial$, where the correspondenc $(r;\sigma;\partial) \longrightarrow (r';\sigma';\partial')$ is as in $(4.13)_3$.)

Now, it is easy to get the compariosn of the sets of the cochains $Z^q(A_\sigma(\tilde{X}_r(P)), H_S)_\partial$ and $Z^q(A_\sigma(X_r(P)), O_X)_\partial$ in Lemma 1.3, which are defined respectively for $C \times C^n$ and $U_0 - D$, from Prop. 4.7$_{1,2}$ and $(4.13)_4$. Thus we have Lemma 1.3, and we also finish the proof of all the lemmas in Chap, I, which is postponed in § 4

---

*) $U_r(P) := \{R \in C^n; d(R,P) < r\}$ (cf. n.4, §1.3).