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On Forming a Series-parallel Graph by Removing Nodes of a Planar Graph

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1. Introduction.

Let P_{SPNR} denote the problem of determining, for a graph $G=(V,E)$ and a nonnegative integer k, whether or not a series-parallel graph can be formed by removing k or fewer nodes of G, where V and E are the sets of nodes and of edges of G, respectively. The authors have already proved in [15] that P_{SPNR} is NP-complete. In this paper, we show that P_{SPNR} remains NP-complete even when the domain is restricted to planar graphs. From now on, let P_{PSPNR} denote P_{SPNR} with the constraint that G is planar.

Technical terms and notations not specified in this paper can be identified in [2],[7],[8] and [9].

In stead of a thorough description of the formal requirements of a proof of NP-completeness, we describe the two steps (i) and (ii) required in proving that a particular problem P_X is NP-complete.

(i) Prove that $P_X \in \mathcal{NP}$, the class of problems that can be solved in polynomial time by a nondeterministic Turing machine.

(ii) Prove that some known NP-complete problem $P_{X'}$ can be polynomially transformed into P_X ($P_{X'} \propto P_X$ for short), in such a way that any polynomial-time algorithm for solving P_X could be used to solve $P_{X'}$ in polynomial time.

We will omit verification of (i) from our proofs since there

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exists an algorithm that determines in polynomial time whether or not a given graph is series-parallel (For example, see [3] or [12]). Thus our proofs will focus on the transformation required by (ii).

2. Preliminaries.

Let $G=(V,E)$ and $X \subset V$. Let us denote

$$E(X) = \{ (u,v) \in E: \{u,v\} \cap X \neq \phi \}$$

for $X = \{v_{i_1}, \ldots, v_{i_x}\}$ ($x = |X|$). We call

$$G-X = (V-X, E-E(X))$$

the graph formed by removing nodes $v_{i_1}, \ldots, v_{i_x}$ (or removing X) of G. Let us denote

$$\delta(G) = \text{Max} \{ \delta_G(v): v \in V \},$$

where $\delta_G(v)$ denotes the node degree of v in G. Let

$$P(u,v) = (V_{P(u,v)}, E_{P(u,v)})$$

denote a path of length $|E_{P(u,v)}| \geq 2$, a subgraph of the graph G in consideration, connecting two nodes u and v of G. $P(u,v)$ is said to be a disjoint path if and only if it satisfies the condition that no node in $V_{P(u,v)} - \{u,v\}$ is (and shall be) contained in any other path except $P(u,v)$ (That is, any node of $P(u,v)$ except u and v is of node degree 2 in any graph in consideration).

For two nodes u and v, let

$$L(u,v) = (V_{L(u,v)}, E_{L(u,v)})$$

denote the graph which consists of $(\alpha+1)$ disjoint paths of length 2 each of which connects u and v, where

$$V_{L(u,v)} = \{ b_i(u,v): i=0, \ldots, \alpha \} \cup \{u,v\}$$

and

$$E_{L(u,v)} = \left\{ (u, b_i(u,v)), \ (v, b_i(u,v)) : i=0, \ldots, \alpha \right\}.$$

L(u,v) is said to be a band of width α connecting u and v.

Remark 1.

Let L(u,v) be a band of width α connecting u and v. Then there exists at least one disjoint path even if α or fewer nodes of L(u,v) except u and v are removed. If we assume $\alpha \geq 2$, then L(u,v)- $\{u\}$ (L(u,v)-$\{v\}$) contains at least three blocks (maximal nonseparable components) each of which consists of a single edge having v (u) as a cutpoint.

A thinning is an operation that deletes one of two multiple edges e_1 and e_2. A shrinking is an operation that contracts one of two edges (u,v) and (v,w) satisfying that v is of node degree 2. A reduction is to repeat a thinning and/or a shrinking a number of times. A graph G is said to be series-parallel (s-p for short) if and only if there exists a reduction that leads G to a single edge. A well-known characterization of s-p graphs has been given by R. J. Duffin in [3].

Theorem 1 [3].

Let G be nonseparable. Then G is s-p if and only if G has no subgraph from which K_4 can be formed by contracting and/or deleting a number of edges.

If G has such a subgraph as mentioned in Theorem 1 then we say that G has a subgraph reducible to K_4 ($G \supset K_4$ for short).

We now turn our attention to the node cover problem. A node cover for G=(V,E) is a subset S of V such that any edge of G is incident upon some node of S. Given a graph G=(V,E) and a non-negative integer k, let P_{NC}, called the node cover problem, denote the problem of determining whether or not G has a node cover

S of size $|S| \leq k$ (We can assume that G is a simple graph). It is well-known that P_{NC} is NP-complete (For example, see [9]).

In this paper we state all lemmas without proofs which are given in [16].

Lemma 1.

P_{C-QPNC}, the node cover problem for simple planar connected cubic graphs, is NP-complete.

3. Forming a series-parallel graph by removing nodes
   of a planar graph.

Throughout this section let k be nonnegative integer and $G=(V,E)$ be a cubic planar connected graph, where

$$V=\left\{v_1,\ldots, v_n\right\} \ (\ n=|V|\ ) \text{ and } E=\left\{e_1,\ldots e_m\right\} \ (\ m=|E|\ ).$$

Represent G on a plane and fix this representation. Let

$$F_G=\left\{f_0, f_1,\ldots, f_{r-1}\right\} \ (\ f_0 \text{ is the infinite face of G }\ )$$

denote the set of all faces of G. We can assume that

$$n \geq 4, \ m \geq 6 \text{ and } r \geq 4.$$

Beginning with the fixed representation of G we construct the planar graphs $G_i=(V_i,E_i)$ (i=1,2) and $G_1'=(V_1',E_1')$ by the following procedures (1), (2), (3) and (4).

(1) Construct the geometric dual $G^*=(V^*,E^*)$ of G, and then determine a spanning tree $T^*=(V_{T^*},E_{T^*})$ of G^*. There exists one to one correspondence between E and E^*. We denote this correspondence by $e^* \in E^*$ for $e \in E$. Let

$$E_T=\left\{e \in E : e^* \in E_{T^*}\right\} .$$

Then $|E_T|=r-1$.

(2) Let G_1 be the graph obtained by replacing each edge $e=(u,v) \in E_T$ with a disjoint path $P(u,v)=(V_{P(u,v)}, E_{P(u,v)})$ of length 4. $P(u,v)$ is called the T-path for e. Let us denote

$$V_{P(u,v)} = \{ v_0 = u, \ v_1, \ v_2, \ v_3, \ v_4 = v \}, \text{ and}$$

$$E_{P(u,v)} = \{ (v_i, v_{i+1}) : i = 0, \ldots, 3 \}$$

(hereafter $V_{P(u,v)}$ and $E_{P(u,v)}$ are used in this sense). Let

$$A_1 = \bigcup_{(u,v) \in E_T} V_{P_a(u,v)},$$

where $V_{P_a(u,v)} = V_{P(u,v)} - \{ u, v \}$.

Then $V_1 = V \oplus A_1$, where $\oplus$ denotes the disjoint union. Let F_{G_1} denote the set of all faces of G_1. Then there exists one to one correspondence between F_G and F_{G_1}, and we denote this correspondence by $f^{(1)} \in F_{G_1}$ for $f \in F_G$.

(3) For each edge $e = (u,v) \in E$, execute the following procedures (i), (ii) and (iii). Let f_i and $f_j \in F_G$ be the two faces (not necessarily distinct) that e touches. If e is a bridge of G then e* is a loop of G*. Since $e* \notin E_{T*}$, we have $e \notin E_T$. That is, if $e \in E_T$ then e is not a bridge of G.

(i) The case when $e \in E - E_T$.

   (i-a) If e is not a bridge of G (then $f_i \neq f_j$), place two bands $L(u^i(e), v^i(e))$ and $L(u^j(e), v^j(e))$ of width $(k+r+2)$ in $f_i^{(1)}$ and $f_j^{(1)}$, respectively.

   (i-b) If e is a bridge of G (then $f_i = f_j$), place two bands $L(u_a^i(e), v_a^i(e))$ and $L(u_b^i(e), v_b^i(e))$ of width $(k+r+2)$ in $f_i^{(1)}$, where we assume that one of them is located across e from the other.

(ii) The case when $e \in E_T$ (then $f_i = f_j$).

Let $V_{P(u,v)}$ be as before. Then place four bands $L(v_t^i(e), v_{t+1}^i(e))$ ($i = 0,1,2,3$) of width $(k+r+2)$ in $f_i^{(1)}$, and similarly four bands $L(v_t^j(e), v_{t+1}^j(e))$ ($j = 0,1,2,3$)

of width (k+r+2) in $f_j^{(1)}$, where $v_0(e) = u^\alpha(e)$ and $v_4(e) = v^\alpha(e)$ for $\alpha = i, j$.

(iii) For each $u \in V$ and each pair of edges $e = (u, v)$ and $g = (u, w)$ ($e, g \in E$ and $v \neq w$), both of which are incident upon u, connect $u^i(e)$ and $u^i(g)$ in terms of a band of width (k+r+2) if and only if both e and g touch a face $f_i^{(1)} \in F_{G_1}$. If e (g) is a bridge of G, then we assume that $u^i(e)$ is set to either $u_a^i(e)$ or $u_b^i(e)$ (either $u_a^i(g)$ or $u_b^i(g)$).

Let G_1' denote the resulting graph. Let V_{C_i} denote the set of nodes placed in $f_i^{(1)} \in F_{G_1}$ in (i), (ii) and (iii) of (2), and let $C_i = (V_{C_i}, E_{C_i})$ denote the subgraph of G_1' induced by V_{C_i}. C_i can be changed into a single circuit if we replace each band of C_i with a disjoint path of length 2. C_i is said to be the inner ring $IR(f_i)$ of f_i (or of $f_i^{(1)}$).

(4) Construct a maximum matching $M \subseteq E$ of G (It is well-known that there exists a polynomial-time algorithm to obtain a maximum matching of a planar graph G (see [4])). And execute the following procedures (i), (ii) and (iii) with respect to M.

(i) For each edge $e = (u, v) \in E_T$, let f_i and f_j ($f_i \neq f_j$) be two faces of G that e touches, where we can assume that one of them is a finite face. Then, for each $v_t \in V_{P_a(u,v)}$ ($t = 1, 2, 3$), connect two pairs of nodes $\{ v_t, v_t^i(e) \}$ and $\{ v_t, v_t^j(e) \}$ by edges $(v_t, v_t^i(e))$ and $(v_t, v_t^j(e))$, respectively. A path of length 2 defined by each pair of these edges is called the c-path. Then we

say that $IR(f_i)$ and $IR(f_j)$ are T-connected. e is said

to be the c-edge of these two inner rings, and then these

two inner rings are said to be the c-supporters of e.

(ii) For each edge $e=(u,v) \in M$, choose arbitrarily a face

of G that e touches. Let, say, $f_i \in F_G$ denote this face.

Let $g=(u,x)$ and $h=(v,y) \in E-M$ ($\{x,y\} \cap \{u,v\} = \emptyset$)

denote the two edges touching f_i. Then connect four

pairs of nodes $\{ u,u^i(e)\}$, $\{ u,u^i(g)\}$, $\{ v,v^i(e)\}$

and $\{ v,v^i(h)\}$ by edges $(u,u^i(e))$, $(u,u^i(g))$ $(v,v^i(e))$

and $(v,v^i(h))$, respectively. e is said to be the b-

edge of $IR(f_i)$, and $IR(f_i)$ to be the supporter of e.

(iii) For each $v \in V-M(V)$ ($M(V)=\{u,v \in V: (u,v) \in M\}$),

choose arbitrarily an edge of E incident upon v and

choose again arbitrarily a face of G that e touches.

Let, say, $f_i \in F_G$ denote this face. Let $g=(v,x)$ and

$h=(v,y) \in E$ ($x \neq y$) denote the two edges touching f_i,

both of which are incident upon v. Then connect two

pairs of nodes $\{ v,v^i(g) \}$ and $\{v,v^i(h)\}$ by edges

$(v,v^i(g))$ and $(v,v^i(h))$, respectively. v is said to be

the b-node of $IR(f_i)$, and $IR(f_i)$ be the supporter of v.

Let G_2 denote the graph constructed by applying the pro-

cedure (4) to G_1'. Since it is well-known that G^*, T^* and M can

be obtained in polynomial time, G_2 can be constructed in poly-

nomial time when G is given as input. Note that G_2 is a non-

separable planar graph. Figure 1 shows an example of the graph

transformation from G to G_2.

Let v be the node mentioned in (iii) of (4). Each $v \in V$ is

connected to exactly one inner ring C_i in terms of a pair of

edges (v,v') and (v,v'') ($v',v'' \in V_{C_i}$) both of which are intro-

duced in either (ii) or (iii) of (4). Hence $\delta_{G_2}(v) = 5$. Let $\Delta(v)$

denote the subgraph of G_2 called the triangle of C_i (or of v),

where it is defined by nodes v, v' and v'', edges (v,v') and (v,v''),

and $E_{L(v',v'')}$. A pair of edges (v,v') and (v,v'') are called the

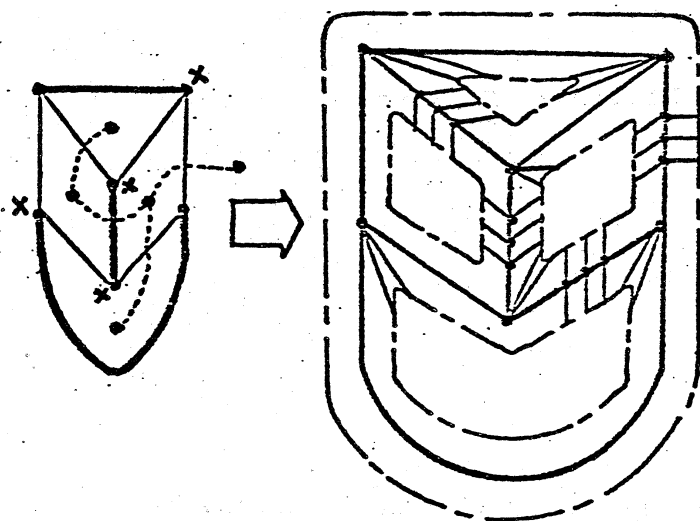Δ -edges of v. We say that v is connected to C_i in terms of $\Delta(v)$.

Figure 1. An example of the graph transformation from G to G_2 (In the figure ●━━━● and ────── , respectively, denote an element of M and an inner ring, and both ●───● and ●--─● edges).

Lemma 2.

G has a node cover of size less than or equal to k if and only if there exists a subset $X \subset V_2$ with $|X| \leq k+r-1$ such that G_2-X is an s-p graph.

Proof. Let $N \subset V$ be a node cover for G with $|N| \leq k$. Define $X_e \subset V_1$ for each $e=(u,v) \in E$ as follows:

$$X_e = \begin{cases} \phi & \text{when } e \notin E_T, \\ \{w\} & \text{when } e \in E_T \text{ and } |\{u,v\} \cap N| = 1, \\ \{v_2\} & \text{when } e \in E_T \text{ and } \{u,v\} \subset N, \end{cases}$$

where if $\{u,v\} \cap N = \{u\}$ ($\{v\}$), then $w=v_3$ ($w=v_1$) for

$V_{P(u,v)} = \{v_0=u, v_1, v_2, v_3, v_4=v\}$. Let

$$X_1 = \bigcup_{e \in E} X_e \quad (= \bigcup_{e \in E_T} X_e)$$

and

$$X = N \cup X_1 .$$

Note that $N \cap X_1 = \emptyset$. Then we have $|X| \leq k+r-1$, since $|E_T| = r-1$. It is easy to see that G_2-X is an s-p graph (see Figure 2).
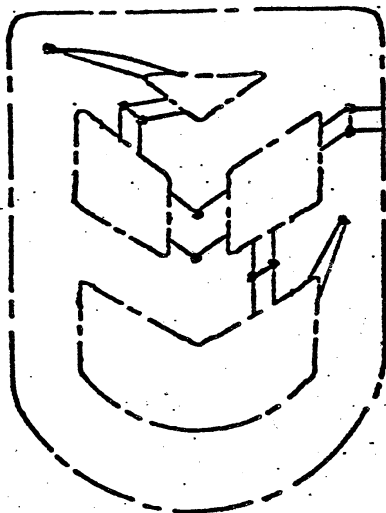
Figure 2. G_2-X for G_2 of Figure 1

(The node cover N that determines

X consists of those nodes marked x

in Figure 1).

Conversely, assume that an s-p graph can be formed by removing (k+r-1) or fewer nodes of G_2. Let $X \subset V_2$ be a minimal set satisfying that G_2-X is s-p and which maximizes, among all such sets, the number of nodes contained in V. By the construction of G_2 (in which any band of each inner ring is of width (k+r+2)), by Remark 1 and by the minimality of X, we can assume that

$$X \subset V_1 .$$

Let

$$N = X \cap V$$

and let $P(u,v)$ be the T-path for arbitrary edge $e=(u,v) \in E_T$. Then

$$|X \cap \{v_1, v_2, v_3\}| \geq 1$$

for $V_{P_a(u,v)} = \{v_1, v_2, v_3\} \subset V_{P(u,v)}$, since otherwise $G_2-X \supset K_4$. Therefore, since we have $|E_T| = r-1$,

$$|N| \leq k.$$

Let $G_3 = (V_3, E_3)$ denote G_2-X for simplicity.

Lemma 10.

G_3 satisfies the following (1) and (2).

(1) $\delta_{G_3}(v)=2$ for $\forall v \in V \cap V_3$.

(2) C_i and C_j are connected only in terms of exactly two c-paths (including the case when there exists an edge connecting these two c-paths) for any pair of distinct inner rings C_i and C_j that are T-connected in G_2.

By Lemma 10, we can prove that N is a node cover for G.

Q.E.D.

Lemma 2 and the fact that G_2 can be constructed in time polynomial in the size of input establish the main theorem.

Theorem 2.

P_{PSPNR} is NP-complete.

We will prove Lemma 10 by way of Lemma 3 through Lemma 9 in the rest of this section.

We have $\delta_{G_3}(v) \geq 2$ for $\forall v \in V_3$. And if $v \in V \cap V_3$ then there exist $\Delta(v)$ and the inner ring to which v is connected in terms of $\Delta(v)$.

By the construction of G_2, any two distinct inner rings C_i and C_j have no node in common. Thus G_3 has a path of length not less than 2 connecting C_i and C_j since $X \subset V_1$ and since G_3 is connected. Two blocks B and B' having a cutpoint u in common are said to be adjacent to each other at u.

Lemma 3.

Any block B of G_3 has at most two other blocks each of which is adjacent to B.

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Let

$$K_\beta = (V_{K_\beta}, E_{K_\beta})$$

denote the graph shown in Figure 3, where

$$V_{K_\beta} = \{u, v, w_1, w_2\}, \text{ and}$$

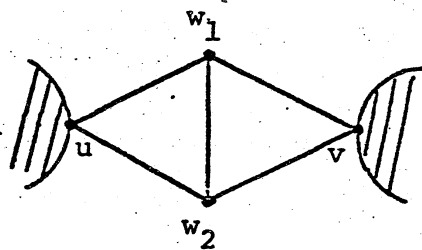$$E_{K_\beta} = \{(u,w_i), (v,w_i): i=1,2\} \cup \{(w_1,w_2)\}.$$

Figure 3. K_β .

Observe that a graph H is not s-p if K_β is a subgraph of H having both u and v as cutpoints of H (any reduction can not be applied to K_β since all of u, v, w_1 and w_2 are of node degree≥ 3). Let

$$\widetilde{K_\beta} = (V_{\widetilde{K_\beta}}, E_{\widetilde{K_\beta}})$$

denote the graph obtained by replacing each edge $(w,w') \in E_{K_\beta}$ with a disjoint path of length not less than 1 (that is, $\widetilde{K_\beta}$ is a graph homeomorphic to K_β). We say that a graph H has a pair of K_β-cutpoints if and only if H has a subgraph isomorphic to $\widetilde{K_\beta}$ having both u and v as cutpoints of H.

Lemma 4.

G_3 has no block which contains a pair of K_β-cutpoints.

Lemma 5.

Assume that G_3 has a triangle $\triangle$ (u) (u $\in$ V). If δ_{G_3} (u) ≥ 3 then u is a cutpoint of G_3.

Lemma 6.

Assume that G_3 has a triangle $\triangle(u)$ ($u \in V$). Let C_i be the inner ring to which u is connected in terms of $\triangle(u)$, $\delta_{G_3}(u)$ ≥ 3 and $B(C_i)$ be the block of G_3 containing C_i. Then $B(C_i)$ has no other cutpoint of G_3 except u.

Let $G_{A_1}=(V_{A_1},E_{A_1})$ ($V_{A_1}=A_1$) denote the subgraph of G_1 induced by A_1, and let $\overline{E}_1=E_1-E_{A_1}$. Then let

$$IR(G_2)=(V_{IR(G_2)},E_{IR(G_2)}),$$

called the inner ring tree, denote the subgraph of G_2 formed by deleting $\overline{E}_1$ from G_2, where $E_{IR(G_2)}=E_2-\overline{E}_1$. Let C and C' be a pair of inner rings that are T-connected in G_2. We say that C and C' are π -connected, $C\pi C'$ for short, if and only if they are connected only in terms of exactly two c-paths (including the case when there exists an edge connecting these two c-paths). For any two distinct inner rings C and C' of a block, a sequence of distinct inner rings $C_{i_0}=C, C_{i_1},..., C_{i_t}=C'$ ($t \geq 1$) of this block is called the π -connected sequence of C and C' if and only if $C_{i_j}\pi C_{i_{j+1}}$ for j=0,..., t-1.

Let D denote the set of all cutpoints of G_3. Then, by the construction of G_2 and by the definition of X, we have

$$D \subset V_1=V \oplus A_1 .$$

Let us denote

$$D=D_V \oplus D_{A_1},$$

where $D_V=D \cap V$ and $D_{A_1}=D \cap A_1$. Then, by Lemmas 3, 5 and 6,

$$|D_V| \leq 2.$$

Let

$$x^{(1)}=x \cup D_V \quad \text{and} \quad G_3^{(1)}=G_2-x^{(1)} \quad (\ =G_3-D_V\ ).$$

Note that $G_3^{(1)}$ is not always connected. Let B_1, B_2,... denote the blocks of $G_3^{(1)}$, and let

$$\mathscr{B}^{(1)}=\{B_1,\ B_2,\ \ldots\ \}\ .$$

Hereafter, let us denote

$$G_3^{(i)}=(V_3^{(i)},E_3^{(i)}) \quad \text{for } i=1,2 \text{ and } 3.$$

Lemma 7.

$G_3^{(1)}$ satisfies the following (1), (2) and (3).

(1) $\delta_{G_3^{(1)}}(v)=2$ for $\forall v \in V \cap V_3^{(1)}$.

(2) $D_{A_1} \subset D^{(1)} \subset A_1$, where $D^{(1)}$ is the set of all cutpoints

of $G_3^{(1)}$.

(3) $\delta_{G_3^{(1)}}(u) \geq 2$ for $\forall u \in V_3^{(1)}$.

Lemma 8 .

For any pair of distinct inner rings C and C' of any block $B_i \in \mathscr{B}^{(1)}$, there exists the π -connected sequence of C and C'.

Let

$$D^{(1)}=\{\ a_1,\ \ldots,\ a_\theta\ \}\ (\ \theta = |D^{(1)}|\ ),$$

where the order of elements of $D^{(1)}$, denoted by the subscript, can arbitrarily fixed. Let $P(u,v)$ denote the T-path containing $a_1 \in D^{(1)}$. Then, clearly,

$$a_1 \in V_{P_a(u,v)}=\{v_1,v_2,v_3\} \subset V_{P(u,v)}=\{u=v_0,v_1,v_2,v_3,v=v_4\}\ .$$

We define $X_{a_1} \subset V_1$ for the following two cases. Note that, for each $a_i \in D^{(1)}$, there exists exactly one T-path containing a_i and that if $a_i \neq a_j$ (a_i, $a_j \in D^{(1)}$) then they are contained in distinct T-paths.

(i) The case when $a_1 = v_1$ (respectively when $a_1 = v_3$).

By the definition of $G_3^{(1)}$,

$$\{ v_0, v_2, v_3 \} \subset X^{(1)} \quad (\{ v_1, v_2, v_4 \} \subset X^{(1)}).$$

Then let

$$X_{a_1} = X^{(1)} - \{ v_2 \}.$$

(ii) The case when $a_1 = v_2$.

By the definition of $G_3^{(1)}$,

$$\{ v_1, v_3 \} \subset X^{(1)}.$$

Note that if $v_0 \notin X^{(1)}$ (if $v_4 \notin X^{(1)}$) then $\delta_{G_3^{(1)}}(v_0) = 2$

($\delta_{G_3^{(1)}}(v_4) = 2$) (that is, v_0 (v_4) is connected to

some inner ring only in terms of $\triangle(v_0)$ ($\triangle(v_4)$).

Then let

$$X_{a_1} = \begin{cases} X^{(1)} - \{ v_1 \} & \text{when } v_0 \in X^{(1)} \text{ and } v_4 \notin X^{(1)}, \\ X^{(1)} - \{ v_3 \} & \text{when } v_0 \notin X^{(1)} \text{ and } v_4 \in X^{(1)}, \\ (X^{(1)} - \{ v_1 \}) \cup \{ v_0 \} & \text{when } \{ v_0, v_4 \} \cap X^{(1)} = \phi, \\ X^{(1)} - \{ v_1 \} & \text{when } \{ v_0, v_4 \} \subset X^{(1)}. \end{cases}$$

Set

$$X_{a_0} = X^{(1)}$$

and, for each $j = 2, \ldots, \theta$, define recursively

$$X_{a_j} \subset V_1,$$

similarily to the definition of X_{a_1}, by replacing $X_{a_{j-2}}$ and $X_{a_{j-1}}$

with $X_{a_{j-1}}$ and X_{a_j}, respectively. Clearly, for each $j = 1, \ldots, \theta$,

we have either

$$| X_{a_{j-1}} | > | X_{a_j} |$$

or

$$| X_{a_{j-1}} \cap V | < | X_{a_j} \cap V | \text{ with } | X_{a_{j-1}} | = | X_{a_j} |.$$

Set

$$X^{(2)} = X_{a_\theta} \quad \text{and} \quad G_3^{(2)} = G_2 - X^{(2)}.$$

Then by the definition of $X^{(2)}$ and by Lemma 8 we have the next lemma.

Lemma 9.

$G_3^{(2)}$ satisfies the following (1), (2) and (3).

(1) Any maximal connected component of $G_3^{(2)}$ is a block of $G_3^{(2)}$.

(2) For any pair of distinct inner rings C and C' of any block of $G_3^{(2)}$, there exists the π-connected sequence of C and C'.

(3) $\delta_{G_3^{(2)}}(v) = 2$ for $\forall v \in V \cap V_3^{(2)}$.

Lemma 9 shows that $G_3^{(2)}$ is a subgraph of the inner ring tree $IR(G_2)$.

For $w \in D_V$, let $B(w)$ denote the block of G_3 containing $\Delta(w)$, B be a block distinct from $B(w)$ and adjacent to $B(w)$ having w as a cutpoint. Let $(w, w^{(i_1)})$ and $(w, w^{(i_2)})$ be the two Δ-edges. Then we can show that any edge incident upon w and distinct from these two Δ-edges is contained in B. Therefore, $G_3^{(1)}$ consists of $|D_V|$ maximal connected components $B^{(1)}(w) = B(w) - \{w\}$ for $w \in D_V$ and another one C_Z which is determined by contracting $B(w)$ to w for $\forall w \in D_V$ in G_3. Hence, by Lemma 9, $G_3^{(2)}$ consists of ($|D_V| + 1$) blocks (or equivalently the same number of maximal connected components). That is, each $B^{(1)}(w)$ of $G_3^{(1)}$ is itself a block $B^{(2)}(w)$ of $G_3^{(2)}$ by Lemmas 6, 7 and 8, and C_Z of $G_3^{(1)}$ is changed into another block $B^{(2)}$ of $G_3^{(2)}$ in the process constructing $G_3^{(2)}$.

Now let us assume that $|D_V| \geq 1$ (In the case when $D_V = \emptyset$, we omit the procedure to construct $X^{(3)}$ described in the following, and set $X^{(3)} = X^{(2)}$ and $G_3^{(3)} = G_3^{(2)}$). Let

$$\mathcal{B}^{(2)} = \{B^{(2)}(w) : w \in D_V\} \cup \{B^{(2)}\} = \{B_1, \ldots, B_{|D_V|+1}\}.$$

Let $B_i \in \mathcal{B}^{(2)}$ be arbitrarily chosen. By the definition of T^* or of $IR(G_2)$, there exist some $B_j \in \mathcal{B}^{(2)} - \{B_i\}$, some inner ring C_i of B_i and some inner ring C_j of B_j such that C_i and C_j are T-connected in G_3. Then we say that B_j is a T-connected block of B_i and vice versa. Let $e = (u,v) \in E_T$ be the c-edge of C_i and C_j, and $P(u,v)$ be the corresponding T-path for e. Then

$$V_{P_a(u,v)} = \{v_1, v_2, v_3\} \subset X^{(2)},$$

where we set $u = v_0$ and $v = v_4$. Note that the following discussion does not depend on whether or not $D_V \cap \{u,v\}$ is empty. Define $X_{ij} \subset V_1$ as follows:

$$X_{ij} = \begin{cases} (X^{(2)} - \{v_1, v_2\}) \cup \{u\} & \text{when } \{u,v\} \cap X^{(2)} = \emptyset, \\ X^{(2)} - \{v_1, v_2\} & \text{when } \{u,v\} \cap X^{(2)} = \{u\}, \\ X^{(2)} - \{v_2, v_3\} & \text{when } \{u,v\} \cap X^{(2)} = \{v\}, \text{ and} \\ X^{(2)} - \{v_1, v_2\} & \text{when } \{u,v\} \subset X^{(2)}. \end{cases}$$

In the case when $|D_V| = 1$ define X_{ij} for $w \in D_V$ by setting $B_i = B^{(2)}(w)$. In the case when $|D_V| = 2$, first define X_{ij} for $w \in D_V$ by setting $B_i = B^{(2)}(w)$. Set $B_s = B^{(2)}(w')$ for $w' \in D_V - \{w\}$ and let $B_t \in \mathcal{B}^{(2)}$ be a T-connected block of B_s. Then define X_{st}, similarly to the definition of X_{ij}, by replacing X_{ij} and $X^{(2)}$ with X_{st} and X_{ij}, respectively. Now we set

$$X^{(3)} = \begin{cases} X_{ij} & \text{when } |D_V| = 1, \text{ and} \\ \\ X_{st} & \text{when } |D_V| = 2, \end{cases}$$

and clarify the relation between X and $X^{(3)}$. First,

$$|X_{ij}| \leq |X^{(2)}| - 1 \text{ and } |X_{st}| \leq |X_{ij}| - 1,$$

so that

$$|X^{(3)}| \leq |X^{(2)}| - |D_V|,$$

where $|D_V| \leq 2$. Secondly,

$$| x | = | x^{(1)} | - | D_V | \quad \text{and} \quad | x^{(1)} | \geq | x^{(2)} | \, ,$$

so that

$$| x | \gtrsim | x^{(3)} | \, .$$

We have

$$| x | = | x^{(3)} |$$

only if we have the following (i) and (ii):

(i) $| x^{(1)} | = | x^{(2)} | \, .$

(ii) $\{ u , v \} \cap x^{(2)} = \phi$ and $\{ u , v \} \cap x_{ij} = \phi$, respectively,

for the c-edges $e = (u , v)$ in the definition of x_{ij} and

of X_{st}.

If

$$| x^{(1)} | = | x^{(2)} |$$

then

$$| x^{(1)} \cap v | < | x^{(2)} \cap v |$$

as mentioned earlier. And (ii) implies that we have

$$| x_{ij} \cap v | = | x^{(2)} \cap v | + 1 \quad \text{and} \quad | x_{st} \cap v | = | x_{ij} \cap v | + 1 ,$$

so that

$$| x^{(3)} \cap v | = | x^{(2)} \cap v | + | D_V | \, .$$

By the definition of $x^{(1)}$, we have

$$| x^{(1)} \cap v | = | x \cap v | + | D_V | \, .$$

Therefore, if

$$| x | = | x^{(3)} |$$

then

$$| x \cap v | < | x^{(3)} \cap v | \, .$$

Let

$$G_3^{(3)} = G_2 - x^{(3)} \, .$$

Then, clearly, we have the following statements (a) and (b).

(a) $G_3^{(3)}$ is a nonseparable subgraph of the inner ring tree

$IR(G_2)$ and satisfies the following (i) and (ii):

(i) $\delta_{G_3^{(3)}}(v) = 2$ for $\forall v \in V_3^{(3)} \cap V$.

(ii) For any pair of distinct inner rings C and C' of G_2, there exists the π -connected sequence of C and C' in $G_3^{(3)}$.

(b) $|X| \geq |X^{(3)}|$, where if $|X| = |X^{(3)}|$ then $|X \cap V| < |X^{(3)} \cap V|$.

$G_3^{(3)}$ is essentially the same form as the G_2-X constructed in the first half of the proof of Lemma 2 , and it is easy to see that $G_3^{(3)}$ is s-p. Now the discussion so far is summarized as follows.

The proof of Lemma 10. Assume that either (1) or (2) of the lemma is false. Then we can define $X^{(3)} \subset V_1$ which is distinct from X and which satisfies the above statements (a) and (b), contradicting our choice of X.   Q.E.D.

4. Concluding remark.

The problems P_{PSPNR} and P_{SPNR} discussed in this paper and in [15], respectively, are not included in the class of problems with properties that are hereditary on induced subgraphs, while a large number of node-deletion NP-complete problems with hereditary properties have been presented by M. S. Krishnamoorthy and N. Deo [11], and by M. Yannakakis [17, 18].

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